



CURVES OF CONSTANT BREADTH ACCORDING TO DARBOUX FRAME

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ABSTRACT. In this paper, we investigate constant breadth curves on a surface according to Darboux frame and give some characterizations of these curves.

1. INTRODUCTION

Since the first introduction of constant breadth curves in the plane by L. Euler in 1778 [3], many researchers focused on this subject and found out a lot of interesting properties about constant breadth curves in the plane [9], [2], [12]. Fujiwara [4] has introduced constant breadth curves, by taking a closed curve whose normal plane at a point P has only one more point Q in common with the curve and for which the distance $d(P, Q)$ is constant.

After the development of cam design, researchers have been shown a strong interest to this subject again and many interesting properties have been discovered. For example Köse has defined a new concept called space curve pair of constant breadth in [7], a pair of unit speed space curves of class C^3 with non-vanishing curvature and torsion in E^3 , which have parallel tangents in opposite directions at corresponding points, and the distance between these points is always constant by using the Frenet frame.

Many authors have been studied spacelike and timelike curves of constant breadth in Minkowski 3-space [5, 10, 13]. Furthermore Akdoğan and Mağden generalized in n dimensional Euclidean space [1].

The characterizations of Köse's paper [6] on constant breadth curves in the space has led us to investigate this topic according to Darboux frame on a surface.

2. BASIC CONCEPTS

Now, we introduce some basic concepts about our study. Let M be an oriented surface and β be a unit speed curve of class C^3 on M . As we know, β has a natural

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frame called Frenet frame $\{T, N, B\}$ with properties below:

$$\begin{aligned} T' &= \varkappa N, \\ N' &= -\varkappa T + \tau B, \\ B' &= -\tau N, \end{aligned} \tag{1}$$

where \varkappa is the curvature, τ is the torsion, T is the unit tangent vector field, N is the principal normal vector field and B is the binormal vector field of the curve β .

By using the unit tangent vector field of the curve β and the unit normal vector field of the surface M on the curve β we define unit vector field g as $g = (n \circ \beta) \times T$, where \times cross product. We will have a new frame called Darboux frame $\{T, g, n \circ \beta\}$. The relations between these two frames can be given as follows:

$$\begin{bmatrix} T \\ g \\ n \circ \beta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2}$$

where α is the angle between the vector fields $n \circ \beta$ and B . If we take the derivatives of T, g, n with respect to s , we will have

$$\begin{bmatrix} T' \\ g' \\ (n \circ \beta)' \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & t_g \\ -k_n & -t_g & 0 \end{bmatrix} \begin{bmatrix} T \\ g \\ n \circ \beta \end{bmatrix} \tag{3}$$

where k_g, k_n and t_g are called the geodesic curvature, the normal curvature and the geodesic torsion respectively. Then, we will have following relations [11].

$$\begin{aligned} k_g &= \varkappa \cos \alpha, \\ k_n &= \varkappa \sin \alpha, \\ t_g &= \tau - \alpha'. \end{aligned} \tag{4}$$

In the differential geometry of surfaces, For a curve $\beta(s)$ lying on a surface, there are following cases:

- i) β is a geodesic curve if and only if $k_g = 0$.
- ii) β is an asymptotic line if and only if $k_n = 0$.
- iii) β is a principal line if and only if $t_g = 0$.

3. CURVES OF CONSTANT BREADTH ACCORDING TO DARBOUX FRAME

Let $\beta(s)$ and $\beta^*(s^*)$ be a pair of unit speed curves of class C^3 with non-vanishing curvature and torsion in E^3 which have parallel tangents in opposite directions at corresponding points and the distance between these points is always constant.

If β lies on a surface, it has Darboux frame in addition to Frenet frame with properties (1), (2), (3) and (4). So we may write for β^*

$$\beta^*(s^*) = \beta(s) + m_1(s)T(s) + m_2(s)g(s) + m_3(s)(n \circ \beta)(s).$$

If we differentiate this equation with respect to s and use (3), we will have

$$(\beta^*)' = \frac{d\beta^*}{ds^*} \frac{ds^*}{ds} = (1 + m'_1 - m_2 k_g - m_3 k_n) T + (m_1 k_g + m'_2 - m_3 t_g) g \quad (5)$$

$$+ (m_1 k_n + m_2 t_g + m'_3) (n \circ \beta)$$

and

$$\frac{d\beta^*}{ds} = \frac{d\beta^*}{ds^*} \frac{ds^*}{ds} = T^* \frac{ds^*}{ds}$$

