Boost Invariant Surfaces with Pointwise 1-Type Gauss Map in Minkowski 4-Space E₁⁴

Ferdag KAHRAMAN AKSOYAK ¹, Yusuf YAYLI ²

¹Erciyes University, Department of Mathematics, Kayseri, Turkey ²Ankara University, Department of Mathematics, Ankara, Turkey

Abstract

In this paper, we study spacelike rotational surfaces which are called boost invariant surfaces in Minkowski 4-space \mathbb{E}_1^4 . We give necessary and sufficient condition for flat spacelike rotational surface to have pointwise 1-type Gauss map. Also, we obtain a characterization for boost invariant marginally trapped surface with pointwise 1-type Gauss map.

Key words: Rotation surface, Gauss map, Pointwise 1-type Gauss map, Marginally trapped surface, Minkowski space.

2000 Mathematics Subject Classification: 53B25; 53C50.

1 Introduction

The notion of finite type mapping was introduced by B.Y. Chen in late 1970's. A pseudo- Riemannian submanifold M of the m-dimensional pseudo-Euclidean space \mathbb{E}_s^m is said to be of finite type if its position vector x can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is, $x = x_0 + x_1 + ... + x_k$, where x_0 is a constant map, $x_1, ..., x_k$ are non-constant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, i = 1, 2, ..., k. If $\lambda_1, \lambda_2, ..., \lambda_k$ are all different, then M is said to be of k-type. This notion of finite type immersions is naturally extended to differentiable maps of M in particular, to Gauss maps of submanifolds [7].

If a submanifold M of a Euclidean space or pseudo-Euclidean space has 1-type Gauss map G, then G satisfies $\Delta G = \lambda (G+C)$ for some $\lambda \in \mathbb{R}$ and some constant vector C. Chen and Piccinni made a general study on compact submanifolds of Euclidean spaces with finite type Gauss map and they proved that a compact hypersurface M of \mathbb{E}^{n+1} has 1-type Gauss map if and only if M is a hypersphere in \mathbb{E}^{n+1} [7].

Hovewer the Laplacian of the Gauss map of some typical well-known surfaces such as a helicoid, a catenoid and right cone in 3-dimensional Euclidean space E^3 and a helicoids of the 1st,2nd and 3rd kind, conjugate Enneper's surface of the

 $^{^{0}}$ E-mail: 1 ferda@erciyes.edu.tr(F. Kahraman Aksoyak); 2 yayli@science.ankara.edu.tr(Y.Yayli)

second kind and B-scrolls in 3- dimensional Minkowski space E_1^3 take a somewhat different form namely,

$$\Delta G = f(G+C) \tag{1}$$

for some non-zero smooth function f on M and some constant vector C. This equation is similar to an eigenvalue problem but the smooth function f is not always constant. So a submanifold M of a pseudo-Euclidean space \mathbb{E}_s^m is said to have pointwise 1-type Gauss map if its Gauss map satisfies (1) for some smooth function f on M and some constant vector C. A submanifold with pointwise 1-type Gauss map is said to be of the first kind if the vector C in (1) is zero vector. Otherwise, the pointwise 1-type Gauss map is said to be of the second kind.

Surfaces in Euclidean space and in pseudo-Euclidean space with pointwise 1-type Gauss map were recently studied in [6], [9], [10], [11], [13], [14], [15], [16], [20], [23], [24]. Also Dursun and Turgay in [12] gave all general rotational surfaces in \mathbb{E}^4 with proper pointwise 1-type Gauss map of the first kind and classified minimal rotational surfaces with proper pointwise 1-type Gauss map of the second kind. Arslan et al. in [3] investigated rotational embedded surface with pointwise 1-type Gauss map. Arslan at el. in [4] gave necessary and sufficent conditions for Vranceanu rotation surface to have pointwise 1-type Gauss map. Yoon in [26] showed that flat Vranceanu rotation surface with pointwise 1-type Gauss map is a Clifford torus and in [25] studied rotation surfaces in the 4dimensional Euclidean space with finite type Gauss map. Kim and Yoon in [21] obtained the complete classification theorems for the flat rotation surfaces with finite type Gauss map and pointwise 1-type Gauss map. The authors in [1] studied flat general rotational surfaces in the 4- dimensional Euclidean space \mathbb{E}^4 with pointwise 1-type Gauss map and they showed that a non-planar flat general rotational surfaces with pointwise 1-type Gauss map is a Lie group if and only if it is a Clifford Torus. Also they gave a characterization for flat general rotation surfaces with pointwise 1-type Gauss map in the 4- dimensional pseudo-Euclidean space \mathbb{E}_2^4 [2].

