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Generalized Bicomplex Numbers and Lie Groups

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Abstract. In this paper, we define the generalized bicomplex numbers and give some algebraic properties of them. Also, we show that some hyperquadrics in $\mathbb{R}^4$ and $\mathbb{R}_2^4$ are Lie groups by using generalized bicomplex number product and obtain Lie algebras of these Lie groups. Moreover, by using tensor product surfaces, we determine some special Lie subgroups of these hyperquadrics.

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1. Introduction

In mathematics, a Lie group is a group which is also a differentiable manifold with the property that the group operations are differentiable. To establish group structure on the surface is quite difficult. Even if the spheres that admit the structure of a Lie group are only the 0-sphere S^0 (real numbers with absolute value 1), the circle S^1 (complex numbers with absolute value 1), the 3-sphere S^3 (the set of quaternions of unit form) and S^7 [3]. A manifold M carrying n linearly independent non-vanishing vector fields is called parallelisable and a Lie group is parallelisable. For even $n > 1$ S^n is not a Lie group because it can not be parallelisable as a differentiable manifold. Thus S^n is parallelisable if and only $n = 0, 1, 3, 7$.

In [5] and [6], Mihai et al. deal with tensor product surfaces of Euclidean planar curves and Lorentzian planar curves, respectively. Özkaldı and Yaylı [7] showed that a hyperquadric P in $\mathbb{R}^4$ is a Lie group by using bicomplex number product. They determined some special subgroups of this Lie group P, by using the tensor product surfaces of Euclidean planar curves. Karakuş and Yaylı [4] showed that a hyperquadric Q in $\mathbb{R}_2^4$ is a Lie group by using bicomplex number product. They changed the rule of tensor product and they gave a new tensor product rule in $\mathbb{R}_2^4$. By means of the tensor product

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surfaces of a Lorentzian plane curve and a Euclidean plane curve, they determined some special subgroups of this Lie group Q. In [1,2], by using curves and surfaces which are obtained by homothetic motion, were obtained some special subgroups of these Lie groups P and Q, respectively. In [8], Ölmez studied on generalized quaternions and applications.

In this paper, we define the generalized bicomplex numbers and give some algebraic properties of them. Also, we show that some hyperquadrics in $\mathbb{R}^4$ and $\mathbb{R}^4_2$ are Lie groups by using generalized bicomplex number product and obtain Lie algebras of these Lie groups. Morever, by means of tensor product surfaces, we determine some special Lie subgroups of these hyperquadrics and obtain left invariant vector fields of these tensor product surfaces which are Lie groups.

2. Preliminaries

Bicomplex number is defined by the basis $\{1, i, j, ij\}$ where i, j, ij satisfy $i^2 = -1$, $j^2 = -1$, $ij = ji$. Thus any bicomplex number x can be expressed as $x = x_1 1 + x_2 i + x_3 j + x_4 ij$, $\forall x_1, x_2, x_3, x_4 \in \mathbb{R}$. We denote the set of bicomplex numbers by C_2. For any $x = x_1 1 + x_2 i + x_3 j + x_4 ij$ and $y = y_1 1 + y_2 i + y_3 j + y_4 ij$ in C_2 the bicomplex number addition is defined as

$$x + y = (x_1 + y_1) + (x_2 + y_2)i + (x_3 + y_3)j + (x_4 + y_4)ij.$$

The multiplication of a bicomplex number $x = x_1 1 + x_2 i + x_3 j + x_4 ij$ by a real scalar λ is defined as

$$\lambda x = \lambda x_1 1 + \lambda x_2 i + \lambda x_3 j + \lambda x_4 ij.$$

With this addition and scalar multiplication, C_2 is a real vector space.

Bicomplex number product, denoted by $\times$, over the set of bicomplex numbers C_2 is given by

$$\begin{aligned} x \times y = {} & (x_1 y_1 - x_2 y_2 - x_3 y_3 + x_4 y_4) + (x_1 y_2 + x_2 y_1 - x_3 y_4 - x_4 y_3)i \\ & + (x_1 y_3 + x_3 y_1 - x_2 y_4 - x_4 y_2)j + (x_1 y_4 + x_4 y_1 + x_2 y_3 + x_3 y_2)ij. \end{aligned}$$

Vector space C_2 together with the bicomplex number product $\times$ is a real algebra [9].

Since the bicomplex algebra is associative, it can be considered in terms of matrices. Consider the set of matrices

$$M = \left\{ \begin{pmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}; \quad x_i \in \mathbb{R}, \quad 1 \leq i \leq 4 \right\}.$$

The set M together with matrix addition and scalar matrix multiplication is a real vector space. Furthermore, the vector space together with matrix product is an algebra.

The transformation

$$g : C_2 \to M$$

given by

$$g(x = x_1 1 + x_2 i + x_3 j + x_4 ij) = \begin{pmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & -x_4 & -x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}$$

is one to one and onto. Morever $\forall x, y \in C_2$ and $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} g(x+y) &= g(x) + g(y) \\ g(\lambda x) &= \lambda g(x) \\ g(xy) &= g(x)g(y). \end{aligned}$$

Thus the algebras C_2 and M are isomorphic.

Let $x \in C_2$. Then x can be expressed as $x = (x_1 + x_2 i) + (x_3 + x_4 i)j$. In that case, there is three different conjugations for bicomplex numbers as follows:

$$\begin{aligned} x^{t_i} &= [(x_1 + x_2 i) + (x_3 + x_4 i)j]^{t_i} = (x_1 - x_2 i) + (x_3 - x_4 i)j \\ x^{t_j} &= [(x_1 + x_2 i) + (x_3 + x_4 i)j]^{t_j} = (x_1 + x_2 i) - (x_3 + x_4 i)j \\ x^{t_{ij}} &= [(x_1 + x_2 i) + (x_3 + x_4 i)j]^{t_{ij}} = (x_1 - x_2 i) - (x_3 - x_4 i)j. \end{aligned}$$

[10]. Then we can write

$$\begin{aligned} x \times x^{t_i} &= (x_1^2 + x_2^2 - x_3^2 - x_4^2) + 2(x_1 x_3 + x_2 x_4)j \\ x \times x^{t_j} &= (x_1^2 - x_2^2 + x_3^2 - x_4^2) + 2(x_1 x_2 + x_3 x_4)i \\ x \times x^{t_{ij}} &= (x_1^2 + x_2^2 + x_3^2 + x_4^2) + 2(x_1 x_4 - x_2 x_3)ij. \end{aligned}$$

3. Generalized Bicomplex Numbers

In this section we define generalized bicomplex numbers and give some algebraic properties of them.

Definition 1. A generalized bicomplex number x is defined by the basis $\{1, i, j, ij\}$ as follows

$$x = x_1 1 + x_2 i + x_3 j + x_4 ij,$$

where x_1, x_2, x_3 and x_4 are real numbers and $i^2 = -\alpha$, $j^2 = -\beta$, $(ij)^2 = \alpha\beta, ij = ji$, $\alpha, \beta \in \mathbb{R}$.

Definition 2. We denote the set of generalized bicomplex numbers by $C_{\alpha\beta}$. For any $x = x_1 1 + x_2 i + x_3 j + x_4 ij$ and $y = y_1 1 + y_2 i + y_3 j + y_4 ij$ in $C_{\alpha\beta}$, the generalized bicomplex number addition is defined as

$$x + y = (x_1 + y_1) + (x_2 + y_2)i + (x_3 + y_3)j + (x_4 + y_4)ij$$

and the multiplication of a generalized bicomplex number $x = x_1 1 + x_2 i + x_3 j + x_4 ij$ by a real scalar λ is defined as

$$\lambda x = \lambda x_1 1 + \lambda x_2 i + \lambda x_3 j + \lambda x_4 ij$$

Corollary 1. *The set of generalized bicomplex numbers $C_{\alpha\beta}$ is a real vector space with this addition and scalar multiplication operations.*

Definition 3. Generalized bicomplex number product, denoted by $\cdot$, over the set of generalized bicomplex numbers $C_{\alpha\beta}$ is given by

$$x \cdot y = (x_1y_1 - \alpha x_2y_2 - \beta x_3y_3 + \alpha\beta x_4y_4) + (x_1y_2 + x_2y_1 - \beta x_3y_4 - \beta x_4y_3)\, i + (x_1y_3 + x_3y_1 - \alpha x_2y_4 - \alpha x_4y_2)\, j + (x_1y_4 + x_4y_1 + x_2y_3 + x_3y_2)\, ij.$$

Theorem 1. *Vector space $C_{\alpha\beta}$ together with the generalized bicomplex product $\cdot$ is a real algebra.*

Proof. $\cdot : C_{\alpha\beta} \times C_{\alpha\beta} \to C_{\alpha\beta}$ $\forall p, q, r \in C_{\alpha\beta}$ and $\lambda \in \mathbb{R}$ satisfy the following conditions

i) $p \cdot (q + r) = p \cdot q + p \cdot r$
ii) $p \cdot (q \cdot r) = (p \cdot q) \cdot r$
iii) $(\lambda p) \cdot q = p \cdot (\lambda q) = \lambda (p \cdot q)$

So, the real vector space $C_{\alpha\beta}$ is a real algebra with generalized bicomplex number product. □

Since the generalized bicomplex algebra is associative, it can be considered in terms of matrices. Consider the set of matrices

$$M_{\alpha\beta} = \left\{ \begin{pmatrix} x_1 & -\alpha x_2 & -\beta x_3 & \alpha\beta x_4 \\ x_2 & x_1 & -\beta x_4 & -\beta x_3 \\ x_3 & -\alpha x_4 & x_1 & -\alpha x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}; \quad x_i \in \mathbb{R}, \quad 1 \le i \le 4 \right\}.$$

The set $M_{\alpha\beta}$ together with matrix addition and scalar matrix multiplication is a real vector space. Furthermore, the vector space together with matrix product is an algebra.

