



## COMMUTATORS OF PARAMETRIC MARCINKIEWICZ INTEGRALS ON GENERALIZED ORLICZ-MORREY SPACES

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ABSTRACT. In this paper, we study the boundedness of the commutators of parametric Marcinkiewicz integral operator with smooth kernel on generalized Orlicz-Morrey spaces.

### 1. INTRODUCTION

Suppose that  $S^{n-1}$  is the unit sphere in  $\mathbb{R}^n$  ( $n \geq 2$ ) equipped with the normalized Lebesgue measure  $d\sigma = d\sigma(x')$ . Let  $\Omega$  be a homogeneous function of degree zero on  $\mathbb{R}^n$  satisfying  $\Omega \in L^1(S^{n-1})$  and the following property

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0,$$

where  $x' = x/|x|$  for any  $x \neq 0$ .

The parametric Marcinkiewicz integral is defined by Hörmander [12] as follows:

$$\mu_{\Omega}^{\rho}(f)(x) = \left( \int_0^{\infty} \left| \frac{1}{t^{\rho}} \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y) dy \right|^2 \frac{dt}{t} \right)^{1/2},$$

where  $0 < \rho < n$ . When  $\rho = 1$ , we simply denote it by  $\mu_{\Omega}(f)$ . The operator  $\mu_{\Omega}(f)$  is defined by Stein in [17].

Let  $b$  be a locally integrable function on  $\mathbb{R}^n$ ; the commutator generated by the parametric Marcinkiewicz integral  $\mu_{\Omega}^{\rho}$  and  $b$  is defined by

$$\mu_{\Omega,b}^{\rho}(f)(x) = \left( \int_0^{\infty} \left| \frac{1}{t^{\rho}} \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} (b(x) - b(y)) f(y) dy \right|^2 \frac{dt}{t} \right)^{1/2}.$$

In [2], Deringöz et al. introduced generalized Orlicz-Morrey spaces as an extension of generalized Morrey spaces. Other definitions of generalized Orlicz-Morrey

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spaces can be found in [14] and [15]. In words of [10], our generalized Orlicz-Morrey space is the third kind and the ones in [14] and [15] are the first kind and the second kind, respectively. According to the examples in [6], one can say that the generalized Orlicz-Morrey spaces of the first kind and the second kind are different and that the second kind and the third kind are different. However, it is not known that relation between the first and the third kind.

Boundedness of commutators of classical operators of harmonic analysis on generalized Orlicz-Morrey spaces were recently studied in various papers, see for example [3, 8, 9]. In this paper, we consider the boundedness of commutator of parametric Marcinkiewicz integral operator on generalized Orlicz-Morrey space of the third kind.

Everywhere in the sequel  $B(x, r)$  is the ball in  $\mathbb{R}^n$  of radius  $r$  centered at  $x$  and  $|B(x, r)| = v_n r^n$  is its Lebesgue measure, where  $v_n$  is the volume of the unit ball in  $\mathbb{R}^n$ . By  $A \lesssim B$  we mean that  $A \leq CB$  with some positive constant  $C$  independent of appropriate quantities. If  $A \lesssim B$  and  $B \lesssim A$ , we write  $A \approx B$  and say that  $A$  and  $B$  are equivalent.

## 2. PRELIMINARIES

We recall the definition of Young functions.

**Definition 1.** A function  $\Phi : [0, \infty) \rightarrow [0, \infty]$  is called a Young function if  $\Phi$  is convex, left-continuous,  $\lim_{r \rightarrow +0} \Phi(r) = \Phi(0) = 0$  and  $\lim_{r \rightarrow \infty} \Phi(r) = \infty$ .

From the convexity and  $\Phi(0) = 0$  it follows that any Young function is increasing. If there exists  $s \in (0, \infty)$  such that  $\Phi(s) = \infty$ , then  $\Phi(r) = \infty$  for  $r \geq s$ . The set of Young functions such that

$$0 < \Phi(r) < \infty \quad \text{for} \quad 0 < r < \infty$$

will be denoted by  $\mathcal{Y}$ . If  $\Phi \in \mathcal{Y}$ , then  $\Phi$  is absolutely continuous on every closed interval in  $[0, \infty)$  and bijective from  $[0, \infty)$  to itself.

