Indian J. Pure Appl. Math., **46**(1): 107-118, February 2015

DOI: 10.1007/s13226-015-0112-0

GENERAL ROTATIONAL SURFACES WITH POINTWISE 1-TYPE GAUSS MAP IN PSEUDO-EUCLIDEAN SPACE $\mathbf{E}_2^4$

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(*Received* 15 *February* 2013; *accepted* 14 *March* 2014)

In this paper, we study general rotational surfaces in the 4- dimensional pseudo-Euclidean space $\mathbb{E}_2^4$ and obtain a characterization of flat general rotation surfaces with pointwise 1-type Gauss map in $\mathbb{E}_2^4$ and give an example of such surfaces.

Key words : Rotation surface, Gauss map, Pointwise 1-type Gauss map , pseudo-Euclidean space.

1. INTRODUCTION

A pseudo-Riemannian submanifold M of the $m-$dimensional pseudo-Euclidean space $\mathbb{E}_s^m$ is said to be of finite type if its position vector x can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is, $x = x_0 + x_1 + ...x_k$, where x_0 is a constant map, $x_1, ..., x_k$ are non-constant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, ..., k$. If λ_1, λ_2,...,λ_k are all different, then M is said to be of $k-$type. This definition was similarly extended to differentiable maps in Euclidean and pseudo-Euclidean space, in particular, to Gauss maps of submanifolds [6].

If a submanifold M of a Euclidean space or pseudo-Euclidean space has 1-type Gauss map G, then G satisfies $\Delta G = \lambda (G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector C. Chen and Piccinni made a general study on compact submanifolds of Euclidean spaces with finite type Gauss map and they proved that a compact hypersurface M of $\mathbb{E}^{n+1}$ has 1-type Gauss map if and only if M is a hypersphere in $\mathbb{E}^{n+1}$ [6].

However the Laplacian of the Gauss map of several surfaces and hypersurfaces such as a helicoids of the 1st, 2nd and 3rd kind, conjugate Enneper's surface of the second kind in 3-dimensional

Minkowski space E_1^3, generalized catenoids, spherical n-cones, hyperbolical n-cones and Enneper's hypersurfaces in E_1^n take the form namely,

$$\Delta G = f\,(G + C) \tag{1}$$

for some smooth function f on M and some constant vector C. A submanifold M of a pseudo-Euclidean space $\mathbb{E}_s^m$ is said to have pointwise 1-type Gauss map if its Gauss map satisfies (1) for some smooth function f on M and some constant vector C. A submanifold with pointwise 1-type Gauss map is said to be of the first kind if the vector C in (1) is zero vector. Otherwise, the pointwise 1-type Gauss map is said to be of the second kind.

Surfaces in Euclidean space and in pseudo-Euclidean space with pointwise 1-type Gauss map were recently studied in [5, 7, 8, 9, 11-15, 17]. Also Dursun and Turgay in [10] gave all general rotational surfaces in $\mathbb{E}^4$ with proper pointwise 1-type Gauss map of the first kind and classified minimal rotational surfaces with proper pointwise 1-type Gauss map of the second kind. Arslan *et al.* in [2] investigated rotational embedded surface with pointwise 1-type Gauss map. Arslan *et al.* in [3] gave necessary and sufficent conditions for Vranceanu rotation surface to have pointwise 1-type Gauss map. Yoon in [20] showed that flat Vranceanu rotation surface with pointwise 1-type Gauss map is a Clifford torus and in [19] studied rotation surfaces in the 4-dimensional Euclidean space with finite type Gauss map. Kim and Yoon in [16] obtained the complete classification theorems for the flat rotation surfaces with finite type Gauss map and pointwise 1-type Gauss map. The authors in [1] studied flat general rotational surfaces in the 4- dimensional Euclidean space $\mathbb{E}^4$ with pointwise 1-type Gauss map and they showed that flat general rotational surfaces with pointwise 1-type Gauss map is a Lie group if and only if it is a Clifford Torus.

In this paper, we study general rotational surfaces in the 4- dimensional pseudo-Euclidean space $\mathbb{E}_2^4$ and obtain a characterization for flat general rotation surfaces with pointwise 1-type Gauss map and give an example of such surfaces.