Theorem 2. *The algebras $C_{\alpha\beta}$ and $M_{\alpha\beta}$ are isomorphic.*

Proof. The transformation

$$h : C_{\alpha\beta} \to M_{\alpha\beta}$$

given by

$$h\,(x = x_1 1 + x_2 i + x_3 j + x_4 ij) = \begin{pmatrix} x_1 & -\alpha x_2 & -\beta x_3 & \alpha\beta x_4 \\ x_2 & x_1 & -\beta x_4 & -\beta x_3 \\ x_3 & -\alpha x_4 & x_1 & -\alpha x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}$$

is one to one and onto. Morever $\forall x, y \in C_{\alpha\beta}$ and $\lambda \in \mathbb{R}$, we have

$$h\,(x + y) = h\,(x) + h\,(y)$$
$$h\,(\lambda x) = \lambda h\,(x)$$
$$h\,(x.y) = h\,(x)\, h\,(y)\,.$$

Thus the algebras $C_{\alpha\beta}$ and $M_{\alpha\beta}$ are isomorphic. □

Remark 1. In Theorem 2, we showed that any generalized bicomplex number is represented a matrix in form 4×4. So generalized bicomplex number addition and product can be expressed as matrix addition and product too, respectively.

Definition 4. Let $x \in C_{\alpha\beta}$. Then x can be expressed as $x = (x_1 + x_2 i) + (x_3 + x_4 i)\, j$. Conjugations of generalized bicomplex numbers with respect to i, j, both i and j are given by

$$\begin{aligned} x^{t_i} &= \left[(x_1 + x_2 i) + (x_3 + x_4 i)\, j\right]^{t_i} = (x_1 - x_2 i) + (x_3 - x_4 i)\, j \\ x^{t_j} &= \left[(x_1 + x_2 i) + (x_3 + x_4 i)\, j\right]^{t_j} = (x_1 + x_2 i) - (x_3 + x_4 i)\, j \\ x^{t_{ij}} &= \left[(x_1 + x_2 i) + (x_3 + x_4 i)\, j\right]^{t_{ij}} = (x_1 - x_2 i) - (x_3 - x_4 i)\, j. \end{aligned}$$

where x^{t_i}, x^{t_j} and $x^{t_{ij}}$ denote conjugations of x with respect to i, j, both i and j respectively. Also we can compute

$$\begin{aligned} x \cdot x^{t_i} &= \left(x_1^2 + \alpha x_2^2 - \beta x_3^2 - \alpha\beta x_4^2\right) + 2\,(x_1 x_3 + \alpha x_2 x_4)\, j \\ x \cdot x^{t_j} &= \left(x_1^2 - \alpha x_2^2 + \beta x_3^2 - \alpha\beta x_4^2\right) + 2\,(x_1 x_2 + \beta x_3 x_4)\, i \\ x \cdot x^{t_{ij}} &= \left(x_1^2 + \alpha x_2^2 + \beta x_3^2 + \alpha\beta x_4^2\right) + 2\,(x_1 x_4 - x_2 x_3)\, ij. \end{aligned}$$

Proposition 1. *Conjugations of generalized bicomplex numbers with respect to i, j, both i and j have following properties*

i) $(\lambda p + \delta q)^{t_k} = \lambda p^{t_k} + \delta q^{t_k}$
ii) $(p^{t_k})^{t_k} = p$
iii) $(p \cdot q)^{t_k} = p^{t_k} \cdot q^{t_k}$,

where $p, q \in C_{\alpha\beta}$, $\lambda, \delta \in \mathbb{R}$ and t_k represent the conjugations with respect to i, j, both i and j.

Proof. The proofs of the properties can be easily seen by directly computation. □

4. Some Hyperquadrics and Lie Groups

In this section we show that some hyperquadrics together with generalized bicomplex number product are Lie groups and find their Lie algebras. We deal with the hyperquadric M_{t_i}

$$M_{t_i} = \left\{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}_v^4 : \; x_1 x_3 + \alpha x_2 x_4 = 0, g_{t_i}(x, x) \neq 0\right\}$$

We consider M_{t_i} as the set of generalized bicomplex numbers

$$M_{t_i} = \left\{x = x_1 1 + x_2 i + x_3 j + x_4 ij \in \mathbb{R}_v^4 : \; x_1 x_3 + \alpha x_2 x_4 = 0, g_{t_i}(x, x) \neq 0\right\}$$

The components of M_{t_i} are easily obtained by representing generalized bicomplex number multiplication in matrix form

$$\tilde{M}_{t_i} = \left\{ x = \begin{pmatrix} x_1 & -\alpha x_2 & -\beta x_3 & \alpha\beta x_4 \\ x_2 & x_1 & -\beta x_4 & -\beta x_3 \\ x_3 & -\alpha x_4 & x_1 & -\alpha x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}, \quad x_1 x_3 + \alpha x_2 x_4 = 0, \; g_{t_i}(x, x) \neq 0 \right\}$$

where g_{t_i} is Euclidean or pseudo-Euclidean metric and it is defined by $g_{t_i} = dx_1^2 + \alpha dx_2^2 - \beta dx_3^2 - \alpha\beta dx_4^2$.

Remark 2. The norm of any element x on the hyperquadric M_{t_i} is given by $N_x = x \cdot x^{t_i} = g_{t_i}(x, x)$.

Now we define the hyperquadric M_{t_j} as

$$M_{t_j} = \left\{ x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4_v : x_1x_2 + \beta x_3x_4 = 0,\ g_{t_j}(x,x) \neq 0 \right\}$$

We consider M_{t_j} as the set of generalized bicomplex numbers

$$M_{t_j} = \left\{ x = x_1 1 + x_2 i + x_3 j + x_4 ij \in \mathbb{R}^4_v : x_1x_2 + \beta x_3x_4 = 0,\ g_{t_j}(x,x) \neq 0 \right\}$$

The components of M_{t_j} are easily obtained by representing generalized bicomplex number multiplication in matrix form

$$\tilde{M}_{t_j} = \left\{ x = \begin{pmatrix} x_1 & -\alpha x_2 & -\beta x_3 & \alpha\beta x_4 \\ x_2 & x_1 & -\beta x_4 & -\beta x_3 \\ x_3 & -\alpha x_4 & x_1 & -\alpha x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix},\ x_1x_2 + \beta x_3x_4 = 0,\ g_{t_j}(x,x) \neq 0 \right\}$$

where g_{t_j} is Euclidean or pseudo-Euclidean metric and it is defined by $g_{t_j} = dx_1^2 - \alpha dx_2^2 + \beta dx_3^2 - \alpha\beta dx_4^2$.

Remark 3. The norm of any element x on the hyperquadric M_{t_j} is given by $N_x = x \cdot x^{t_j} = g_{t_j}(x,x)$.

We define the hyperquadric $M_{t_{ij}}$

$$M_{t_{ij}} = \left\{ x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4_v : x_1x_4 - x_2x_3 = 0,\ g_{t_{ij}}(x,x) \neq 0 \right\}$$

We consider $M_{t_{ij}}$ as the set of generalized bicomplex numbers

$$M_{t_{ij}} = \left\{ x = x_1 1 + x_2 i + x_3 j + x_4 ij \in \mathbb{R}^4_v : x_1x_4 - x_2x_3 = 0,\ g_{t_{ij}}(x,x) \neq 0 \right\}$$

The components of $M_{t_{ij}}$ are easily obtained by representing generalized bicomplex number multiplication in matrix form

$$\tilde{M}_{t_{ij}} = \left\{ x = \begin{pmatrix} x_1 & -\alpha x_2 & -\beta x_3 & \alpha\beta x_4 \\ x_2 & x_1 & -\beta x_4 & -\beta x_3 \\ x_3 & -\alpha x_4 & x_1 & -\alpha x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix},\ x_1x_4 - x_2x_3 = 0,\ g_{t_{ij}}(x,x) \neq 0 \right\}$$

where $g_{t_{ij}}$ is Euclidean or pseudo-Euclidean metric and it is defined by $g_{t_{ij}} = dx_1^2 + \alpha dx_2^2 + \beta dx_3^2 + \alpha\beta dx_4^2$.

Remark 4. The norm of any element x on the hyperquadric $M_{t_{ij}}$ is given by $N_x = x \cdot x^{t_{ij}} = g_{t_{ij}}(x,x)$.