As we know $\langle T, T^* \rangle = -1$. Then

$$-\frac{ds^*}{ds} = 1 + m'_1 - m_2 k_g - m_3 k_n.$$

So we find from (5),

$$\begin{aligned} m'_1 &= m_2 k_g + m_3 k_n - 1 - \frac{ds^*}{ds}, \\ m'_2 &= m_3 t_g - m_1 k_g, \\ m'_3 &= -m_1 k_n - m_2 t_g. \end{aligned} \quad (6)$$

Let us denote the angle between the tangents at the points $\beta(s)$ and $\beta(s + \Delta s)$ with $\Delta\theta$. If we denote the vector $T(s + \Delta s) - T(s)$ with ΔT , we know $\lim_{\Delta s \rightarrow 0} \frac{\Delta T}{\Delta s} = \lim_{\Delta s \rightarrow 0} \frac{\Delta\theta}{\Delta s} = \frac{d\theta}{ds} = \varkappa$. We called the angle of contingency to the angle $\Delta\theta$ [12]. Let us denote the differentiation with respect to θ with "·". By using the equation $\frac{d\theta}{ds} = \varkappa$, we can write (6) as follows:

$$\begin{aligned} \dot{m}_1 &= \rho(m_2 k_g + m_3 k_n) - f(\theta), \\ \dot{m}_2 &= \rho(m_3 t_g - m_1 k_g), \\ \dot{m}_3 &= \rho(-m_1 k_n - m_2 t_g), \end{aligned} \quad (7)$$

where $\rho = \frac{1}{\varkappa}$, $\rho^* = \frac{1}{\varkappa^*}$ and $\rho + \rho^* = f(\theta)$.

Now we investigate curves of constant breadth according to Darboux frame for some special cases:

3.1. Case (For geodesic curves). Let β be non straight line geodesic curve on a surface. Then $k_g = \varkappa \cos \alpha = 0$ and $\varkappa \neq 0$, we get $\cos \alpha = 0$. So it implies that $k_n = \varkappa$, $t_g = \tau$. By using (7), we have following differential equation system

$$\begin{aligned} \dot{m}_1 &= m_3 - f(\theta), \\ \dot{m}_2 &= m_3 \varphi, \\ \dot{m}_3 &= -m_1 - m_2 \varphi, \end{aligned} \quad (8)$$

where $\varphi = \frac{\tau}{\varkappa}$. By using (8), we obtain a differential equation as follows:

$$\left(\ddot{m}_1 + \dot{f} \right) - \frac{d\varphi}{d\theta} \frac{1}{\varphi} \left(\ddot{m}_1 + m_1 + \dot{f} \right) + (1 + \varphi^2) \dot{m}_1 + \varphi^2 f = 0. \quad (9)$$

We assume that (β^*, β) a pair of curve is constant breadth , then

$$\|\beta^* - \beta\|^2 = m_1^2 + m_2^2 + m_3^2 = \text{constant}$$

which implies that

$$m_1\dot{m}_1 + m_2\dot{m}_2 + m_3\dot{m}_3 = 0. \quad (10)$$

By combining (8) and (10) then we get

$$m_1 f(\theta) = 0.$$

3.1.1. *Case $f(\theta) = 0$.* We assume that $f(\theta) = 0$. By using (9), we get

$$\ddot{m}_1 - \frac{d\varphi}{d\theta} \frac{1}{\varphi} (\dot{m}_1 + m_1) + (1 + \varphi^2) \dot{m}_1 = 0. \quad (11)$$

If β is a helix curve then $\varphi = \varphi_0 = \text{constant}$. From (11), we have

$$\ddot{m}_1 + (1 + \varphi_0^2) \dot{m}_1 = 0$$

whose the solution is

$$m_1 = \frac{1}{\sqrt{1 + \varphi_0^2}} \left(c_1 \sin \left(\sqrt{1 + \varphi_0^2} \theta \right) - c_2 \cos \left(\sqrt{1 + \varphi_0^2} \theta \right) \right) + c_3,$$

where c_1 , c_2 and c_3 are real constants. By using (8), we can find as $m_2 = -\frac{1}{\varphi_0} (m_1 + \dot{m}_1)$ and $m_3 = \dot{m}_1$. In that case we can compute m_2 and m_3 as follows:

$$m_2 = \frac{\varphi_0}{\sqrt{1 + \varphi_0^2}} \left(c_1 \sin \left(\sqrt{1 + \varphi_0^2} \theta \right) - c_2 \cos \left(\sqrt{1 + \varphi_0^2} \theta \right) \right) + \varphi_0 c_3$$

and

$$m_3 = c_1 \cos \left(\sqrt{1 + \varphi_0^2} \theta \right) + c_2 \sin \left(\sqrt{1 + \varphi_0^2} \theta \right).$$

If $f(\theta) = 0$ then from (8), it can easily seen that the vector $d = m_1(s)T(s) + m_2(s)g(s) + m_3(s)(n \circ \beta)(s)$ is a constant vector. In that case the curve β^* is the translation of the curve β along the vector d .