2. PRELIMINARIES

Let E_s^m be the $m-$dimensional pseudo-Euclidean space with signature $(s, m-s)$. Then the metric tensor g in E_s^m has the form

$$g = \sum_{i=1}^{m-s} (dx_i)^2 - \sum_{i=m-s+1}^{m} (dx_i)^2$$

where $(x_1, ..., x_m)$ is a standard rectangular coordinate system in E_s^m.

Let M be an $n-$dimensional pseudo-Riemannian submanifold of a $m-$dimensional pseudo-Euclidean space $\mathbb{E}_s^m$. We denote Levi-Civita connections of $\mathbb{E}_s^m$ and M by $\tilde{\nabla}$ and ∇, respectively. Let e_1,..., e_n, e_{n+1},..., e_m be an adapted local orthonormal frame in $\mathbb{E}_s^m$ such that e_1,..., e_n are tangent to M and e_{n+1},..., e_m normal to M. We use the following convention on the ranges of indices: $1 \leq i, j, k, ... \leq n$, $n+1 \leq r, s, t, ... \leq m$, $1 \leq A, B, C, ... \leq m$.

Let ω_A be the dual-1 form of e_A defined by $\omega_A(X) = \langle e_A, X \rangle$ and $\varepsilon_A = \langle e_A, e_A \rangle = \pm 1$. Also, the connection forms ω_{AB} are defined by

$$de_A = \sum_B \varepsilon_B \omega_{AB} e_B, \quad \omega_{AB} + \omega_{BA} = 0.$$

Then we have

$$\tilde{\nabla}_{e_k}^{e_i} = \sum_{j=1}^{n} \varepsilon_j \omega_{ij}(e_k) e_j + \sum_{r=n+1}^{m} \varepsilon_r h_{ik}^r e_r \tag{2}$$

and

$$\tilde{\nabla}_{e_k}^{e_s} = -\sum_{j=1}^{n} \varepsilon_j h_{kj}^s e_j + \sum_{r=n+1}^{m} \varepsilon_r \omega_{sr}(e_k) e_r, D_{e_k}^{e_s} = \sum_{r=n+1}^{m} \varepsilon_r \omega_{sr}(e_k) e_r, \tag{3}$$

where D is the normal connection, h_{ik}^r the coefficients of the second fundamental form h.

If we define a covariant differention $\tilde{\nabla}h$ of the second fundamental form h on the direct sum of the tangent bundle and the normal bundle $TM \oplus T^{\perp}M$ of M by

$$\left(\tilde{\nabla}_X h\right)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

for any vector fields X, Y and Z tangent to M. Then we have the Codazzi equation

$$\left(\tilde{\nabla}_X h\right)(Y, Z) = \left(\tilde{\nabla}_Y h\right)(X, Z) \tag{4}$$

and the Gauss equation is given by

$$\langle R(X,Y)Z, W \rangle = \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle \tag{5}$$

where the vectors X, Y, Z and W are tangent to M and R is the curvature tensor associated with ∇. The curvature tensor R associated with ∇ is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

For any real function f on M the Laplacian Δf of f is given by

$$\Delta f = -\varepsilon_i \sum_i \left(\tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} f - \tilde{\nabla}_{\nabla_{e_i}^{e_i}} f \right). \tag{6}$$

Let us now define the Gauss map G of a submanifold M into $G(n,m)$ in $\wedge^n \mathbb{E}_s^m$, where $G(n,m)$ is the Grassmannian manifold consisting of all oriented $n-$planes through the origin of $\mathbb{E}_s^m$ and $\wedge^n \mathbb{E}_s^m$ is the vector space obtained by the exterior product of n vectors in $\mathbb{E}_s^m$. Let $e_{i_1} \wedge ... \wedge e_{i_n}$ and $f_{j_1} \wedge ... \wedge f_{j_n}$ be two vectors of $\wedge^n \mathbb{E}_s^m$, where $\{e_1, ..., e_m\}$ and $\{f_1, ..., f_m\}$ are orthonormal bases of $\mathbb{E}_s^m$. Define an indefinite inner product $\langle , \rangle$ on $\wedge^n \mathbb{E}_s^m$ by

$$\langle e_{i_1} \wedge ... \wedge e_{i_n}, f_{j_1} \wedge ... \wedge f_{j_n} \rangle = \det\left(\langle e_{i_l}, f_{j_k} \rangle\right).$$

Therefore, for some positive integer t, we may identify $\wedge^n \mathbb{E}_s^m$ with some Euclidean space $\mathbb{E}_t^N$ where $N = \begin{pmatrix} m \\ n \end{pmatrix}$. The map $G : M \to G(n,m) \subset E_t^N$ defined by $G(p) = (e_1 \wedge ... \wedge e_n)(p)$ is called the Gauss map of M, that is, a smooth map which carries a point p in M into the oriented $n-$plane in $\mathbb{E}_s^m$ obtained from parallel translation of the tangent space of M at p in $\mathbb{E}_s^m$.