On the other hand, trapped surfaces, introduced by Penrose in 1965, have a fundamental role in the study of the singularity theorems in General Relativity. If the mean curvature vector of a surface in E_1^4 is timelike everywhere, it is called trapped surfaces; if the mean curvature vector is always null (the mean curvature vector is proportional to one of the null normals), the surface is called marginally trapped surface. Since the mean curvature of such spacelike surface H satisfy ||H|| = 0, in mathematical literature these surfaces are called quasi-minimal. In general relativity, marginally trapped surfaces are used the study of the surfaces of black hole.

S.Haesen and M. Ortega in [18] and [19] classified marginally trapped surfaces which are invariant under a spacelike rotations and boost transformations in Minkowski 4-space. Also B. Y. Chen classify marginally trapped Lorentzian flat surfaces and biharmonic surfaces in the Pseudo Euclidean space E_2^4 [8]. Milou-

sheva in [22] studied marginally trapped surface with pointwise 1-type Gauss map in Minkowski 4-space and proved that marginally trapped surface is of pointwise 1-type Gauss map if and only if it has parallel mean curvature vector field.

In this paper, we study spacelike surfaces which are invariant under boost transformation (hyperbolic rotations) in Minkowski 4-space. We give a characterization of flat spacelike rotational surface with pointwise 1-type Gauss map. Also we obtain a characterization for boost invariant marginally trapped surface with pointwise 1-type Gauss map and give an example of such surfaces.

2 Preliminaries

Let E_s^m be the m-dimensional pseudo-Euclidean space with signature (s, m-s). Then the metric tensor g in E_s^m has the form

$$g = \sum_{i=1}^{m-s} (dx_i)^2 - \sum_{i=m-s+1}^{m} (dx_i)^2$$

where $(x_1, ..., x_m)$ is a standard rectangular coordinate system in E_s^m .

Let M be an n-dimensional pseudo-Riemannian submanifold of a m-dimensional pseudo-Euclidean space \mathbb{E}^m_s . We denote Levi-Civita connections of \mathbb{E}^m_s and M by $\tilde{\nabla}$ and ∇ , respectively. Let $e_1,...,e_n,e_{n+1},...,e_m$ be an adapted local orthonormal frame in \mathbb{E}^m_s such that $e_1,...,e_n$ are tangent to M and $e_{n+1},...,e_m$ normal to M. We use the following convention on the ranges of indices: $1 \leq i,j,k,... \leq n,$ $n+1 \leq r,s,t,... \leq m,$ $1 \leq A,B,C,... \leq m$.

Let ω_A be the dual-1 form of e_A defined by $\omega_A(X) = \langle e_A, X \rangle$ and $\varepsilon_A = \langle e_A, e_A \rangle = \pm 1$. Also, the connection forms ω_{AB} are defined by

$$de_A = \sum_B \varepsilon_B \omega_{AB} e_B, \quad \omega_{AB} + \omega_{BA} = 0$$

Then we have

$$\tilde{\nabla}_{e_k}^{e_i} = \sum_{j=1}^n \varepsilon_j \omega_{ij} (e_k) e_j + \sum_{r=n+1}^m \varepsilon_r h_{ik}^r e_r$$

and

$$\tilde{\nabla}_{e_k}^{e_s} = -\sum_{j=1}^n \varepsilon_j h_{kj}^s e_j + \sum_{r=n+1}^m \varepsilon_r \omega_{sr}(e_k) e_r, \qquad D_{e_k}^{e_s} = \sum_{r=n+1}^m \omega_{sr}(e_k) e_r, \qquad (2)$$

where D is the normal connection, h_{ik}^r the coefficients of the second fundamental form h. The mean curvature vector H of M in \mathbb{E}_s^m is defined by

$$H = \frac{1}{n} \sum_{s=n+1}^{m} \sum_{i=1}^{n} \varepsilon_{i} \varepsilon_{s} h_{ii}^{s} e_{s}$$

and the Gaussian curvature K of M is given by

$$K = \sum_{s=n+1}^{m} \varepsilon_s \left(h_{11}^s h_{22}^s - h_{12}^s h_{21}^s \right)$$