For a Young function  $\Phi$  and  $0 \leq s \leq \infty$ , let

$$\Phi^{-1}(s) = \inf\{r \geq 0 : \Phi(r) > s\}.$$

If  $\Phi \in \mathcal{Y}$ , then  $\Phi^{-1}$  is the usual inverse function of  $\Phi$ . We note that

$$\Phi(\Phi^{-1}(r)) \leq r \leq \Phi^{-1}(\Phi(r)) \quad \text{for } 0 \leq r < \infty.$$

It is well known that

$$r \leq \Phi^{-1}(r) \tilde{\Phi}^{-1}(r) \leq 2r \quad \text{for } r \geq 0, \tag{2.1}$$

where  $\tilde{\Phi}(r)$  is defined by

$$\tilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0, \infty)\} & , \quad r \in [0, \infty) \\ \infty & , \quad r = \infty. \end{cases}$$

A Young function  $\Phi$  is said to satisfy the  $\Delta_2$ -condition, denoted also as  $\Phi \in \Delta_2$ , if

$$\Phi(2r) \leq k\Phi(r) \text{ for } r > 0$$

for some  $k > 1$ . If  $\Phi \in \Delta_2$ , then  $\Phi \in \mathcal{Y}$ . A Young function  $\Phi$  is said to satisfy the  $\nabla_2$ -condition, denoted also by  $\Phi \in \nabla_2$ , if

$$\Phi(r) \leq \frac{1}{2k}\Phi(kr), \quad r \geq 0,$$

for some  $k > 1$ .

**Definition 2.** (*Orlicz Space*). For a Young function  $\Phi$ , the set

$$L^\Phi(\mathbb{R}^n) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < \infty \text{ for some } k > 0 \right\}$$

is called Orlicz space. If  $\Phi(r) = r^p$ ,  $1 \leq p < \infty$ , then  $L^\Phi(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ . If  $\Phi(r) = 0$ , ( $0 \leq r \leq 1$ ) and  $\Phi(r) = \infty$ , ( $r > 1$ ), then  $L^\Phi(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$ . The space  $L^\Phi_{\text{loc}}(\mathbb{R}^n)$  is defined as the set of all functions  $f$  such that  $f\chi_B \in L^\Phi(\mathbb{R}^n)$  for all balls  $B \subset \mathbb{R}^n$ .

$L^\Phi(\mathbb{R}^n)$  is a Banach space with respect to the norm

$$\|f\|_{L^\Phi(\mathbb{R}^n)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right)dx \leq 1 \right\}.$$

By elementary calculations we have the following.

**Lemma 1.** Let  $\Phi$  be a Young function and  $B$  be a set in  $\mathbb{R}^n$  with finite Lebesgue measure. Then

$$\|\chi_B\|_{L^\Phi(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}(|B|^{-1})}.$$

In the next sections where we prove our main estimates, we use the following lemma.

**Lemma 2.** [2] For a Young function  $\Phi$ , the following inequality is valid

$$\int_{B(x,r)} |f(y)|dy \leq 2|B(x,r)|\Phi^{-1}(|B(x,r)|^{-1})\|f\|_{L^\Phi(B(x,r))},$$

where  $\|f\|_{L^\Phi(B(x,r))} = \|f\chi_B\|_{L^\Phi(\mathbb{R}^n)}$ .

Various versions of generalized Orlicz-Morrey spaces were introduced in [14], [15] and [2]. We used the definition of [2] which runs as follows.

**Definition 3.** Let  $\varphi(x,r)$  be a positive measurable function on  $\mathbb{R}^n \times (0, \infty)$  and  $\Phi$  be any Young function. We denote by  $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$  the generalized Orlicz-Morrey space, the space of all functions  $f \in L^\Phi_{\text{loc}}(\mathbb{R}^n)$  for which

$$\|f\|_{\mathcal{M}^{\Phi,\varphi}} = \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x,r)^{-1}\Phi^{-1}(|B(x,r)|^{-1})\|f\|_{L^\Phi(B(x,r))} < \infty.$$

We recall the definition of the space of  $BMO(\mathbb{R}^n)$ .

**Definition 4.** Suppose that  $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ , let

$$\|b\|_* = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |b(y) - b_{B(x, r)}| dy,$$

where

$$b_{B(x, r)} = \frac{1}{|B(x, r)|} \int_{B(x, r)} b(y) dy.$$

Define

$$BMO(\mathbb{R}^n) = \{b \in L^1_{\text{loc}}(\mathbb{R}^n) : \|b\|_* < \infty\}.$$

To prove our theorems, we need the following lemmas.