3. Flat Rotation Surfaces with Pointwise 1-Type Gauss Map in E_2^4

In this section, we study the flat rotation surfaces with pointwise 1-type Gauss map in the 4-dimensional pseudo-Euclidean space E_2^4. Let M_1 and M_2 be the rotation surfaces in E_2^4 defined by

$$\varphi(t,s) = \begin{pmatrix} \cosh t & 0 & 0 & \sinh t \\ 0 & \cosh t & \sinh t & 0 \\ 0 & \sinh t & \cosh t & 0 \\ \sinh t & 0 & 0 & \cosh t \end{pmatrix} \begin{pmatrix} 0 \\ x(s) \\ 0 \\ y(s) \end{pmatrix},$$

$$M_1 : \ \varphi(t,s) = (y(s)\sinh t, x(s)\cosh t, x(s)\sinh t, y(s)\cosh t) \tag{7}$$

and

$$\varphi(t,s) = \begin{pmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -\sin t \\ 0 & 0 & \sin t & \cos t \end{pmatrix} \begin{pmatrix} x(s) \\ 0 \\ y(s) \\ 0 \end{pmatrix}$$

$$M_2 : \ \varphi(t,s) = (x(s)\cos t, x(s)\sin t, y(s)\cos t, y(s)\sin t) \tag{8}$$

where the profile curve of M_1 (resp. the profile curve of M_2) is unit speed curve, that is, $(x'(s))^2 - (y'(s))^2 = 1$. We choose a moving frame e_1, e_2, e_3, e_4 such that e_1, e_2 are tangent to M_1 and e_3, e_4 are normal to M_1 and choose a moving frame $\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4$ such that $\bar{e}_1, \bar{e}_2$ are tangent to M_2 and

$\bar{e}_3, \bar{e}_4$ are normal to M_2 which are given by the following:

$$\begin{aligned}
e_1 &= \frac{1}{\sqrt{\varepsilon_1 \left(y^2(s) - x^2(s)\right)}} \left(y(s)\cosh t, x(s)\sinh t, x(s)\cosh t, y(s)\sinh t\right) \\
e_2 &= \left(y'(s)\sinh t, x'(s)\cosh t, x'(s)\sinh t, y'(s)\cosh t\right) \\
e_3 &= \left(x'(s)\sinh t, y'(s)\cosh t, y'(s)\sinh t, x'(s)\cosh t\right) \\
e_4 &= \frac{1}{\sqrt{\varepsilon_1 \left(y^2(s) - x^2(s)\right)}} \left(x(s)\cosh t, y(s)\sinh t, y(s)\cosh t, x(s)\sinh t\right)
\end{aligned}$$

and

$$\begin{aligned}
\bar{e}_1 &= \frac{1}{\sqrt{\varepsilon_1 \left(y^2(s) - x^2(s)\right)}} \left(-x(s)\sin t, x(s)\cos t, -y(s)\sin t, y(s)\cos t\right) \\
\bar{e}_2 &= \left(x'(s)\cos t, x'(s)\sin t, y'(s)\cos t, y'(s)\sin t\right) \\
\bar{e}_3 &= \left(y'(s)\cos t, y'(s)\sin t, x'(s)\cos t, x'(s)\sin t\right) \\
\bar{e}_4 &= \frac{1}{\sqrt{\varepsilon_1 \left(y^2(s) - x^2(s)\right)}} \left(y(s)\sin t, -y(s)\cos t, x(s)\sin t, -x(s)\cos t\right)
\end{aligned}$$

where $\varepsilon_1 \left(y^2(s) - x^2(s)\right) > 0$, $\varepsilon_1 = \pm 1$. Then it is easily seen that