3.1.2. *Case $m_1 = 0$.* We assume that $m_1 = 0$, then by using (9), we get

$$\ddot{f} - \frac{d\varphi}{d\theta} \frac{1}{\varphi} \dot{f} + \varphi^2 f = 0. \quad (12)$$

If β is a helix curve, then $\varphi = \varphi_0 = \text{constant}$. From (12) we obtain

$$\ddot{f} + \varphi_0^2 f = 0$$

whose the solution is

$$f(\theta) = c_1 \cos(\varphi_0 \theta) + c_2 \sin(\varphi_0 \theta).$$

Since $m_1 = 0$, (8) implies that $m_3 = f(\theta)$ and $m_2 = -\frac{\dot{f}(\theta)}{\varphi_0}$. In that case we can compute m_2 and m_3 as follows:

$$m_2 = c_1 \sin(\varphi_0 \theta) - c_2 \cos(\varphi_0 \theta)$$

and

$$m_3 = c_1 \cos(\varphi_0 \theta) + c_2 \sin(\varphi_0 \theta).$$

Theorem 1. *Let β be a geodesic curve and a helix curve. Let (β, β^*) be a pair of constant breadth curve. In that case β^* can be expressed as one of the following cases:*

i)

$$\beta^*(s^*) = \beta(s) + m_1(s)T(s) - \frac{1}{\varphi_0} (m_1(s) + \ddot{m}_1(s))g(s) + \dot{m}_1(s)(n \circ \beta)(s),$$

$$\text{where } m_1 = \frac{1}{\sqrt{1+\varphi_0^2}} \left(c_1 \sin(\sqrt{1+\varphi_0^2}\theta) - c_2 \cos(\sqrt{1+\varphi_0^2}\theta) \right) + c_3.$$

ii)

$$\beta^*(s^*) = \beta(s) - \frac{\dot{f}(\theta)}{\varphi_0} g(s) + f(\theta)(n \circ \beta)(s),$$

$$\text{where } f(\theta) = c_1 \cos(\varphi_0 \theta) + c_2 \sin(\varphi_0 \theta).$$

3.2. Case (For asymptotic lines). Let β be non straight line asymptotic line on a surface. Then $k_n = \varkappa \sin \alpha = 0$ and $\varkappa \neq 0$, we have $\sin \alpha = 0$. So we get $k_g = \varepsilon \varkappa$, $t_g = \tau$, where $\varepsilon = \pm 1$. By using (7), we have following differential equation system

$$\begin{aligned} \dot{m}_1 &= \varepsilon m_2 - f(\theta), \\ \dot{m}_2 &= m_3 \varphi - \varepsilon m_1, \\ \dot{m}_3 &= -m_2 \varphi, \end{aligned} \quad (13)$$

where $\varphi = \frac{\tau}{\varkappa}$. By using (13), we obtain a differential equation as follows:

$$\left(\ddot{m}_1 + \dot{f} \right) - \frac{d\varphi}{d\theta} \frac{1}{\varphi} \left(\ddot{m}_1 + m_1 + \dot{f} \right) + (1 + \varphi^2) \dot{m}_1 + \varphi^2 f = 0. \quad (14)$$

We assume that (β^*, β) a pair of curve is constant breadth then

$$\|\beta^* - \beta\|^2 = m_1^2 + m_2^2 + m_3^2 = \text{constant}$$

which implies that

$$m_1 \dot{m}_1 + m_2 \dot{m}_2 + m_3 \dot{m}_3 = 0. \quad (15)$$

By combining (13) and (15) then we get

$$m_1 f(\theta) = 0.$$

3.2.1. Case $f(\theta) = 0$. We assume that $f(\theta) = 0$. By using (14), we get

$$\ddot{m}_1 - \frac{d\varphi}{d\theta} \frac{1}{\varphi} (\ddot{m}_1 + m_1) + (1 + \varphi^2) \dot{m}_1 = 0. \quad (16)$$