**Lemma 3.** [13] Let  $b \in BMO(\mathbb{R}^n)$ . Then there is a constant  $C > 0$  such that

$$|b_{B(x, r)} - b_{B(x, t)}| \leq C \|b\|_* \ln \frac{t}{r} \quad \text{for } 0 < 2r < t,$$

where  $C$  is independent of  $b$ ,  $x$ ,  $r$ , and  $t$ .

**Lemma 4.** [8, 11] Let  $b \in BMO(\mathbb{R}^n)$  and  $\Phi$  be a Young function with  $\Phi \in \Delta_2$ , then

$$\|b\|_* \approx \sup_{x \in \mathbb{R}^n, r > 0} \Phi^{-1}(|B(x, r)|^{-1}) \|b(\cdot) - b_{B(x, r)}\|_{L^\Phi(B(x, r))}. \quad (2.2)$$

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w^* g(t) := \int_t^\infty \left(1 + \ln \frac{s}{t}\right) g(s) w(s) ds, \quad 0 < t < \infty,$$

where  $w$  is a weight.

**Lemma 5.** Let  $v_1, v_2$  and  $w$  be weights on  $(0, \infty)$  and  $v_1(t)$  be bounded outside a neighborhood of the origin. The inequality

$$\text{ess sup}_{t > 0} v_2(t) H_w^* g(t) \leq C \text{ess sup}_{t > 0} v_1(t) g(t) \quad (2.3)$$

holds for some  $C > 0$  for all non-negative and non-decreasing  $g$  on  $(0, \infty)$  if and only if

$$B := \text{ess sup}_{t > 0} v_2(t) \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \frac{w(s) ds}{\text{ess sup}_{s < \tau < \infty} v_1(\tau)} < \infty. \quad (2.4)$$

Moreover, the value  $C = B$  is the best constant for (2.3).

Note that, Lemma 5 is proved analogously to [7, Theorem 3.1].

**Remark 1.** In (2.3) and (2.4) it is assumed that  $\frac{1}{\infty} = 0$  and  $0 \cdot \infty = 0$ .

## 3. MAIN RESULTS

The following result concerning the boundedness of commutator of parametric Marcinkiewicz integral operator  $\mu_{\Omega,b}^\rho$  on  $L^p$  is known.

**Theorem A.** [16] *Suppose that  $1 < p, q < \infty$ ,  $\Omega \in L^q(S^{n-1})$ ,  $0 < \rho < n$  and  $b \in BMO(\mathbb{R}^n)$ . Then, there is a constant  $C$  independent of  $f$  such that*

$$\|\mu_{\Omega,b}^\rho(f)\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}.$$

The following interpolation result is from [5].

**Lemma 6.** *Let  $T$  be a sublinear operator of weak type  $(p, p)$  for any  $p \in (1, \infty)$ . Then  $T$  is bounded on  $L^\Phi(\mathbb{R}^n)$ , where  $\Phi$  is a Young function satisfying that  $\Phi \in \Delta_2 \cap \nabla_2$ .*

As a consequence of Lemma 6 and Theorem A, we get the following result.

**Corollary 1.** *Let  $\Phi$  be a Young function,  $b \in BMO(\mathbb{R}^n)$ ,  $\Omega \in L^q(S^{n-1})$  ( $q > 1$ ) and  $0 < \rho < n$ . If  $\Phi \in \Delta_2 \cap \nabla_2$ , then  $\mu_{\Omega,b}^\rho$  is bounded on  $L^\Phi(\mathbb{R}^n)$ .*

The following lemma is a generalization of the [1, Lemma 4.3] for Orlicz spaces.

**Lemma 7.** *Let  $\Phi$  be a Young function with  $\Phi \in \Delta_2 \cap \nabla_2$ ,  $b \in BMO(\mathbb{R}^n)$ ,  $B = B(x_0, r)$  and  $\Omega \in L^\infty(S^{n-1})$ . Then*

$$\|\mu_{\Omega,b}^\rho f\|_{L^\Phi(B)} \lesssim \frac{\|b\|_*}{\Phi^{-1}(|B|^{-1})} \int_r^\infty \left(1 + \ln \frac{t}{r}\right) \|f\|_{L^\Phi(B(x_0,t))} \Phi^{-1}(|B(x_0,t)|^{-1}) \frac{dt}{t} \quad (3.1)$$

holds for any ball  $B$ ,  $0 < \rho < n$ , and for all  $f \in L_{\text{loc}}^\Phi(\mathbb{R}^n)$ .