$$\begin{aligned}
\langle e_1, e_1 \rangle &= -\langle e_4, e_4 \rangle = \varepsilon_1, \quad \langle e_2, e_2 \rangle = -\langle e_3, e_3 \rangle = 1 \\
-\langle \bar{e}_1, \bar{e}_1 \rangle &= \langle \bar{e}_4, \bar{e}_4 \rangle = \varepsilon_1, \quad \langle \bar{e}_2, \bar{e}_2 \rangle = -\langle \bar{e}_3, \bar{e}_3 \rangle = 1
\end{aligned}$$

we have the dual 1-forms as:

$$\omega_1 = \varepsilon_1 \sqrt{\varepsilon_1 \left(y^2(s) - x^2(s)\right)} dt \quad \text{and} \quad \omega_2 = ds \tag{9}$$

and

$$\bar{\omega}_1 = -\varepsilon_1 \sqrt{\varepsilon_1 \left(y^2(s) - x^2(s)\right)} dt \quad \text{and} \quad \bar{\omega}_2 = ds. \tag{10}$$

By a direct computation we have components of the second fundamental form and the connection forms as:

$$\begin{aligned}
h_{11}^3 &= b(s),\ h_{12}^3 = 0,\ h_{22}^3 = c(s) \qquad (11) \\
h_{11}^4 &= 0,\ h_{12}^4 = b(s),\ h_{22}^4 = 0
\end{aligned}$$

$$\begin{aligned}
\bar{h}_{11}^3 &= -b(s),\ \bar{h}_{12}^3 = 0,\ \bar{h}_{22}^3 = c(s) \qquad (12) \\
\bar{h}_{11}^4 &= 0,\ \bar{h}_{12}^4 = b(s),\ \bar{h}_{22}^4 = 0
\end{aligned}$$

$$\begin{aligned}\omega_{12} &= \varepsilon_1 a(s)\omega_1, \quad \omega_{13} = \varepsilon_1 b(s)\omega_1, \quad \omega_{14} = b(s)\omega_2 \\ \omega_{23} &= c(s)\omega_2, \quad \omega_{24} = \varepsilon_1 b(s)\omega_1, \quad \omega_{34} = \varepsilon_1 a(s)\omega_1\end{aligned} \tag{13}$$

$$\begin{aligned}\bar{\omega}_{12} &= \varepsilon_1 a(s)\bar{\omega}_1, \quad \bar{\omega}_{13} = \varepsilon_1 b(s)\bar{\omega}_1, \quad \bar{\omega}_{14} = b(s)\bar{\omega}_2 \\ \bar{\omega}_{23} &= c(s)\bar{\omega}_2, \quad \bar{\omega}_{24} = -\varepsilon_1 b(s)\bar{\omega}_1, \quad \bar{\omega}_{34} = -\varepsilon_1 a(s)\bar{\omega}_1.\end{aligned} \tag{14}$$

By covariant differentiation with respect to e_1 and e_2 (resp. $\bar{e}_1$ and $\bar{e}_2$) a straightforward calculation gives:

$$\begin{aligned}\tilde{\nabla}_{e_1} e_1 &= a(s)e_2 - b(s)e_3 \\ \tilde{\nabla}_{e_2} e_1 &= -\varepsilon_1 b(s)e_4 \\ \tilde{\nabla}_{e_1} e_2 &= -\varepsilon_1 a(s)e_1 - \varepsilon_1 b(s)e_4 \\ \tilde{\nabla}_{e_2} e_2 &= -c(s)e_3 \\ \tilde{\nabla}_{e_1} e_3 &= -\varepsilon_1 b(s)e_1 - \varepsilon_1 a(s)e_4 \\ \tilde{\nabla}_{e_2} e_3 &= -c(s)e_2 \\ \tilde{\nabla}_{e_1} e_4 &= -b(s)e_2 + a(s)e_3 \\ \tilde{\nabla}_{e_2} e_4 &= -\varepsilon_1 b(s)e_1\end{aligned} \tag{15}$$