If β is a helix curve then $\varphi = \varphi_0 = \text{constant}$. From (16), we have

$$\ddot{m}_1 + (1 + \varphi_0^2) \dot{m}_1 = 0$$

whose the solution is

$$m_1 = \frac{1}{\sqrt{1+\varphi_0^2}} \left(c_1 \sin \left(\sqrt{1+\varphi_0^2} \theta \right) - c_2 \cos \left(\sqrt{1+\varphi_0^2} \theta \right) \right) + c_3,$$

where c_1, c_2 and c_3 are real constants. By using (13), we can find as $m_2 = \varepsilon \dot{m}_1$ and $m_3 = \varepsilon \frac{1}{\varphi_0} (m_1 + \ddot{m}_1)$. In that case we can compute m_2 and m_3 as follows:

$$m_2 = \varepsilon \left(c_1 \cos \left(\sqrt{1+\varphi_0^2} \theta \right) + c_2 \sin \left(\sqrt{1+\varphi_0^2} \theta \right) \right)$$

and

$$m_3 = -\frac{\varepsilon \varphi_0}{\sqrt{1+\varphi_0^2}} \left(c_1 \sin \left(\sqrt{1+\varphi_0^2} \theta \right) - c_2 \cos \left(\sqrt{1+\varphi_0^2} \theta \right) \right) - \varepsilon \varphi_0 c_3.$$

If $f(\theta) = 0$ then from (13), it can easily seen that the vector $d = m_1(s)T(s) + m_2(s)g(s) + m_3(s)(n \circ \beta)(s)$ is a constant vector. In that case the curve β^* is the translation of the curve β along the vector d .

3.2.2. *Case $m_1 = 0$.* We assume that $m_1 = 0$. Then by using (14), we get

$$\ddot{f} - \frac{d\varphi}{d\theta} \frac{1}{\varphi} \dot{f} + \varphi^2 f = 0 \tag{17}$$

If β is a helix curve, then $\varphi = \varphi_0 = \text{constant}$. From (17) we obtain

$$\ddot{f} + \varphi_0^2 f = 0$$

whose the solution is

$$f(\theta) = c_1 \cos(\varphi_0 \theta) + c_2 \sin(\varphi_0 \theta)$$

Since $m_1 = 0$, (13) implies that $m_2 = \varepsilon f(\theta)$ and $m_3 = \varepsilon \frac{\dot{f}(\theta)}{\varphi_0}$. In that case we can compute m_2 and m_3 as follows:

$$m_2 = \varepsilon (c_1 \cos(\varphi_0 \theta) + c_2 \sin(\varphi_0 \theta))$$

and

$$m_3 = \varepsilon (-c_1 \sin(\varphi_0 \theta) + c_2 \cos(\varphi_0 \theta)).$$

Theorem 2. *Let β be an asymptotic line and a helix curve. Let (β, β^*) be a pair of constant breadth curve. In that case β^* can be expressed as one of the following cases:*

i)

$$\beta^*(s^*) = \beta(s) + m_1(s)T(s) + \varepsilon \dot{m}_1(s)g(s) + \frac{\varepsilon}{\varphi_0} (m_1(s) + \ddot{m}_1(s))(n \circ \beta)(s),$$

where $m_1 = \frac{1}{\sqrt{1+\varphi_0^2}} \left(c_1 \sin \left(\sqrt{1+\varphi_0^2} \theta \right) - c_2 \cos \left(\sqrt{1+\varphi_0^2} \theta \right) \right) + c_3$.

ii)

$$\beta^*(s^*) = \beta(s) + \varepsilon f(\theta)g(s) + \varepsilon \frac{\dot{f}(\theta)}{\varphi_0} (n \circ \beta)(s),$$

where $f(\theta) = c_1 \cos(\varphi_0\theta) + c_2 \sin(\varphi_0\theta)$.