Theorem 3. *The set of M_{t_i} with generalized bicomplex number product is a Lie group.*

Proof. $\tilde{M}_{t_i}$ differentiable manifold and at the same time $\tilde{M}_{t_i}$ is a group with group operation given by matrix multiplication. The group function is given by

$$\begin{aligned} . : \tilde{M}_{t_i} \times \tilde{M}_{t_i} &\to \tilde{M}_{t_i} \\ (x,y) &\to x.y^{-1}, \end{aligned}$$

where y^{-1} is obtained as a element of M_{t_i} as follows:

$$y^{-1} = \frac{y^{t_i}}{N_y} = \frac{1}{y_1^2 + \alpha y_2^2 - \beta y_3^2 - \alpha\beta y_4^2.} \left(y_1, -y_2, y_3, -y_4 \right).$$

Since the transformation h is an isomorphism $(M_{t_i}, \cdot)$ is a Lie group. □

We denote the set of all unit generalized bicomplex numbers x on M_{t_i} by $M_{t_i}^*$. $M_{t_i}^*$ is defined as

$$M_{t_i}^* = \{x \in M_{t_i} : g_{t_i}(x, x) = 1\}$$

or

$$M_{t_i}^* = \left\{x \in M_{t_i} : x_1^2 + \alpha x_2^2 - \beta x_3^2 - \alpha\beta x_4^2 = 1\right\}$$

$M_{t_i}^*$ is a group with the group operation of generalized bicomplex multiplication. So we can give the following corollary.

Corollary 2. *$M_{t_i}^*$ is 2-dimensional Lie subgroup of M_{t_i}.*

Theorem 4. *The Lie algebra of Lie group M_{t_i} is $sp\{X_1, X_2, X_4\}$ such that left invariant vector fields X_1, X_2, X_4 are given by*

$$\begin{aligned} X_1 &= (x_1, x_2, x_3, x_4) \\ X_2 &= (-\alpha x_2, x_1, -\alpha x_4, x_3) \\ X_4 &= (\alpha\beta x_4, -\beta x_3, -\alpha x_2, x_1) \end{aligned}$$

Proof. Let us find the Lie algebra of Lie group M_{t_i}. Let

$$a(t) = a_1(t)1 + a_2(t)i + a_3(t)j + a_4(t)ij$$

be a curve on M_{t_i} such that $a(0) = 1$, i.e. $a_1(0) = 1$, $a_m(0) = 0$ for $m = 2, 3, 4$. Differentiation of the equation

$$a_1(t)a_3(t) + \alpha a_2(t)a_4(t) = 0$$

yields the equation

$$a_1'(t)a_3(t) + a_1(t)a_3'(t) + \alpha a_2'(t)a_4(t) + \alpha a_2(t)a_4'(t) = 0$$

Substituting $t = 0$, we obtain $a_3'(0) = 0$. The Lie algebra is thus constituted by vectors of the form $\zeta = \zeta_m \left(\frac{\partial}{\partial a_m}\right)\Big|_{a=1}$ where $m = 1, 2, 4$. The vector ζ is formally written in the form $\zeta = \zeta_1 + \zeta_2 j + \zeta_4 ij$. Let us find the left invariant vector field X on M_{t_i} for which $X|_{a=1} = \zeta$. Let $b(t)$ be a curve on M_{t_i} such that $b(0) = 1$, $b'(0) = \zeta$. Then $L_x(b(t)) = xb(t)$ is the left translation of the curve $b(t)$ by the generalized bicomplex number x. Let L_x^* be the differentiation of L_x left translation. In that case $L_x^*(b'(0)) = x\zeta$. In particular, denote by X_m those left invariant vector fields on M_{t_i} for which

$$X_m|_{a=1} = \frac{\partial}{\partial a_m}\bigg|_{a=1}$$

where $m = 1, 2, 4$. These three vector fields are represented at the point $a = 1$ by the generalized bicomplex units $1, i, ij$. For the components of these vector fields at the point $x = x_1 1 + x_2 i + x_3 j + x_4 ij$, we have $(X_1)_x = x1$, $(X_2)_x = xi$ and $(X_4)_x = xij$.

$$\begin{aligned} X_1 &= (x_1, x_2, x_3, x_4), \\ X_2 &= (-\alpha x_2, x_1, -\alpha x_4, x_3), \\ X_4 &= (\alpha\beta x_4, -\beta x_3, -\alpha x_2, x_1), \end{aligned}$$

where all the partial derivatives are at the point x. □

Corollary 3. *The Lie algebra of Lie group* $M_{t_i}^*$ *is* $sp\{X_2, X_4\}$.

Theorem 5. *The set of* M_{t_j} *together with generalized bicomplex number product is a Lie group.*

Corollary 4. $M_{t_j}^*$ *is 2-dimensional Lie subgroup of* M_{t_j}.

Theorem 6. *The Lie algebra of Lie group* M_{t_j} *is* $sp\{X_1, X_3, X_4\}$ *such that left invariant vector fields* X_1, X_3, X_4 *are given by*

$$\begin{aligned} X_1 &= (x_1, x_2, x_3, x_4), \\ X_3 &= (-\beta x_3, -\beta x_4, x_1, x_2), \\ X_4 &= (\alpha\beta x_4, -\beta x_3, -\alpha x_2, x_1). \end{aligned}$$

Corollary 5. *The Lie algebra of Lie group* $M_{t_j}^*$ *is* $sp\{X_3, X_4\}$

Theorem 7. *The set of* $M_{t_{ij}}$ *together with generalized bicomplex number product is a Lie group.*

Corollary 6. $M_{t_{ij}}^*$ *is 2-dimensional Lie subgroup of* $M_{t_{ij}}$.

Theorem 8. *The Lie algebra of Lie group* $M_{t_{ij}}$ *is* $sp\{X_1, X_2, X_3\}$ *such that left invariant vector fields* X_1, X_2, X_3 *are given by*

$$\begin{aligned} X_1 &= (x_1, x_2, x_3, x_4), \\ X_2 &= (-\alpha x_2, x_1, -\alpha x_4, x_3), \\ X_3 &= (-\beta x_3, -\beta x_4, x_1, x_2). \end{aligned}$$

Corollary 7. *The Lie algebra of Lie group* $M_{t_{ij}}^*$ *is* $sp\{X_2, X_3\}$.

5. Tensor Product Surfaces and Lie Groups

In this section we define the tensor product surfaces on the hyperquadrics M_{t_i}, M_{t_j} and $M_{t_{ij}}$. By means of tensor product surfaces, we determine some special subgroups of these Lie groups M_{t_i}, M_{t_j} and $M_{t_{ij}}$ in $\mathbb{R}^4$ and $\mathbb{R}^4_2$.

5.1. Tensor Product Surfaces on M_{t_i} Hyperquadric and Some Special Lie Subgroups

In this subsection, we change the definition of tensor product as follows:

Let $\gamma : \mathbb{R} \to \mathbb{R}^2_k\ (+\ -\alpha\beta)$ and $\delta : \mathbb{R} \to \mathbb{R}^2_t\ (+\ \alpha)$ be planar curves in Euclidean or Lorentzian space. Put $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ and $\delta(s) = (\delta_1(s), \delta_2(s))$. Let us define their tensor product as

$$\begin{aligned} &f = \gamma \otimes \delta : \mathbb{R}^2 \to \mathbb{R}^4_v \quad (+\alpha - \beta - \alpha\beta), \\ &f(t,s) = (\gamma_1(t)\delta_1(s), \gamma_1(t)\delta_2(s), -\alpha\gamma_2(t)\delta_2(s), \gamma_2(t)\delta_1(s)). \end{aligned} \tag{1}$$

Tensor product surface given by (1) is a surface on M_{t_i} hyperquadric. Tangent vector fields of $f(t,s)$ can be easily computed as

$$\begin{aligned} \frac{\partial f}{\partial t} &= (\gamma_1'(t)\delta_1(s), \gamma_1'(t)\delta_2(s), -\alpha\gamma_2'(t)\delta_2(s), \gamma_2'(t)\delta_1(s)) \\ \frac{\partial f}{\partial s} &= (\gamma_1(t)\delta_1'(s), \gamma_1(t)\delta_2'(s), -\alpha\gamma_2(t)\delta_2'(s), \gamma_2(t)\delta_1'(s)). \end{aligned} \tag{2}$$

By using (2), we have

$$g_{11} = g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right) = g_1(\gamma', \gamma')g_2(\delta, \delta)$$
$$g_{12} = g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right) = g_1(\gamma, \gamma')g_2(\delta, \delta')$$
$$g_{22} = g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\right) = g_1(\gamma, \gamma)g_2(\delta', \delta'),$$

where $g_1 = dx_1^2 - \alpha\beta dx_2^2$ and $g_2 = dx_1^2 + \alpha dx_2^2$ are the metrics of $\mathbb{R}_k^2$ and $\mathbb{R}_t^2$, respectively.