and

$$\begin{aligned}\tilde{\nabla}_{\bar{e}_1} \bar{e}_1 &= -a(s)\bar{e}_2 + b(s)\bar{e}_3 \\ \tilde{\nabla}_{\bar{e}_2} \bar{e}_1 &= \varepsilon_1 b(s)\bar{e}_4 \\ \tilde{\nabla}_{\bar{e}_1} \bar{e}_2 &= -\varepsilon_1 a(s)\bar{e}_1 + \varepsilon_1 b(s)\bar{e}_4 \\ \tilde{\nabla}_{\bar{e}_2} \bar{e}_2 &= -c(s)\bar{e}_3 \\ \tilde{\nabla}_{\bar{e}_1} \bar{e}_3 &= -\varepsilon_1 b(s)\bar{e}_1 + \varepsilon_1 a(s)\bar{e}_4 \\ \tilde{\nabla}_{\bar{e}_2} \bar{e}_3 &= -c(s)\bar{e}_2 \\ \tilde{\nabla}_{\bar{e}_1} \bar{e}_4 &= -b(s)\bar{e}_2 + a(s)\bar{e}_3 \\ \tilde{\nabla}_{\bar{e}_2} \bar{e}_4 &= \varepsilon_1 b(s)\bar{e}_1\end{aligned} \tag{16}$$

where

$$a(s) = \frac{x(s)x'(s) - y(s)y'(s)}{\varepsilon_1 \left(y^2(s) - x^2(s)\right)} \tag{17}$$

$$b(s) = \frac{x(s)y'(s) - x'(s)y(s)}{\varepsilon_1 \left(y^2(s) - x^2(s)\right)} \tag{18}$$

$$c(s) = x''(s)y'(s) - x'(s)y''(s) \tag{19}$$

The Gaussian curvature K of M_1 and $\bar{K}$ that of M_2 are respectively given by

$$K = \varepsilon_1 b^2(s) - b(s)c(s) \tag{20}$$

and

$$\bar{K} = b(s)c(s) - \varepsilon_1 b^2(s) \tag{21}$$

If the surfaces M_1 or M_2 is flat, then (20) and (21) imply

$$b(s)c(s) - \varepsilon_1 b^2(s) = 0. \tag{22}$$

Furthermore, after some computations we obtain Gauss and Codazzi equations for both surfaces M_1 and M_2

$$\varepsilon_1 a^2(s) - a'(s) = b(s)c(s) - \varepsilon_1 b^2(s) \tag{23}$$

and

$$b'(s) = 2\varepsilon_1 a(s)b(s) - a(s)c(s) \tag{24}$$

respectively.

By using (6), (15), (16) and straight-forward computations, the Laplacians ΔG and $\Delta\bar{G}$ of the Gauss map G and $\bar{G}$ can be expressed as

$$\begin{aligned} \Delta G &= -\left(3b^2(s) + c^2(s)\right)(e_1 \wedge e_2) + \left(2a(s)b(s) - \varepsilon_1 a(s)c(s) + c'(s)\right)(e_1 \wedge e_3) \\ &\quad + \left(3a(s)b(s) - \varepsilon_1 b'(s)\right)(e_2 \wedge e_4) + 2\left(\varepsilon_1 b(s)c(s) - b^2(s)\right)(e_3 \wedge e_4) \end{aligned} \tag{25}$$

$$\begin{aligned} \Delta\bar{G} &= -\left(3b^2(s) + c^2(s)\right)(\bar{e}_1 \wedge \bar{e}_2) + \left(2a(s)b(s) - \varepsilon_1 a(s)c(s) + c'(s)\right)(\bar{e}_1 \wedge \bar{e}_3) \\ &\quad + \left(-3a(s)b(s) + \varepsilon_1 b'(s)\right)(\bar{e}_2 \wedge \bar{e}_4) + 2\left(b^2(s) - \varepsilon_1 b(s)c(s)\right)(\bar{e}_3 \wedge \bar{e}_4). \end{aligned} \tag{26}$$

Now we investigate the flat rotation surfaces in E^4_2 with the pointwise 1-type Gauss map satisfying (1).

Suppose that the rotation surface M_1 given by the parametrization (7) is a flat rotation surface. From (20), we obtain that $b(s) = 0$ or $\varepsilon_1 b(s) - c(s) = 0$. We assume that $\varepsilon_1 b(s) - c(s) \neq 0$. Then $b(s)$ is equal to zero and (24) implies that $a(s)c(s) = 0$. Since $\varepsilon_1 b(s) - c(s) \neq 0$, it implies that $c(s)$ is not equal to zero. Then we obtain as $a(s) = 0$. In that case, by using (17) and (18) we obtain that