Also normal curvature tensor \mathbb{R}^D of M in $\mathbb{E}^{m=n+2}_s$ is given by

$$R^{D}(e_j, e_k; e_r, e_s) = \sum_{i=1}^{n} \varepsilon_i \left(h_{ik}^r h_{ij}^s - h_{ij}^r h_{ik}^s \right)$$
(3)

We recall that a surface M in \mathbb{E}_1^4 is called extremal surface if its mean curvature vector vanishes. If its Gaussian curvature vanishes, the surface M is called flat surface. If its normal curvature tensor R^D vanishes identically then a surface M in \mathbb{E}_1^4 is said to have flat normal bundle.

For any real function f on M the Laplacian Δf of f is given by

$$\Delta f = -\varepsilon_i \sum_{i} \left(\tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} f - \tilde{\nabla}_{\nabla_{e_i}^{e_i}} f \right) \tag{4}$$

Let us now define the Gauss map G of a submanifold M into G(n,m) in $\wedge^n \mathbb{E}^m_s$, where G(n,m) is the Grassmannian manifold consisting of all oriented n-planes through the origin of \mathbb{E}^m_s and $\wedge^n \mathbb{E}^m_s$ is the vector space obtained by the exterior product of n vectors in \mathbb{E}^m_s . Let $e_{i_1} \wedge ... \wedge e_{i_n}$ and $f_{j_1} \wedge ... \wedge f_{j_n}$ be two vectors of $\wedge^n \mathbb{E}^m_s$, where $\{e_1, ..., e_m\}$ and $\{f_1, ..., f_m\}$ are orthonormal bases of \mathbb{E}^m_s . Define an indefinite inner product \langle , \rangle on $\wedge^n \mathbb{E}^m_s$ by

$$\langle e_{i_1} \wedge ... \wedge e_{i_n}, f_{j_1} \wedge ... \wedge f_{j_n} \rangle = \det (\langle e_{i_l}, f_{j_k} \rangle).$$

Therefore, for some positive integer t, we may identify $\wedge^n \mathbb{E}^m_s$ with some Euclidean space \mathbb{E}^N_t where $N = \binom{m}{n}$. The map $G: M \to G(n,m) \subset E^N_t$ defined by $G(p) = (e_1 \wedge \ldots \wedge e_n)(p)$ is called the Gauss map of M, that is, a smooth map which carries a point p in M into the oriented n-plane in \mathbb{E}^m_s obtained from parallel translation of the tangent space of M at p in \mathbb{E}^m_s .

3 Boost Invariant Surfaces with Pointwise 1-Type Gauss Map in E_1^4

In this section, we consider spacelike surfaces in the Minkowski space E_1^4 which are invariant under the following subgroup of direct, linear isometries of E_1^4 :

$$G = \left\{ \begin{pmatrix} \cos t & -\sin t & 0 & 0\\ \sin t & \cos t & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\},\,$$

well-known as boost isometries.

$$\varphi(t,s) = \begin{pmatrix} \cos t & -\sin t & 0 & 0\\ \sin t & \cos t & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1(s) \\ 0 \\ \alpha_3(s) \\ \alpha_4(s) \end{pmatrix}$$

$$M: \varphi(t,s) = (\alpha_1(s)\cosh t, \alpha_1(s)\sinh t, \alpha_3(s), \alpha_4(s))$$
(5)

where the profile curve of M is unit speed spacelike curve, that is, $-(\alpha'_1(s))^2 + (\alpha'_3(s))^2 + (\alpha'_4(s))^2 = 1$. We choose a moving frame e_1, e_2, e_3, e_4 such that e_1, e_2 are tangent to M and e_3, e_4 are normal to M which are given by the following:

$$e_{1} = (\alpha'_{1}(s) \cosh t, \alpha'_{1}(s) \sinh t, \alpha'_{3}(s), \alpha'_{4}(s))$$

$$e_{2} = (\sinh t, \cosh t, 0, 0)$$

$$e_{3} = \frac{1}{\sqrt{1 + (\alpha'_{1}(s))^{2}}} ((1 + (\alpha'_{1}(s))^{2}) \cosh t, (1 + (\alpha'_{1}(s))^{2}) \sinh t$$

$$, \alpha'_{1}(s)\alpha'_{3}(s), \alpha'_{1}(s)\alpha'_{4}(s)$$

$$e_{4} = \frac{1}{\sqrt{1 + (\alpha'_{1}(s))^{2}}} (0, 0, -\alpha'_{4}(s), \alpha'_{3}(s))$$