Remark 5. Tensor product surface given by (1) is a surface in $\mathbb{R}^4$ or $\mathbb{R}_2^4$ according to the case of α and β. If we take as $\alpha = \beta = 1$, we obtain a tensor product surface of a Lorentzian plane curve and a Euclidean plane curve in $\mathbb{R}_2^4$. If we take as $\alpha = 1$, $\beta = -1$, we obtain a tensor product surface of two Euclidean plane curves in $\mathbb{R}^4$. If we take as $\alpha = -1$, $\beta = 1$, we obtain a tensor product surface of a Euclidean plane curve and a Lorentzian plane curve in $\mathbb{R}_2^4$. If we take as $\alpha = -1$, $\beta = -1$ we obtain a tensor product surface of two Lorentzian plane curves in $\mathbb{R}_2^4$.

Now we investigate Lie group structure of tensor product surfaces given by the parametrization (1) in $\mathbb{R}^4$ or $\mathbb{R}_2^4$ according to above cases. Morever we obtain left invariant vector fields of the tensor product surface that has the structure of Lie group.

5.1.1. Case I: $\alpha = \beta = 1$.

Proposition 2. *Let* $\gamma : \mathbb{R} \to \mathbb{R}_1^2$ $(+\ -)$ *be a hyperbolic spiral and* $\delta : \mathbb{R} \to \mathbb{R}^2$ $(+\ +)$ *be a spiral with the same parameter, i.e.* $\gamma(t) = e^{at}(\cosh t, \sinh t)$ *and* $\delta(t) = e^{bt}(\cos t, \sin t)$ $(a, b \in \mathbb{R})$. *Their tensor product is a one parameter subgroup of Lie group* M_{t_i}.

Proof. We obtain

$$\varphi(t) = \gamma(t) \otimes \delta(t) = e^{(a+b)t}(\cosh t \cos t, \cosh t \sin t, -\sinh t \sin t, \sinh t \cos t)$$

It can be easily seen that

$$\varphi(t_1) \cdot \varphi(t_2) = \varphi(t_1 + t_2)$$

for all t_1, t_2. Also $\varphi^{-1}(t) = \varphi(-t)$. Hence $(\varphi(t), \cdot)$ is a one parameter Lie subgroup of Lie group $(M_{t_i}, \cdot)$. □

Corollary 8. *Let* $\gamma : \mathbb{R} \to \mathbb{R}_1^2$ *be a Lorentzian circle centered at O and* $\delta : \mathbb{R} \to \mathbb{R}^2$ *be circle centered at O with the same parameter, i.e.,* $\gamma(t) = (\cosh t, \sinh t)$ *and* $\delta(t) = (\cos t, \sin t)$. *Then their tensor product is a one parameter subgroup of Lie group* $M_{t_i}^*$.

Proof. Since $g_{t_i}(\gamma(t) \otimes \delta(t), \gamma(t) \otimes \delta(t)) = 1$, it follows that $\gamma(t) \otimes \delta(t) \subset M_{t_i}^*$. If we take as $a = b = 0$ in Proposition 2, we obtain that γ is a Lorentzian circle centered at O and δ is a circle centered at O. Then their tensor product is a one parameter Lie subgroup in Lie group $M_{t_i}^*$. □

Proposition 3. *Let $\varphi(t)$ be tensor product of a Lorentzian circle centered at O and a circle centered at O with the same parameter. Then the left invariant vector field on $\varphi(t)$ is $X = X_2 + X_4$, where X_2 and X_4 are left invariant vector fields on $M_{t_i}^*$.*

Proof. Since $\varphi(t)$ is tensor product of a Lorentzian circle centered at O and a circle centered at O with the same parameter we write

$$\varphi(t) = (\cosh t \cos t, \cosh t \sin t, -\sinh t \sin t, \sinh t \cos t)$$

$\varphi(0) = (1, 0, 0, 0) = e$ and $\varphi'(0) = (0, 1, 0, 1) = X_e$. Then

$$\begin{aligned} L_g^*(X_e) &= g \cdot X_e = (x_1 1 + x_2 i + x_3 j + x_4 ij) \cdot (i + ij) \\ &= X_2 + X_4 \end{aligned}$$

This completes the proof. □

Proposition 4. *Let $\gamma : \mathbb{R} \to \mathbb{R}_1^2$, $\gamma(t) = e^{at}(\cosh t, \sinh t)$ be a hyperbolic spiral and $\delta : \mathbb{R} \to \mathbb{R}^2$ $\delta(s) = e^{bs}(\cos s, \sin s)$ be a spiral $(a, b \in \mathbb{R})$. Then their tensor product is 2-dimensional Lie subgroup of $M_{t_{ij}}$.*

Proof. By using tensor product rule given by (1), we get

$$f(t, s) = e^{at+bs}(\cosh t \cos s, \cosh t \sin s, -\sinh t \sin s, \sinh t \cos s)$$

Every point of $f(t, s)$ is on the hyperquadric M_{t_i}. Since $f(t, s)$ is both subgroup and submanifold of a Lie group M_{t_i}, it is a 2-dimensional Lie subgroup of $M_{t_{ij}}$. □

Proposition 5. *Let $\gamma : \mathbb{R} \to \mathbb{R}_1^2$, $\gamma(t) = (\cosh t, \sinh t)$ be a Lorentzian circle centered at O and $\delta : \mathbb{R} \to \mathbb{R}^2$ $\delta(s) = (\cos s, \sin s)$ be a circle centered at O $(a, b \in \mathbb{R})$. Then their tensor product is 2-dimensional Lie subgroup of $M_{t_i}^*$.*

Proof. In Proposition 4 taking $a = b = 0$, we can see that the tensor product surface $f(t, s) \subset M_{t_i}^*$. Hence, it is 2-dimensional Lie subgroup of $M_{t_i}^*$. □

Proposition 6. *Let $\gamma : \mathbb{R} \to \mathbb{R}_1^2$, $\gamma(t) = (\cosh t, \sinh t)$ be a Lorentzian circle at centered O and $\delta : \mathbb{R} \to \mathbb{R}^2$ $\delta(s) = (\cos s, \sin s)$ be a circle at centered O $(a, b \in \mathbb{R})$. Then, the left invariant vector fields on tensor product surface $f(t, s) = \gamma(t) \otimes \delta(s)$ are X_2 and X_4 which are the left invariant vector fields on $M_{t_i}^*$.*

Proof. The unit element of 2-dimensional Lie subgroup is the point $e = f(0, 0)$. Let us find the left invariant vector fields on $f(t, s)$ to the vectors

$$u_1 = \left.\frac{\partial}{\partial t}\right|_e \quad \textit{and} \quad u_2 = \left.\frac{\partial}{\partial s}\right|_e$$

for the vector u_1 we obtain

$$\begin{aligned} L_g^*(u_1) &= g \cdot u_1 = (x_1 1 + x_2 i + x_3 j + x_4 ij) \cdot ij \\ &= X_4 \end{aligned}$$

Analogously, for the vector u_2 we obtain left invariant vector field X_2. □

5.1.2. Case II: $\alpha = 1, \beta = -1$.

Proposition 7. *Let* $\gamma : \mathbb{R} \to \mathbb{R}^2\ (+\,+)$ *and* $\delta : \mathbb{R} \to \mathbb{R}^2\ (+\,+)$ *be two spirals with the same parameter, i.e.* $\gamma(t) = e^{at}(\cos t, \sin t)$ *and* $\delta(t) = e^{bt}(\cos t, \sin t)$ $(a, b \in \mathbb{R})$. *Then their tensor product is a one parameter subgroup of Lie group* M_{t_i}.

Proof. We obtain

$$\varphi(t) = \gamma(t) \otimes \delta(t) = e^{(a+b)t}\left(\cos^2 t, \cos t \sin t, -\sin^2 t, \cos t \sin t\right)$$

It can be easily seen that

$$\varphi(t_1) \cdot \varphi(t_2) = \varphi(t_1 + t_2)$$

for all t_1, t_2. Also $\varphi^{-1}(t) = \varphi(-t)$. Hence $(\varphi(t), \cdot)$ is a one parameter Lie subgroup of Lie group $(M_{t_i}, \cdot)$. □

Corollary 9. *Let* $\gamma : \mathbb{R} \to \mathbb{R}^2$ *and* $\delta : \mathbb{R} \to \mathbb{R}^2$ *be two circles centered at O with the same parameter, i.e.,* $\gamma(t) = (\cos t, \sin t)$ *and* $\delta(t) = (\cos t, \sin t)$. *Then their tensor product is a one parameter subgroup of Lie group* $M^*_{t_i}$.

Proposition 8. *Let* $\varphi(t)$ *be tensor product of two circles centered at O with the same parameter. Then the left invariant vector field on* $\varphi(t)$ *is* $X = X_2 + X_4$, *where* X_2 *and* X_4 *are left invariant vector fields on* $M^*_{t_i}$.

Proposition 9. *Let* $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = e^{at}(\cos t, \sin t)$ *and* $\delta : \mathbb{R} \to \mathbb{R}^2$ $\delta(s) = e^{bs}(\cos s, \sin s)$ *be two spirals* $(a, b \in \mathbb{R})$. *Then their tensor product is 2-dimensional Lie subgroup of* $M_{t_{ij}}$.