$\alpha(s) = (0, x(s), 0, y(s))$ is a constant vector. This is a contradiction. Therefore $\varepsilon_1 b(s) = c(s)$ for all s. From (23), we get

$$\varepsilon_1 a^2(s) - a'(s) = 0 \tag{27}$$

whose the trivial solution and non-trivial solution

$$a(s) = 0$$

and

$$a(s) = \frac{1}{-\varepsilon_1 s + c},$$

respectively. We assume that $a(s) = 0$. By (24) $b = b_0$ is a constant and $c = \varepsilon_1 b_0$. In that case by using (17), (18) and (19), x and y satisfy the following differential equations

$$x^2(s) - y^2(s) = \mu \quad \mu \text{ is a constant,} \tag{28}$$

$$x(s)y'(s) - x'(s)y(s) = -\varepsilon_1 b_0 \mu, \tag{29}$$

$$x''(s)y'(s) - x'(s)y''(s) = \varepsilon_1 b_0. \tag{30}$$

From (28) we may put

$$x(s) = \frac{1}{2}\varepsilon\left(\mu_2 e^{\theta(s)} + \mu_1 e^{-\theta(s)}\right), \quad y(s) = \frac{1}{2}\varepsilon\left(\mu_2 e^{\theta(s)} - \mu_1 e^{-\theta(s)}\right), \tag{31}$$

where $\theta(s)$ is some smooth function, $\varepsilon = \pm 1$ and $\mu = \mu_1\mu_2$. Differentiating (31) with respect to s, we have

$$x'(s) = \theta'(s)y(s), \quad y'(s) = \theta'(s)x(s). \tag{32}$$

By substituting (31) and (32) into (29), we get

$$\theta(s) = -\varepsilon_1 b_0 s + d, \quad d = const.$$

And since the curve α is a unit speed curve, we have

$$b_0^2 \mu = -1.$$

Since $\mu = -\frac{1}{b_0^2}$, $y^2(s) - x^2(s) > 0$. In that case we obtain that $\varepsilon_1 = 1$. Then we can write components of the curve α as:

$$\begin{aligned} x(s) &= \frac{1}{2}\varepsilon\left(\mu_2 e^{(-b_0 s + d)} + \mu_1 e^{-(-b_0 s + d)}\right), \\ y(s) &= \frac{1}{2}\varepsilon\left(\mu_2 e^{(-b_0 s + d)} - \mu_1 e^{-(-b_0 s + d)}\right), \ \mu_1\mu_2 = -\frac{1}{b_0^2}. \end{aligned} \tag{33}$$

On the other hand, by using (25) we can rewrite the Laplacian of the Gauss map G with $a(s) = 0$ and $b = c = b_0$ as follows:

$$\Delta G = -4b_0^2 (e_1 \wedge e_2)$$

that is, the flat surface M is pointwise 1-type Gauss map with the function $f = -4b_0^2$ and $C = 0$. Even if it is a pointwise 1-type Gauss map of the first kind.

Now we assume that $a(s) = \frac{1}{-\varepsilon_1 s + c}$. By using $c(s) = \varepsilon_1 b(s)$ and (24) we get

$$b'(s) = \varepsilon_1 a(s) b(s) \tag{34}$$

or we can write

$$\frac{b'(s)}{b(s)} = \frac{\varepsilon_1}{-\varepsilon_1 s + c},$$

whose the solution

$$b(s) = \frac{\lambda}{|-\varepsilon_1 s + c|}, \quad \lambda \text{ is a constant.} \tag{35}$$

By using (25) we can rewrite the Laplacian of the Gauss map G with the equations $c(s) = \varepsilon_1 b(s)$, $b'(s) = \varepsilon_1 a(s) b(s)$ and $a'(s) = \varepsilon_1 a^2(s)$

$$\Delta G = -4b^2(s)(e_1 \wedge e_2) + 2a(s)b(s)(e_1 \wedge e_3) + 2a(s)b(s)(e_2 \wedge e_4). \tag{36}$$

We suppose that the flat rotational surface M_1 has pointwise 1-type Gauss map. From (1) and (36), we get

$$-4\varepsilon_1 b^2(s) = f\varepsilon_1 + f\langle C, e_1 \wedge e_2\rangle \tag{37}$$

$$-2\varepsilon_1 a(s)b(s) = f\langle C, e_1 \wedge e_3\rangle \tag{38}$$

$$-2\varepsilon_1 a(s)b(s) = f\langle C, e_2 \wedge e_4\rangle \tag{39}$$

Then, we have

$$\langle C, e_1 \wedge e_4\rangle = 0, \;\; \langle C, e_2 \wedge e_3\rangle = 0, \;\; \langle C, e_3 \wedge e_4\rangle = 0 \tag{40}$$