3.3. Case (For principal line). We assume that β is a principal line. Then we have $t_g = 0$ and it implies that $\tau = \alpha'$. By using (7), we get

$$\begin{aligned} \dot{m}_1 &= m_2 \cos \alpha + m_3 \sin \alpha - f(\theta), \\ \dot{m}_2 &= -m_1 \cos \alpha, \\ \dot{m}_3 &= -m_1 \sin \alpha. \end{aligned} \quad (18)$$

By using (18), we obtain following differential equation

$$(\ddot{m}_1 + \dot{m}_1) + (\dot{m}_1 + f) \dot{\alpha}^2 - \left(\sin \alpha \int m_1 \cos \alpha d\theta - \cos \alpha \int m_1 \sin \alpha d\theta \right) \ddot{\alpha} + \ddot{f} = 0. \quad (19)$$

Since $t_g = 0$ we obtain $\dot{\alpha} = \frac{\tau}{\kappa}$. We assume that (β^*, β) a pair of curve is constant breadth. In that case

$$\|\beta^* - \beta\|^2 = m_1^2 + m_2^2 + m_3^2 = \text{constant}$$

which implies that

$$m_1 \dot{m}_1 + m_2 \dot{m}_2 + m_3 \dot{m}_3 = 0. \quad (20)$$

By combining (18) and (20) then we get

$$m_1 f(\theta) = 0.$$

3.3.1. Case $f(\theta) = 0$. We assume that $f(\theta) = 0$. By using (19), we get

$$(\ddot{m}_1 + \dot{m}_1) + \dot{m}_1 \dot{\alpha}^2 - \left(\sin \alpha \int m_1 \cos \alpha d\theta - \cos \alpha \int m_1 \sin \alpha d\theta \right) \ddot{\alpha} = 0. \quad (21)$$

If β is a helix curve then $\dot{\alpha} = \frac{\tau}{\kappa} = \text{constant}$. From (21), we have

$$\ddot{m}_1 + (1 + \dot{\alpha}^2) \dot{m}_1 = 0.$$

Then we get

$$m_1(s) = \frac{1}{\sqrt{1 + \dot{\alpha}^2}} \left(c_1 \sin \left(\sqrt{1 + \dot{\alpha}^2} \theta \right) - c_2 \cos \left(\sqrt{1 + \dot{\alpha}^2} \theta \right) \right) + c_3,$$

where c_1, c_2 and c_3 are real constants. By using (18) we obtain

$$\begin{aligned} m_2 &= - \int m_1 \cos \alpha d\theta, \\ m_3 &= - \int m_1 \sin \alpha d\theta, \end{aligned}$$

where $\alpha = \int \tau ds$.

If $f(\theta) = 0$ then from (18), it can easily seen that the vector $d = m_1(s)T(s) + m_2(s)g(s) + m_3(s)(n \circ \beta)(s)$ is a constant vector. In that case the curve β^* is the translation of the curve β along the vector d .

Theorem 3. *Let β be a principal line and a helix curve. Let (β, β^*) be a pair of constant breadth curve such that $\langle \beta^* - \beta, T \rangle = m_1 \neq 0$. In that case β^* can be expressed as:*

$$\beta^* = \beta + m_1(s)T(s) - \left(\int m_1(s) \cos \alpha d\theta \right) g(s) - \left(\int m_1(s) \sin \alpha d\theta \right) (n \circ \beta)(s),$$

where $m_1(s) = \frac{1}{\sqrt{1+\dot{\alpha}^2}} \left(c_1 \sin \left(\sqrt{1+\dot{\alpha}^2} \theta \right) - c_2 \cos \left(\sqrt{1+\dot{\alpha}^2} \theta \right) \right) + c_3$.

3.3.2. *Case $m_1 = 0$.* If $m_1 = 0$, then from (19)

$$\ddot{f} + \dot{\alpha}^2 f = 0, \tag{22}$$

where $\dot{\alpha} = \frac{\tau}{\rho}$. On the other hand since $m_1 = 0$ from (18) we have $m_2 = c_2 = \text{constant}$, $m_3 = c_3 = \text{constant}$ and

$$f = c_2 \cos \alpha + c_3 \sin \alpha. \tag{23}$$

By combining (22) and (23)

$$\ddot{\alpha} (-c_2 \sin \alpha + c_3 \cos \alpha) = 0.$$

In that case, if $\ddot{\alpha} = 0$ then we obtain that $\dot{\alpha} = \frac{\tau}{\rho} = \text{constant}$. β becomes a helix curve. If $-c_2 \sin \alpha + c_3 \cos \alpha = 0$ then we have $\alpha = \text{constant}$. This means that β is a planar curve.

Theorem 4. *Let β be a principal line. Let (β, β^*) be a pair of constant breadth curve such that $\langle \beta^* - \beta, T \rangle = m_1 = 0$. In that case β is a helix curve or a planar curve and β^* can be expressed as:*

$$\beta^* = \beta + c_2 g(s) + c_3 (n \circ \beta)(s)$$

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