Proof. By using tensor product rule given by (1), we get

$$f(t, s) = e^{at+bs}(\cos t \cos s, \cos t \sin s, -\sin t \sin s, \sin t \cos s)$$

Every point of $f(t, s)$ is on the hyperquadric M_{t_i}. Since $f(t, s)$ is both subgroup and submanifold of a Lie group M_{t_i}, it is a 2-dimensional Lie subgroup of $M_{t_{ij}}$. □

Corollary 10. *Let* $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (\cos t, \sin t)$ *and* $\delta : \mathbb{R} \to \mathbb{R}^2$ $\delta(s) = (\cos s, \sin s)$ *be two circles centered at O* $(a, b \in \mathbb{R})$. *Then their tensor product is 2-dimensional Lie subgroup of* $M^*_{t_i}$.

Proposition 10. *Let* $\gamma : \mathbb{R} \to \mathbb{R}^2_1$, $\gamma(t) = (\cos t, \sin t)$ *and* $\delta : \mathbb{R} \to \mathbb{R}^2$ $\delta(s) = (\cos s, \sin s)$ *be two circles centered at O* $(a, b \in \mathbb{R})$. *Then, the left invariant vector fields on tensor product surface* $f(t, s) = \gamma(t) \otimes \delta(s)$ *are* X_2 *and* X_4 *which are the left invariant vector fields on* $M^*_{t_i}$.

5.1.3. Case III: $\alpha = -1, \beta = -1$.

Proposition 11. *Let* $\gamma : \mathbb{R} \to \mathbb{R}^2_1\ (+\,-)$ *and* $\delta : \mathbb{R} \to \mathbb{R}^2_1\ (+\,-)$ *be two hyperbolic spirals with the same parameter, i.e.* $\gamma(t) = e^{at}(\cosh t, \sinh t)$ *and* $\delta(t) = e^{bt}(\cosh t, \sinh t)$ $(a, b \in \mathbb{R})$. *Then their tensor product is a one parameter subgroup of Lie group* M_{t_i}.

Proof. We obtain

$$\varphi(t) = \gamma(t) \otimes \delta(t) = e^{(a+b)t}\left(\cosh^2 t, \cosh t \sinh t, \sinh^2 t, \cosh t \sinh t\right)$$

It can be easily seen that

$$\varphi(t_1) \cdot \varphi(t_2) = \varphi(t_1 + t_2)$$

for all t_1, t_2. Also $\varphi^{-1}(t) = \varphi(-t)$. Hence $(\varphi(t), \cdot)$ is a one parameter Lie subgroup of Lie group $(M_{t_i}, \cdot)$. □

Corollary 11. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2_1$ and $\delta : \mathbb{R} \to \mathbb{R}^2_1$ be two Lorentzian circles centered at O with the same parameter, i.e., $\gamma(t) = (\cosh t, \sinh t)$ and $\delta(t) = (\cosh t, \sinh t)$. Then their tensor product is a one parameter subgroup of Lie group $M^*_{t_i}$.*

Proposition 12. *Let $\varphi(t)$ be tensor product of two Lorentzian circles centered at O with the same parameter. Then the left invariant vector field on $\varphi(t)$ is $X = X_2 + X_4$, where X_2 and X_4 are left invariant vector fields on $M^*_{t_i}$.*

Proposition 13. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2_1$, $\gamma(t) = e^{at}(\cosh t, \sinh t)$ and $\delta : \mathbb{R} \to \mathbb{R}^2_1$ $\delta(s) = e^{bs}(\cosh s, \sinh s)$ be two spirals $(a, b \in \mathbb{R})$. Then their tensor product is 2-dimensional Lie subgroup of M_{t_i}.*

Proof. By using tensor product rule given by (1), we get

$$f(t, s) = e^{at+bs}(\cosh t \cosh s, \cosh t \sinh s, -\sinh t \sinh s, \sinh t \cosh s)$$

Every point of $f(t, s)$ is on the hyperquadric M_{t_i}. Since $f(t, s)$ is both subgroup and submanifold of a Lie group M_{t_i}, it is a 2-dimensional Lie subgroup of M_{t_i}. □

Corollary 12. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2_1$, $\gamma(t) = (\cosh t, \sinh t)$ and $\delta : \mathbb{R} \to \mathbb{R}^2_1$ $\delta(s) = (\cosh s, \sinh s)$ be two Lorentzian circles centered at O $(a, b \in \mathbb{R})$. Then their tensor product is 2-dimensional Lie subgroup of $M^*_{t_i}$.*

Proposition 14. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2_1$, $\gamma(t) = (\cosh t, \sinh t)$ and $\delta : \mathbb{R} \to \mathbb{R}^2_1$ $\delta(s) = (\cosh s, \sinh s)$ be two Lorentzian circles centered at O $(a, b \in \mathbb{R})$. Then, the left invariant vector fields on tensor product surface $f(t, s) = \gamma(t) \otimes \delta(s)$ are X_2 and X_4 which are the left invariant vector fields on $M^*_{t_i}$.*

Remark 6. If we take as $\alpha = \beta = 1$ in the hyperquadric M_{t_i}, the hyperquadric M_{t_i} coincides the hyperquadric Q in the paper studied by Karakuş Ö. and Yaylı [4], where the hyperquadric Q is given by

$$Q = \left\{x = (x_1, x_2, x_3, x_4) \neq 0 \quad x_1 x_3 + x_2 x_4 = 0,\ x_1^2 + x_2^2 - x_3^2 - x_4^2 \neq 0\right\}.$$

In [4], they showed that the hyperquadric Q is a Lie group by using bicomplex number product. Also they defined a new tensor product surface in $\mathbb{R}^4_2$ as follows:

$$(\gamma \otimes \delta)(t, s) = (\gamma_1(t)\delta_1(s), \gamma_1(t)\delta_2(s), -\gamma_2(t)\delta_2(s), \gamma_2(t)\delta_1(s))$$

where $\gamma : \mathbb{R} \to \mathbb{R}^2_1$ and $\delta : \mathbb{R} \to \mathbb{R}^2$ is respectively, a Lorentzian plane curve and a Euclidean plane curve. By means of this tensor product surfaces, they determined some special Lie subgroups of this Lie group Q. So, the case I in

Sect. 5.1 coincides the paper studied by Karakuş and Yaylı [4]. Hence, it can be considered that the Sect. 5.1 is a generalization of that study.

5.2. Tensor Product Surfaces on M_{t_j} Hyperquadric and Some Special Lie Subgroups

In this subsection, we change the definition of tensor product as follows:

Let $\gamma : \mathbb{R} \to \mathbb{R}^2_k\ (+\ -\alpha\beta)$ and $\delta : \mathbb{R} \to \mathbb{R}^2_t\ (+\ \beta)$ be planar curves in Euclidean or Lorentzian space. Put $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ and $\delta(s) = (\delta_1(s), \delta_2(s))$. Let us define their tensor product as

$$\begin{aligned} &f = \gamma \otimes \delta : \mathbb{R}^2 \to \mathbb{R}^4_v \quad (+\ \alpha\ -\beta\ -\alpha\beta), \\ &f(t,s) = (\gamma_1(t)\delta_1(s), -\beta\gamma_2(t)\delta_2(s), \gamma_1(t)\delta_2(s), \gamma_2(t)\delta_1(s)) \end{aligned} \tag{3}$$

tensor product surface given by (3) is a surface on M_{t_j} hyperquadric. Tangent vector fields of $f(t,s)$ can be easily computed as

$$\begin{aligned} \frac{\partial f}{\partial t} &= (\gamma_1'(t)\delta_1(s), -\beta\gamma_2'(t)\delta_2(s), \gamma_1'(t)\delta_2(s), \gamma_2'(t)\delta_1(s)) \\ \frac{\partial f}{\partial s} &= (\gamma_1(t)\delta_1'(s), -\beta\gamma_2(t)\delta_2'(s), \gamma_1(t)\delta_2'(s), \gamma_2(t)\delta_1'(s)) \end{aligned} \tag{4}$$

By using (4), we have

$$\begin{aligned} g_{11} &= g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right) = g_1(\gamma', \gamma')g_2(\delta, \delta) \\ g_{12} &= g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right) = g_1(\gamma, \gamma')g_2(\delta, \delta') \\ g_{22} &= g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\right) = g_1(\gamma, \gamma)g_2(\delta', \delta') \end{aligned}$$

where $g_1 = dx_1^2 - \alpha\beta dx_2^2$ and $g_2 = dx_1^2 + \beta dx_2^2$ are the metrics of $\mathbb{R}^2_k$ and $\mathbb{R}^2_t$, respectively.