By using (38) and (39) we obtain

$$\langle C, e_1 \wedge e_3\rangle = \langle C, e_2 \wedge e_4\rangle \tag{41}$$

By differentiating the first equation in (40) with respect to e_1 and by using the third equation in (40) and (41), we get

$$2a(s)\langle C, e_1 \wedge e_3\rangle - b(s)\langle C, e_1 \wedge e_2\rangle = 0 \tag{42}$$

Combining $(37),(38)$ and (42) we then have

$$f=4\left(a^2(s)-b^2(s)\right)$$

that is, a smooth function f depends only on s. By differentiating f with respect to s and by using (34) and (27), we get

$$f'=2\varepsilon_1 a(s)f \tag{43}$$

By differentiating (38) with respect to s and by using $(15),(27),(34),(37)$ and (38) we have

$$a^2b=0$$

or from (35) we can write

$$\lambda a^3=0$$

Since $a(s)\neq 0$, it follows that $\lambda=0$. Then we obtain that $b=c=0$. Then the surface M_1 is a totally geodesic.

Thus we can give the following theorems.

***Theorem* 1** — *Let M_1 be the flat rotation surface given by the parametrization (7). Then M_1 has pointwise 1-type Gauss map if and only if M_1 is either totally geodesic or parametrized by*

$$\varphi(t,s)=\begin{pmatrix}\frac{1}{2}\varepsilon\left(\mu_2e^{(-b_0s+d)}-\mu_1e^{-(-b_0s+d)}\right)\sinh t,\\ \frac{1}{2}\varepsilon\left(\mu_2e^{(-b_0s+d)}+\mu_1e^{-(-b_0s+d)}\right)\cosh t,\\ \frac{1}{2}\varepsilon\left(\mu_2e^{(-b_0s+d)}+\mu_1e^{-(-b_0s+d)}\right)\sinh t,\\ \frac{1}{2}\varepsilon\left(\mu_2e^{(-b_0s+d)}-\mu_1e^{-(-b_0s+d)}\right)\cosh t\end{pmatrix},\quad \mu_1\mu_2=-\frac{1}{b_0^2}. \tag{44}$$

where b_0, μ_1, μ_2 and d are real constants.

Example 1: Let M_1 be the flat rotation surface with pointwise 1-type Gauss map given by the parametrization (44). If we take as $b_0=-1$, $\mu_1=-1$, $\mu_2=1$, $d=0$ and $\varepsilon=1$, then we obtain a surface as follows:

$$\varphi(t,s)=(\cosh s\sinh t,\sinh s\cosh t,\sinh s\sinh t,\cosh s\cosh t).$$

This surface is the product of two plane hyperbolas.

***Theorem* 2** — *Let M_2 be the flat rotation surface given by the parametrization (8). Then M_2 has pointwise 1-type Gauss map if and only if M_2 is either totally geodesic or parametrized by*

$$\varphi(t,s)=\begin{pmatrix}\frac{1}{2}\varepsilon\left(\mu_2e^{(-b_0s+d)}+\mu_1e^{-(-b_0s+d)}\right)\cos t,\\ \frac{1}{2}\varepsilon\left(\mu_2e^{(-b_0s+d)}+\mu_1e^{-(-b_0s+d)}\right)\sin t,\\ \frac{1}{2}\varepsilon\left(\mu_2e^{(-b_0s+d)}-\mu_1e^{-(-b_0s+d)}\right)\cos t,\\ \frac{1}{2}\varepsilon\left(\mu_2e^{(-b_0s+d)}-\mu_1e^{-(-b_0s+d)}\right)\sin t\end{pmatrix},\quad \mu_1\mu_2=-\frac{1}{b_0^2}. \tag{45}$$

Example 2 : Let M_2 be the flat rotation surface with pointwise 1-type Gauss map given by the parametrization (45). If we take as $b_0 = -1$, $\mu_1 = -1$, $\mu_2 = 1$, $d = 0$ and $\varepsilon = 1$, then we obtain a surface as follows:

$$\varphi(t, s) = (\sinh s \cos t, \sinh s \sin t, \cosh s \cos t, \cosh s \sin t) .$$

This surface is the product of a plane circle and a plane hyperbola.