*Proof.* For  $B = B(x_0, r)$  and  $2B = B(x_0, 2r)$  write  $f = f_1 + f_2$  with  $f_1 = f\chi_{2B}$  and  $f_2 = f\chi_{\tau_{(2B)}}$ . Hence

$$\left\| \mu_{\Omega,b}^\rho f \right\|_{L^\Phi(B)} \leq \left\| \mu_{\Omega,b}^\rho f_1 \right\|_{L^\Phi(B)} + \left\| \mu_{\Omega,b}^\rho f_2 \right\|_{L^\Phi(B)}.$$

Since  $L^\infty(S^{n-1}) \not\subseteq L^q(S^{n-1})$  for  $1 \leq q < \infty$ , from the boundedness of  $\mu_{\Omega,b}^\rho$  in  $L^\Phi(\mathbb{R}^n)$  provided by Corollary 1, it follows that

$$\left\| \mu_{\Omega,b}^\rho f_1 \right\|_{L^\Phi(B)} \leq \left\| \mu_{\Omega,b}^\rho f_1 \right\|_{L^\Phi(\mathbb{R}^n)} \lesssim \|b\|_* \|f_1\|_{L^\Phi(\mathbb{R}^n)} = \|b\|_* \|f\|_{L^\Phi(2B)}. \quad (3.2)$$

It's clear that  $x \in B$ ,  $y \in {}^c(2B)$  implies  $\frac{1}{2}|x_0 - y| \leq |x - y| \leq \frac{3}{2}|x_0 - y|$ . Then by the Minkowski inequality and conditions on  $\Omega$ , we get

$$\begin{aligned} |\mu_{\Omega,b}^\rho(f_2)(x)| &\leq \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}} |f_2(y)| |b(x) - b(y)| \left( \int_{|x-y|}^{\infty} \frac{dt}{t^{1+2\rho}} \right)^{1/2} dy \\ &\lesssim \int_{{}^c(2B)} \frac{|\Omega(|x-y|\frac{x-y}{|x-y|})| |f(y)|}{|x-y|^n} |b(x) - b(y)| dy \\ &\lesssim \int_{{}^c(2B)} \frac{|f(y)|}{|x_0 - y|^n} |b(x) - b(y)| dy. \end{aligned}$$

Then

$$\begin{aligned} \|\mu_{\Omega,b}^\rho f_2\|_{L^\Phi(B)} &\lesssim \left\| \int_{{}^c(2B)} \frac{|b(y) - b(\cdot)|}{|x_0 - y|^n} |f(y)| dy \right\|_{L^\Phi(B)} \\ &\lesssim \left\| \int_{{}^c(2B)} \frac{|b(y) - b_B|}{|x_0 - y|^n} |f(y)| dy \right\|_{L^\Phi(B)} + \left\| \int_{{}^c(2B)} \frac{|b(\cdot) - b_B|}{|x_0 - y|^n} |f(y)| dy \right\|_{L^\Phi(B)} \\ &= I_1 + I_2. \end{aligned}$$

For the term  $I_1$  we have

$$\begin{aligned} I_1 &\approx \frac{1}{\Phi^{-1}(|B|^{-1})} \int_{{}^c(2B)} \frac{|b(y) - b_B|}{|x_0 - y|^n} |f(y)| dy \\ &\approx \frac{1}{\Phi^{-1}(|B|^{-1})} \int_{{}^c(2B)} |b(y) - b_B| |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy \\ &= \frac{1}{\Phi^{-1}(|B|^{-1})} \int_{2r}^{\infty} \int_{2r \leq |x_0-y| \leq t} |b(y) - b_B| |f(y)| dy \frac{dt}{t^{n+1}} \\ &\lesssim \frac{1}{\Phi^{-1}(|B|^{-1})} \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y) - b_B| |f(y)| dy \frac{dt}{t^{n+1}}. \end{aligned}$$

Hence

$$\begin{aligned} I_1 &\lesssim \frac{1}{\Phi^{-1}(|B|^{-1})} \int_{2r}^{\infty} \int_{B(x_0,t)} |b(y) - b_{B(x_0,t)}| |f(y)| dy \frac{dt}{t^{n+1}} \\ &\quad + \frac{1}{\Phi^{-1}(|B|^{-1})} \int_{2r}^{\infty} |b_{B(x_0,r)} - b_{B(x_0,t)}| \int_{B(x_0,t)} |f(y)| dy \frac{dt}{t^{n+1}}. \end{aligned}$$