Remark 7. Tensor product surface given by (3) is a surface in $\mathbb{R}^4$ or $\mathbb{R}^4_2$ according to the case of α and β. If we take as $\alpha = \beta = 1$, we obtain a tensor product surface of a Lorentzian plane curve and a Euclidean plane curve in $\mathbb{R}^4_2$. If we take as $\alpha = 1$, $\beta = -1$, we obtain a tensor product surface of a Euclidean plane curve and a Lorentzian plane curve in $\mathbb{R}^4_2$. If we take as $\alpha = -1$, $\beta = 1$, we obtain a tensor product surface of two Euclidean plane curves in $\mathbb{R}^4$. If we take as $\alpha = -1$, $\beta = -1$ we obtain a tensor product surface of two Lorentzian plane curves in $\mathbb{R}^4_2$.

Now we investigate Lie group structure of tensor product surfaces given by the parametrization (3) in $\mathbb{R}^4$ or $\mathbb{R}^4_2$ according to above cases. Morever we obtain left invariant vector fields of the tensor product surface that has the structure of Lie group.

5.2.1. Case I $\alpha = \beta = 1$.

Proposition 15. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2_1\ (+\ -)$ be a hyperbolic spiral and $\delta : \mathbb{R} \to \mathbb{R}^2$ $(+\ +)$ be a spiral with the same parameter, i.e. $\gamma(t) = e^{at}(\cosh t, \sinh t)$ and $\delta(t) = e^{bt}(\cos t, \sin t)$ $(a, b \in \mathbb{R})$. Their tensor product is a one parameter subgroup of Lie group M_{t_j}.*

Proof. We obtain

$$\varphi(t) = \gamma(t) \otimes \delta(t) = e^{(a+b)t}(\cosh t \cos t, -\sinh t \sin t, \cosh t \sin t, \sinh t \cos t)$$

It can be easily seen that

$$\varphi(t_1) \cdot \varphi(t_2) = \varphi(t_1 + t_2)$$

for all t_1, t_2. Also $\varphi^{-1}(t) = \varphi(-t)$. Hence $(\varphi(t), \cdot)$ is a one parameter Lie subgroup of Lie group $(M_{t_j}, \cdot)$. □

Corollary 13. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2_1$ be a Lorentzian circle centered at O and $\delta : \mathbb{R} \to \mathbb{R}^2$ be circle centered at O with the same parameter, i.e., $\gamma(t) = (\cosh t, \sinh t)$ and $\delta(t) = (\cos t, \sin t)$. Then their tensor product is a one parameter subgroup of Lie group $M^*_{t_j}$.*

Proposition 16. *Let $\varphi(t)$ be tensor product of a Lorentzian circle centered at O and a circle centered at O with the same parameter. Then the left invariant vector field on $\varphi(t)$ is $X = X_3 + X_4$, where X_3 and X_4 are left invariant vector fields on $M^*_{t_j}$.*

Proposition 17. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2_1$, $\gamma(t) = e^{at}(\cosh t, \sinh t)$ be a hyperbolic spiral and $\delta : \mathbb{R} \to \mathbb{R}^2$ $\delta(s) = e^{bs}(\cos s, \sin s)$ be a spiral $(a, b \in \mathbb{R})$. Then their tensor product is 2-dimensional Lie subgroup of M_{t_j}.*

Proof. By using tensor product rule given by (3), we get

$$f(t,s) = e^{at+bs}(\cosh t \cos s, -\sinh t \sin s, \cosh t \sin s, \sinh t \cos s)$$

Every point of $f(t,s)$ is on the hyperquadric M_{t_j}. Since $f(t,s)$ is both subgroup and submanifold of a Lie group M_{t_j}, it is a 2-dimensional Lie subgroup of M_{t_j}. □

Proposition 18. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2_1$, $\gamma(t) = (\cosh t, \sinh t)$ be a Lorentzian circle and $\delta : \mathbb{R} \to \mathbb{R}^2$ $\delta(s) = (\cos s, \sin s)$ be a circle $(a, b \in \mathbb{R})$. Then their tensor product is 2-dimensional Lie subgroup of $M^*_{t_j}$.*

Proposition 19. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2_1$, $\gamma(t) = (\cosh t, \sinh t)$ be a Lorentzian circle at centered O and $\delta : \mathbb{R} \to \mathbb{R}^2$ $\delta(s) = (\cos s, \sin s)$ be a circle at centered O $(a, b \in \mathbb{R})$. Then, the left invariant vector fields on tensor product surface $f(t,s) = \gamma(t) \otimes \delta(s)$ are X_3 and X_4 which are the left invariant vector fields on $M^*_{t_j}$.*

5.2.2. Case II $\alpha = -1, \beta = 1$.

Proposition 20. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ $(+\,+)$ and $\delta : \mathbb{R} \to \mathbb{R}^2$ $(+\,+)$ be two spirals with the same parameter, i.e. $\gamma(t) = e^{at}(\cos t, \sin t)$ and $\delta(t) = e^{bt}(\cos t, \sin t)$ $(a, b \in \mathbb{R})$. Then their tensor product is a one parameter subgroup of Lie group M_{t_j}.*

Proof. We obtain

$$\varphi(t) = \gamma(t) \otimes \delta(t) = e^{(a+b)t}\left(\cos^2 t, -\sin^2 t, \cos t \sin t, \cos t \sin t\right)$$

It can be easily seen that

$$\varphi(t_1) \cdot \varphi(t_2) = \varphi(t_1 + t_2)$$

for all t_1, t_2. Also $\varphi^{-1}(t) = \varphi(-t)$. Hence $(\varphi(t), \cdot)$ is a one parameter Lie subgroup of Lie group $\left(M_{t_j}, \cdot\right)$. □

Corollary 14. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ and $\delta : \mathbb{R} \to \mathbb{R}^2$ be two circles centered at O with the same parameter, i.e., $\gamma(t) = (\cos t, \sin t)$ and $\delta(t) = (\cos t, \sin t)$. Then their tensor product is a one parameter subgroup of Lie group $M^*_{t_j}$.*

Proposition 21. *Let $\varphi(t)$ be tensor product of two circles centered at O with the same parameter. Then the left invariant vector field on $\varphi(t)$ is $X = X_3 + X_4$, where X_3 and X_4 are left invariant vector fields on $M^*_{t_j}$.*

Proposition 22. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = e^{at}(\cos t, \sin t)$ and $\delta : \mathbb{R} \to \mathbb{R}^2$ $\delta(s) = e^{bs}(\cos s, \sin s)$ be two spirals $(a, b \in \mathbb{R})$. Then their tensor product is 2-dimensional Lie subgroup of M_{t_j}.*

Proof. By using tensor product rule given by (3), we get

$$f(t,s) = e^{at+bs}(\cos t \cos s, -\sin t \sin s, \cos t \sin s, , \sin t \cos s)$$

Every point of $f(t,s)$ is on the hyperquadric M_{t_j}. Since $f(t,s)$ is both subgroup and submanifold of a Lie group M_{t_j}, it is a 2-dimensional Lie subgroup of M_{t_j}. □

Corollary 15. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (\cos t, \sin t)$ and $\delta : \mathbb{R} \to \mathbb{R}^2$ $\delta(s) = (\cos s, \sin s)$ be two circles centered at O $(a, b \in \mathbb{R})$. Then their tensor product is 2-dimensional Lie subgroup of $M^*_{t_j}$.*

Proposition 23. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2_1$, $\gamma(t) = (\cos t, \sin t)$ and $\delta : \mathbb{R} \to \mathbb{R}^2$ $\delta(s) = (\cos s, \sin s)$ be two circles centered at O $(a, b \in \mathbb{R})$. Then, the left invariant vector fields on tensor product surface $f(t,s) = \gamma(t) \otimes \delta(s)$ are X_3 and X_4 which are the left invariant vector fields on $M^*_{t_j}$.*

5.2.3. Case III $\boldsymbol{\alpha = -1, \beta = -1}$.

Proposition 24. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2_1$ $(+\,-)$ and $\delta : \mathbb{R} \to \mathbb{R}^2_1$ $(+\,-)$ be two hyperbolic spirals with the same parameter, i.e. $\gamma(t) = e^{at}(\cosh t, \sinh t)$ and $\delta(t) = e^{bt}(\cosh t, \sinh t)$ $(a, b \in \mathbb{R})$. Then their tensor product is a one parameter subgroup of Lie group M_{t_j}.*

Proof. We obtain

$$\varphi(t) = \gamma(t) \otimes \delta(t) = e^{(a+b)t}\left(\cosh^2 t, \sinh^2 t, \cosh t \sinh t, \cosh t, \sinh t\right)$$

It can be easily seen that

$$\varphi(t_1) \cdot \varphi(t_2) = \varphi(t_1 + t_2)$$

for all t_1, t_2. Also $\varphi^{-1}(t) = \varphi(-t)$. Hence $(\varphi(t), \cdot)$ is a one parameter Lie subgroup of Lie group $\left(M_{t_j}, \cdot\right)$. □

Corollary 16. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2_1$ and $\delta : \mathbb{R} \to \mathbb{R}^2_1$ be two Lorentzian circles centered at O with the same parameter, i.e., $\gamma(t) = (\cosh t, \sinh t)$ and $\delta(t) = (\cosh t, \sinh t)$. Then their tensor product is a one parameter subgroup of Lie group $M^*_{t_j}$.*