Corollary 1 — Let M be non-totally geodesic flat general rotation surface given by the parametrization (7) or (8). If M has pointwise 1-type Gauss map then the Gauss map G on M is of 1-type.

REFERENCES

1. F. Aksoyak and Y. Yaylı, Flat Rotational Surfaces with Pointwise 1-Type Gauss Map in E^4, arXiv. 1302.2804, 2013.

2. K. Arslan, B. K. Bayram, B. Bulca, Y. H. Kim, C. Murathan and G. Öztürk, Rotational Embeddings in E^4 with Pointwise 1-Type Gauss Map, *Turk. J. Math.*, **35** (2011), 493-499.

3. K. Arslan, B. K. Bayram, Y. H. Kim, C. Murathan and G. Öztürk, Vranceanu Surface in E^4 with Pointwise 1-Type Gauss Map, *Indian J. Pure. Appl. Math.*, **42** (2011), 41-51.

4. K. Arslan, B. Bulca, B. Kılıç, Y. H. Kim, C. Murathan and G. Öztürk, Tensor Product Surfaces with Pointwise 1-Type Gauss Map, *Bull. Korean Math. Soc.*, **48** (2011), 601-609.

5. B. Y. Chen, M. Choi and Y. H. Kim, Surfaces of Revolution with Pointwise 1-Type Gauss Map, *J. Korean Math.*, **42** (2005), 447-455.

6. B. Y. Chen and P. Piccinni, Submanifolds with Finite Type-Gauss map, *Bull. Austral. Math. Soc.*, **35** (1987), 161-186.

7. M. Choi and Y. H. Kim, Characterization of the Helicoid as Ruled Surfaces with Pointwise 1-Type Gauss Map, *Bull. Korean Math. Soc.*, **38** (2001), 753-761.

8. M. Choi, D. S. Kim and Y. H. Kim, Helicoidal Surfaces with Pointwise 1-Type Gauss Map, *J. Korean Math. Soc.*, **46** (2009), 215-223.

9. M. Choi and Kim, Y. H. and D. W. Yoon, Classification of Ruled Surfaces with Pointwise 1-Type Gauss Map in Minkowski 3-Space, *Taiwanese J. Math.*, **15** (2011), 1141-1161.

10. U. Dursun and N. C. Turgay, General Rotational Surfaces in Euclidean Space E^4 with Pointwise 1-Type Gauss Map, *Math. Commun.*, **17** (2012), 71-81.

11. U. Dursun, Hypersurfaces with Pointwise 1-Type Gauss Map in Lorentz-Minkowski Space, *Proc. Est. Acad. Sci.*, **58** (2009), 146-161.

12. U. Dursun and E. Coşkun, Flat Surfaces in the Minkowski Space E_1^3 with Pointwise 1-Type Gauss Map, *Turk. J. Math.*, **36** (2012), 613-629.

13. U. Dursun and G. G. Arsan, Surfaces in the Euclidean Space E^4 with Pointwise 1-Type Gauss Map, *Hacet. J. Math. Stat.*, **40** (2011), 617-625.

14. U. Dursun and N. C. Turgay, On Spacelike Surfaces in Minkowski 4-Space with Pointwise 1-Type Gauss Map of Second Type, *Balkan J. Geom. App.*, **17** (2012), 34-45.

15. Y. H. Kim and D. W. Yoon, Ruled Surfaces with Pointwise 1-Type Gauss Map, *J. Geom. Phys.*, **34** (2000), 191-205.

16. Y. H. Kim and D. W. Yoon, Classification of Rotation Surfaces in Pseudo Euclidean Space, *J. Korean Math.*, **41** (2004), 379-396.

17. A. Niang, Rotation Surfaces with 1-Type Gauss Map, *Bull. Korean Math. Soc.*, **42** (2005), 23-27.

18. A. Niang, On Rotation Surfaces in the Minkowski 3- Dimensional Space with Pointwise 1-Type Gauss Map, *J. Korean Math. Soc.*, **41** (2004), 1007-1021.

19. D. W. Yoon, Rotation Surfaces with Finite Type Gauss Map in E^4, *Indian J. Pure. Appl. Math.*, **32** (2001), 1803-1808.

20. D. W. Yoon, Some Properties of The Clifford Torus as Rotation Surface, *Indian J. Pure. Appl. Math.*, **34** (2003), 907-915.