Applying Hölder's inequality, by (2.1) and Lemmas 2, 3 and 4 we get

$$\begin{aligned} I_1 &\lesssim \frac{1}{\Phi^{-1}(|B|^{-1})} \int_{2r}^{\infty} \|b(\cdot) - b_{B(x_0,t)}\|_{L^{\tilde{\Phi}}(B(x_0,t))} \|f\|_{L^\Phi(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &\quad + \frac{1}{\Phi^{-1}(|B|^{-1})} \int_{2r}^{\infty} |b_{B(x_0,r)} - b_{B(x_0,t)}| \|f\|_{L^\Phi(B(x_0,t))} \Phi^{-1}(|B(x_0,t)|^{-1}) \frac{dt}{t} \end{aligned}$$

$$\lesssim \frac{\|b\|_*}{\Phi^{-1}(|B|^{-1})} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L^\Phi(B(x_0,t))} \Phi^{-1}(|B(x_0,t)|^{-1}) \frac{dt}{t}.$$

For  $I_2$  we obtain

$$I_2 \approx \|b(\cdot) - b_B\|_{L^\Phi(B)} \int_{\mathfrak{c}(2B)} \frac{|f(y)|}{|x_0 - y|^n} dy.$$

By Lemma 4, we get

$$I_2 \lesssim \frac{\|b\|_*}{\Phi^{-1}(|B|^{-1})} \int_{\mathfrak{c}(2B)} \frac{|f(y)|}{|x_0 - y|^n} dy. \quad (3.3)$$

To the remaining integral we use the same trick as above in the estimation of  $I_1$  :

$$\begin{aligned} \int_{\mathfrak{c}(2B)} \frac{|f(y)|}{|x_0 - y|^n} dy &= n \int_{\mathfrak{c}(2B)} |f(y)| \int_{|x_0 - y|}^{\infty} \frac{dt}{t^{n+1}} dy \\ &= n \int_{2r}^{\infty} \int_{2r \leq |x_0 - y| < t} |f(y)| dy \frac{dt}{t^{n+1}} \leq n \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| dy \frac{dt}{t^{n+1}} \end{aligned}$$

and by Lemma 2 we then get

$$\int_{\mathfrak{c}(2B)} \frac{|f(y)|}{|x_0 - y|^n} dy \lesssim \int_{2r}^{\infty} \|f\|_{L^\Phi(B(x_0,t))} \Phi^{-1}(|B(x_0,t)|^{-1}) \frac{dt}{t}. \quad (3.4)$$

Therefore, by (3.4) and (3.3) we have

$$I_2 \lesssim \frac{\|b\|_*}{\Phi^{-1}(|B|^{-1})} \int_{2r}^{\infty} \|f\|_{L^\Phi(B(x_0,t))} \Phi^{-1}(|B(x_0,t)|^{-1}) \frac{dt}{t}$$

gathering estimates for  $I_1$  and  $I_2$ , we obtain

$$\|\mu_{\Omega,b}^\rho f_2\|_{L^\Phi(B)} \lesssim \frac{\|b\|_*}{\Phi^{-1}(|B|^{-1})} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L^\Phi(B(x_0,t))} \Phi^{-1}(|B(x_0,t)|^{-1}) \frac{dt}{t}. \quad (3.5)$$

Now collect the estimates (3.2) and (3.5):

$$\begin{aligned} \|\mu_{\Omega,b}^\rho f\|_{L^\Phi(B)} &\lesssim \|b\|_* \|f\|_{L^\Phi(2B)} \\ &\quad + \frac{\|b\|_*}{\Phi^{-1}(|B|^{-1})} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L^\Phi(B(x_0,t))} \Phi^{-1}(|B(x_0,t)|^{-1}) \frac{dt}{t}. \end{aligned}$$

To finalize the proof, it remains to note that the first term here may be estimated in the form similar to the second one:

$$\|f\|_{L^\Phi(2B)} \leq \frac{C}{\Phi^{-1}(|B|^{-1})} \int_{2r}^{\infty} \|f\|_{L^\Phi(B(x_0,t))} \Phi^{-1}(|B(x_0,t)|^{-1}) \frac{dt}{t}. \quad (3.6)$$