Proposition 25. *Let $\varphi(t)$ be tensor product of two Lorentzian circles centered at O with the same parameter. Then the left invariant vector field on $\varphi(t)$ is $X = X_3 + X_4$, where X_3 and X_4 are left invariant vector fields on $M^*_{t_j}$.*

Proposition 26. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2_1$, $\gamma(t) = e^{at}(\cosh t, \sinh t)$ and $\delta : \mathbb{R} \to \mathbb{R}^2_1$ $\delta(s) = e^{bs}(\cosh s, \sinh s)$ be two spirals $(a, b \in \mathbb{R})$. Then their tensor product is 2-dimensional Lie subgroup of M_{t_j}.*

Proof. By using tensor product rule given by (3), we get

$$f(t,s) = e^{at+bs}(\cosh t \cosh s, \sinh t \sinh s, \cosh t \sinh s, \sinh t \cosh s)$$

Every point of $f(t,s)$ is on the hyperquadric M_{t_j}. Since $f(t,s)$ is both subgroup and submanifold of a Lie group M_{t_j}, it is a 2-dimensional Lie subgroup of M_{t_j}. □

Corollary 17. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2_1$, $\gamma(t) = (\cosh t, \sinh t)$ and $\delta : \mathbb{R} \to \mathbb{R}^2_1$ $\delta(s) = (\cosh s, \sinh s)$ be two Lorentzian circles centered at O $(a, b \in \mathbb{R})$. Then their tensor product is 2-dimensional Lie subgroup of $M^*_{t_j}$.*

Proposition 27. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2_1$, $\gamma(t) = (\cosh t, \sinh t)$ and $\delta : \mathbb{R} \to \mathbb{R}^2_1$ $\delta(s) = (\cosh s, \sinh s)$ be two Lorentzian circles centered at O $(a, b \in \mathbb{R})$. Then, the left invariant vector fields on tensor product surface $f(t,s) = \gamma(t) \otimes \delta(s)$ are X_3 and X_4 which are the left invariant vector fields on $M^*_{t_j}$.*

5.3. Tensor Product Surfaces on $M_{t_{ij}}$ Hyperquadric and Some Special Lie Subgroups

In this subsection, we use the definition of tensor product given by Mihai.

Let $\gamma : \mathbb{R} \to \mathbb{R}^2_k\ (+\,\beta)$ and $\delta : \mathbb{R} \to \mathbb{R}^2_t\ (+\,\alpha)$ be planar curves in Euclidean or Lorentzian space. Put $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ and $\delta(s) = (\delta_1(s), \delta_2(s))$.

$$\begin{aligned} &f = \gamma \otimes \delta : \mathbb{R}^2 \to \mathbb{R}^4_v\ (+\,\alpha\,\beta\,\alpha\beta), \\ &f(t,s) = (\gamma_1(t)\delta_1(s), \gamma_1(t)\delta_2(s), \gamma_2(t)\delta_1(s), \gamma_2(t)\delta_2(s)) \end{aligned} \tag{5}$$

tensor product surface given by (5) is a surface on $M_{t_{ij}}$ hyperquadric. Tangent vector fields of $f(t,s)$ can be easily computed as

$$\begin{aligned} \frac{\partial f}{\partial t} &= (\gamma_1'(t)\delta_1(s), \gamma_1'(t)\delta_2(s), \gamma_2'(t)\delta_1(s), \gamma_2'(t)\delta_2(s)) \\ \frac{\partial f}{\partial s} &= (\gamma_1(t)\delta_1'(s), \gamma_1(t)\delta_2'(s), \gamma_2(t)\delta_1'(s), \gamma_2(t)\delta_2'(s)) \end{aligned} \tag{6}$$

By using (6), we have

$$\begin{aligned} g_{11} &= g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right) = g_1(\gamma', \gamma')g_2(\delta, \delta) \\ g_{12} &= g\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right) = g_1(\gamma, \gamma')g_2(\delta, \delta') \\ g_{22} &= g\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\right) = g_1(\gamma, \gamma)g_2(\delta', \delta') \end{aligned}$$

where $g_1 = dx_1^2 + \beta dx_2^2$ and $g_2 = dx_1^2 + \alpha dx_2^2$ are the metrics of $\mathbb{R}^2_k$ and $\mathbb{R}^2_t$, respectively.

Remark 8. Tensor product surface given by (5) is a surface in $\mathbb{R}^4$ or $\mathbb{R}^4_2$ according to the case of α and β. If we take as $\alpha = \beta = 1$, we obtain a tensor product surface of two Euclidean plane curves in $\mathbb{R}^4$. If we take as $\alpha = 1$, $\beta = -1$, we obtain a tensor product surface of a Lorentzian plane curve and a Euclidean plane curve in $\mathbb{R}^4_2$. If we take as $\alpha = -1$, $\beta = 1$, we obtain a tensor product surface of a Euclidean plane curve and a Lorentzian plane curve in $\mathbb{R}^4_2$. If we take as $\alpha = -1$, $\beta = -1$ we obtain a tensor product surface of two Lorentzian plane curves in $\mathbb{R}^4_2$.

Now we investigate Lie group structure of tensor product surfaces given by the parametrization (5) in $\mathbb{R}^4$ or $\mathbb{R}^4_2$ according to above cases. Morever we obtain left invariant vector fields of the tensor product surface that has the structure of Lie group.

5.3.1. Case I $\alpha = \beta = 1$.

Proposition 28. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ $(+\,+)$ and $\delta : \mathbb{R} \to \mathbb{R}^2$ $(+\,+)$ be two spirals with the same parameter, i.e. $\gamma(t) = e^{at}(\cos t, \sin t)$ and $\delta(t) = e^{bt}(\cos t, \sin t)$ $(a, b \in \mathbb{R})$. Then their tensor product is a one parameter subgroup of Lie group $M_{t_{ij}}$.*

Proof. We obtain

$$\varphi(t) = \gamma(t) \otimes \delta(t) = e^{(a+b)t}\left(\cos^2 t, \cos t \sin t, \cos t \sin t, \sin^2 t\right)$$

It can be easily seen that

$$\varphi(t_1) \cdot \varphi(t_2) = \varphi(t_1 + t_2)$$

for all t_1, t_2. Also $\varphi^{-1}(t) = \varphi(-t)$. Hence $(\varphi(t), \cdot)$ is a one parameter Lie subgroup of Lie group $\left(M_{t_{ij}}, \cdot\right)$. □

Corollary 18. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ and $\delta : \mathbb{R} \to \mathbb{R}^2$ be two circles centered at O with the same parameter, i.e., $\gamma(t) = (\cos t, \sin t)$ and $\delta(t) = (\cos t, \sin t)$. Then their tensor product is a one parameter subgroup of Lie group $M^*_{t_{ij}}$.*

Proposition 29. *Let $\varphi(t)$ be tensor product of two circles centered at O with the same parameter. Then the left invariant vector field on $\varphi(t)$ is $X = X_2 + X_3$, where X_2 and X_3 are left invariant vector fields on $M^*_{t_{ij}}$.*

Proposition 30. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = e^{at}(\cos t, \sin t)$ and $\delta : \mathbb{R} \to \mathbb{R}^2$ $\delta(s) = e^{bs}(\cos s, \sin s)$ be two spirals $(a, b \in \mathbb{R})$. Then their tensor product is 2-dimensional Lie subgroup of $M_{t_{ij}}$.*

Proof. By using tensor product rule given by (5), we get

$$f(t, s) = e^{at+bs}(\cos t \cos s, \cos t \sin s, \sin t \cos s, \sin t \sin s)$$

Every point of $f(t, s)$ is on the hyperquadric $M_{t_{ij}}$. Since $f(t, s)$ is both subgroup and submanifold of a Lie group $M_{t_{ij}}$, it is a 2-dimensional Lie subgroup of $M_{t_{ij}}$. □

Corollary 19. *Let $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (\cos t, \sin t)$ and $\delta : \mathbb{R} \to \mathbb{R}^2$ $\delta(s) = (\cos s, \sin s)$ be two circles centered at O $(a, b \in \mathbb{R})$. Then their tensor product is 2-dimensional Lie subgroup of $M^*_{t_{ij}}$.*

Proposition 31. *Let* $\gamma : \mathbb{R} \to \mathbb{R}_1^2$, $\gamma(t) = (\cos t, \sin t)$ *and* $\delta : \mathbb{R} \to \mathbb{R}^2$ $\delta(s) = (\cos s, \sin s)$ *be two circles centered at* O $(a, b \in \mathbb{R})$. *Then, the left invariant vector fields on tensor product surface* $f(t,s) = \gamma(t) \otimes \delta(s)$ *are* X_2 *and* X_3 *which are the left invariant vector fields on* $M_{t_{ij}}^*$.

5.3.2. Case II $\alpha = 1, \beta = -1$.

Proposition 32. *Let* $\gamma : \mathbb{R} \to \mathbb{R}_1^2$ $(+\,-)$ *be a hyperbolic spiral and* $\delta : \mathbb{R} \to \mathbb{R}^2$ $(+\,+)$ *be a spiral with the same parameter, i.e.* $\gamma(t) = e^{at}(\cosh t, \sinh t)$ *and* $\delta(t) = e^{bt}(\cos t, \sin t)$ $(a, b \in \mathbb{R})$. *Their tensor product is a one parameter subgroup of Lie group* $M_{t_{ij}}$.