To prove (3.6), observe that since  $\Phi^{-1}$  is concave and nonnegative it follows that  $\Phi^{-1}(u) \leq \frac{u}{v} \Phi^{-1}(v)$  for  $u \geq v$ , whence

$$\Phi^{-1}(|B|^{-1}) \leq \frac{t^n}{r^n} \Phi^{-1}(|B(x_0,t)|^{-1}), \quad r \leq t.$$

Then

$$\Phi^{-1}(|B|^{-1}) = n\Phi^{-1}(|B|^{-1})(2r)^n \int_{2r}^{\infty} \frac{dt}{t^{n+1}} \lesssim \int_{2r}^{\infty} \Phi^{-1}(|B(x_0, t)|^{-1}) \frac{dt}{t},$$

from which (3.6) follows by the monotonicity of the norm  $\|f\|_{L^\Phi(B(x_0, t))}$  with respect to  $t$ , and this completes the proof.  $\square$

**Theorem 1.** *Let  $0 < \rho < n$ ,  $b \in BMO(\mathbb{R}^n)$ ,  $\Phi$  be any Young function,  $\varphi_1, \varphi_2$  and  $\Phi$  satisfy the condition*

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \left(\operatorname{ess\,inf}_{t < s < \infty} \frac{\varphi_1(x, s)}{\Phi^{-1}(|B(x, s)|^{-1})}\right) \Phi^{-1}(|B(x, t)|^{-1}) \frac{dt}{t} \leq C \varphi_2(x, r), \quad (3.7)$$

where  $C$  does not depend on  $x$  and  $r$ . Let also  $\Omega \in L^\infty(S^{n-1})$ . If  $\Phi \in \Delta_2 \cap \nabla_2$ , then the operator  $\mu_{\Omega, b}^\rho$  is bounded from  $\mathcal{M}^{\Phi, \varphi_1}(\mathbb{R}^n)$  to  $\mathcal{M}^{\Phi, \varphi_2}(\mathbb{R}^n)$ .

*Proof.* The proof follows from the Lemmas 5 and 7. We can also give the following alternative proof for Theorem 1 by inspiring the ideas in [4].

Since  $f \in \mathcal{M}^{\Phi, \varphi_1}(\mathbb{R}^n)$  and  $\varphi_1, \varphi_2$  and  $\Phi$  satisfy the condition (3.7), we have

$$\begin{aligned} & \int_r^\infty \left(1 + \ln \frac{t}{r}\right) \|f\|_{L^\Phi(B(x, t))} \Phi^{-1}(|B(x, t)|^{-1}) \frac{dt}{t} \\ & \approx \int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\|f\|_{L^\Phi(B(x, t))}}{\operatorname{ess\,inf}_{t < s < \infty} \frac{\varphi_1(x, s)}{\Phi^{-1}(|B(x, s)|^{-1})}} \operatorname{ess\,inf}_{t < s < \infty} \frac{\varphi_1(x, s)}{\Phi^{-1}(|B(x, s)|^{-1})} \Phi^{-1}(|B(x, t)|^{-1}) \frac{dt}{t} \\ & \lesssim \|f\|_{\mathcal{M}^{\Phi, \varphi_1}} \int_r^\infty \left(1 + \ln \frac{t}{r}\right) \operatorname{ess\,inf}_{t < s < \infty} \frac{\varphi_1(x, s)}{\Phi^{-1}(|B(x, s)|^{-1})} \Phi^{-1}(|B(x, t)|^{-1}) \frac{dt}{t} \\ & \lesssim \|f\|_{\mathcal{M}^{\Phi, \varphi_1}} \varphi_2(x, r). \end{aligned}$$

Then from (3.1) we get

$$\begin{aligned} \|\mu_{\Omega, b}^\rho\|_{\mathcal{M}^{\Phi, \varphi_2}} &= \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(x, r)^{-1} \Phi^{-1}(|B(x, r)|^{-1}) \|\mu_{\Omega, b}^\rho\|_{L^\Phi(B(x, r))} \\ &\lesssim \|b\|_* \sup_{x \in \mathbb{R}^n, r > 0} \varphi_2(x, r)^{-1} \int_r^\infty \left(1 + \ln \frac{t}{r}\right) \|f\|_{L^\Phi(B(x, t))} \Phi^{-1}(|B(x, t)|^{-1}) \frac{dt}{t} \\ &\lesssim \|b\|_* \|f\|_{\mathcal{M}^{\Phi, \varphi_1}} \end{aligned}$$

$\square$

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