Proof. We obtain

$$\varphi(t) = \gamma(t) \otimes \delta(t) = e^{(a+b)t}(\cosh t \cos t, \cosh t \sin t, \sinh t \cos t, \sinh t \sin t)$$

It can be easily seen that

$$\varphi(t_1) \cdot \varphi(t_2) = \varphi(t_1 + t_2)$$

for all t_1, t_2. Also $\varphi^{-1}(t) = \varphi(-t)$. Hence $(\varphi(t), \cdot)$ is a one parameter Lie subgroup of Lie group $\left(M_{t_{ij}}, \cdot\right)$. □

Corollary 20. *Let* $\gamma : \mathbb{R} \to \mathbb{R}_1^2$ *be a Lorentzian circle centered at* O *and* $\delta : \mathbb{R} \to \mathbb{R}^2$ *be circle centered at* O *with the same parameter, i.e.,* $\gamma(t) = (\cosh t, \sinh t)$ *and* $\delta(t) = (\cos t, \sin t)$. *Then their tensor product is a one parameter subgroup of Lie group* $M_{t_{ij}}^*$.

Proposition 33. *Let* $\varphi(t)$ *be tensor product of a Lorentzian circle centered at* O *and a circle centered at* O *with the same parameter. Then the left invariant vector field on* $\varphi(t)$ *is* $X = X_2 + X_3$, *where* X_2 *and* X_3 *are left invariant vector fields on* $M_{t_{ij}}^*$.

Proposition 34. *Let* $\gamma : \mathbb{R} \to \mathbb{R}_1^2$, $\gamma(t) = e^{at}(\cosh t, \sinh t)$ *be a hyperbolic spiral and* $\delta : \mathbb{R} \to \mathbb{R}^2$ $\delta(s) = e^{bs}(\cos s, \sin s)$ *be a spiral* $(a, b \in \mathbb{R})$. *Then their tensor product is 2-dimensional Lie subgroup of* $M_{t_{ij}}$.

Proof. By using tensor product rule given by (5), we get

$$f(t,s) = e^{at+bs}(\cosh t \cos s, \cosh t \sin s, \sinh t \cos s, \sinh t \sin s)$$

Every point of $f(t,s)$ is on the hyperquadric $M_{t_{ij}}$. Since $f(t,s)$ is both subgroup and submanifold of a Lie group $M_{t_{ij}}$, it is a 2-dimensional Lie subgroup of $M_{t_{ij}}$. □

Proposition 35. *Let* $\gamma : \mathbb{R} \to \mathbb{R}_1^2$, $\gamma(t) = (\cosh t, \sinh t)$ *be a Lorentzian circle and* $\delta : \mathbb{R} \to \mathbb{R}^2$ $\delta(s) = (\cos s, \sin s)$ *be a circle* $(a, b \in \mathbb{R})$. *Then their tensor product is 2-dimensional Lie subgroup of* $M_{t_{ij}}^*$.

Proposition 36. *Let* $\gamma : \mathbb{R} \to \mathbb{R}_1^2$, $\gamma(t) = (\cosh t, \sinh t)$ *be a Lorentzian circle at centered* O *and* $\delta : \mathbb{R} \to \mathbb{R}^2$ $\delta(s) = (\cos s, \sin s)$ *be a circle at centered* O $(a, b \in \mathbb{R})$. *Then, the left invariant vector fields on tensor product surface* $f(t,s) = \gamma(t) \otimes \delta(s)$ *are* X_2 *and* X_3 *which are the left invariant vector fields on* $M_{t_{ij}}^*$.

5.3.3. Case III $\alpha = -1, \beta = -1$.

Proposition 37. *Let* $\gamma : \mathbb{R} \to \mathbb{R}_1^2$ $(+\,-)$ *and* $\delta : \mathbb{R} \to \mathbb{R}_1^2$ $(+\,-)$ *be two hyperbolic spirals with the same parameter, i.e.* $\gamma(t) = e^{at}(\cosh t, \sinh t)$ *and* $\delta(t) = e^{bt}(\cosh t, \sinh t)$ $(a, b \in \mathbb{R})$. *Then their tensor product is a one parameter subgroup of Lie group* $M_{t_{ij}}$.

Proof. We obtain

$$\varphi(t) = \gamma(t) \otimes \delta(t) = e^{(a+b)t}\left(\cosh^2 t, \cosh t \sinh t, \cosh t, \sinh t, \sinh^2 t\right)$$

It can be easily seen that

$$\varphi(t_1) \cdot \varphi(t_2) = \varphi(t_1 + t_2)$$

for all t_1, t_2. Also $\varphi^{-1}(t) = \varphi(-t)$. Hence $(\varphi(t), \cdot)$ is a one parameter Lie subgroup of Lie group $\left(M_{t_{ij}}, \cdot\right)$. □

Corollary 21. *Let* $\gamma : \mathbb{R} \to \mathbb{R}_1^2$ *and* $\delta : \mathbb{R} \to \mathbb{R}_1^2$ *be two Lorentzian circles centered at O with the same parameter, i.e.,* $\gamma(t) = (\cosh t, \sinh t)$ *and* $\delta(t) = (\cosh t, \sinh t)$. *Then their tensor product is a one parameter subgroup of Lie group* $M_{t_{ij}}^*$.

Proposition 38. *Let* $\varphi(t)$ *be tensor product of two Lorentzian circles centered at O with the same parameter. Then the left invariant vector field on* $\varphi(t)$ *is* $X = X_2 + X_3$, *where* X_2 *and* X_3 *are left invariant vector fields on* $M_{t_{ij}}^*$.

Proposition 39. *Let* $\gamma : \mathbb{R} \to \mathbb{R}_1^2$, $\gamma(t) = e^{at}(\cosh t, \sinh t)$ *and* $\delta : \mathbb{R} \to \mathbb{R}_1^2$ $\delta(s) = e^{bs}(\cosh s, \sinh s)$ *be two spirals* $(a, b \in \mathbb{R})$. *Then their tensor product is 2-dimensional Lie subgroup of* $M_{t_{ij}}$.

Proof. By using tensor product rule given by (5), we get

$$f(t, s) = e^{at+bs}(\cosh t \cosh s, \cosh t \sinh s, \sinh t \cosh s, , \sinh t \sinh s)$$

Every point of $f(t, s)$ is on the hyperquadric $M_{t_{ij}}$. Since $f(t, s)$ is both subgroup and submanifold of a Lie group $M_{t_{ij}}$, it is a 2-dimensional Lie subgroup of $M_{t_{ij}}$. □

Corollary 22. *Let* $\gamma : \mathbb{R} \to \mathbb{R}_1^2$, $\gamma(t) = (\cosh t, \sinh t)$ *and* $\delta : \mathbb{R} \to \mathbb{R}_1^2$ $\delta(s) = (\cosh s, \sinh s)$ *be two Lorentzian circles centered at O* $(a, b \in \mathbb{R})$. *Then their tensor product is 2-dimensional Lie subgroup of* $M_{t_{ij}}^*$.

Proposition 40. *Let* $\gamma : \mathbb{R} \to \mathbb{R}_1^2$, $\gamma(t) = (\cosh t, \sinh t)$ *and* $\delta : \mathbb{R} \to \mathbb{R}_1^2$ $\delta(s) = (\cosh s, \sinh s)$ *be two Lorentzian circles centered at O* $(a, b \in \mathbb{R})$. *Then, the left invariant vector fields on tensor product surface* $f(t, s) = \gamma(t) \otimes \delta(s)$ *are* X_2 *and* X_3 *which are the left invariant vector fields on* $M_{t_{ij}}^*$.

Remark 9. If we take as $\alpha = \beta = 1$ in the hyperquadric $M_{t_{ij}}$, the hyperquadric $M_{t_{ij}}$ coincides the hyperquadric P in the paper studied by Özkaldı and Yaylı [7], where the hyperquadric P is given by

$$P = \{x = (x_1, x_2, x_3, x_4) \neq 0; \quad x_1 x_4 = x_2 x_3\}.$$

In [7], they showed that the hyperquadric P is a Lie group by using bicomplex number product. Also they use the tensor product surface in $\mathbb{R}^4$ as follows:

$$(\gamma \otimes \delta)(t, s) = (\gamma_1(t)\delta_1(s), \gamma_1(t)\delta_2(s), \gamma_2(t)\delta_1(s), \gamma_2(t)\delta_2(s))$$

where $\gamma : \mathbb{R} \to \mathbb{R}^2$ and $\delta : \mathbb{R} \to \mathbb{R}^2$ are Euclidean plane curves. They determined some special Lie subgroups of this Lie group P by using the tensor product surfaces of Euclidean planar curves. So the case I in Sect. 5.1.2 coincides the paper studied by Özkaldı and Yaylı [7]. Hence, it can be considered that the Sect. 5.1.2 is a generalization of that study.

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