Then it is easily seen that

$$\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = \langle e_4, e_4 \rangle = 1, \ \langle e_3, e_3 \rangle = -1$$

we have the dual 1-forms as:

$$\omega_1 = ds$$
 and $\omega_2 = \alpha_1(s)dt$ (6)

By a direct computation we have components of the second fundamental form and the connection forms as:

$$h_{11}^3 = -c(s), h_{12}^3 = 0, h_{22}^3 = -b(s)$$
 (7)
 $h_{11}^4 = d(s), h_{12}^4 = 0, h_{22}^4 = 0$

$$\omega_{12} = a(s)b(s)\omega_{2}, \quad \omega_{13} = -c(s)\omega_{1}, \quad \omega_{14} = d(s)\omega_{1}
\omega_{23} = -b(s)\omega_{2}, \quad \omega_{24} = 0, \quad \omega_{34} = a(s)d(s)\omega_{1}$$
(8)

By covariant differentiation with respect to e_1 and e_2 a straightforward calculation gives:

$$\tilde{\nabla}_{e_{1}}e_{1} = c(s)e_{3} + d(s)e_{4}
\tilde{\nabla}_{e_{2}}e_{1} = a(s)b(s)e_{2}
\tilde{\nabla}_{e_{1}}e_{2} = 0
\tilde{\nabla}_{e_{2}}e_{2} = -a(s)b(s)e_{1} + b(s)e_{3}
\tilde{\nabla}_{e_{1}}e_{3} = c(s)e_{1} + a(s)d(s)e_{4}
\tilde{\nabla}_{e_{2}}e_{3} = b(s)e_{2}
\tilde{\nabla}_{e_{1}}e_{4} = -d(s)e_{1} + a(s)d(s)e_{3}
\tilde{\nabla}_{e_{2}}e_{4} = 0$$
(9)

where

$$a(s) = \frac{\alpha_1'(s)}{\sqrt{1 + (\alpha_1'(s))^2}}$$
 (10)

$$b(s) = \frac{\sqrt{1 + (\alpha_1'(s))^2}}{\alpha_1(s)} \tag{11}$$

$$c(s) = \frac{\alpha_1''(s)}{\sqrt{1 + (\alpha_1'(s))^2}}$$
(12)

$$d(s) = \frac{-\alpha_3''(s)\alpha_4'(s) + \alpha_4''(s)\alpha_3'(s)}{\sqrt{1 + (\alpha_1'(s))^2}}$$
(13)

The Gaussian curvature K of M is given by

$$K = -b(s)c(s) \tag{14}$$

The mean curvature H of M is given by

$$H = \frac{1}{2} \left(-h_1 e_3 + h_2 e_4 \right) \quad h_1 = -\left(b + c \right) \text{ and } h_2 = d \tag{15}$$

By using (4), (9) and straight-forward computation, the Laplacian ΔG of the Gauss map G can be expressed as

$$\Delta G = A(s) (e_1 \wedge e_2) + B(s) (e_2 \wedge e_3) + D(s) (e_2 \wedge e_4)$$
(16)

where

$$A(s) = d^{2}(s) - b^{2}(s) - c^{2}(s)$$
(17)

$$B(s) = b'(s) + c'(s) + a(s)d^{2}(s)$$
(18)

$$D(s) = d'(s) + a(s)d(s)(b(s) + c(s))$$
(19)

Theorem 1. Let M be the flat rotation surface given by the parametrization (5). Then M has pointwise 1-type Gauss map if and only if the profile curve of M is parametrized by

$$\alpha_{1}(s) = a_{1}$$

$$\alpha_{3}(s) = \frac{1}{a_{2}} \left(1 + a_{1}^{2}\right)^{\frac{1}{2}} \cos\left(a_{2}s + a_{3}\right)$$

$$\alpha_{4}(s) = -\frac{1}{a_{2}} \left(1 + a_{1}^{2}\right)^{\frac{1}{2}} \sin\left(a_{2}s + a_{3}\right)$$
(20)

or

$$\alpha_{1}(s) = b_{1}s + b_{2}$$

$$\alpha_{3}(s) = \int (1 + b_{1}^{2})^{\frac{1}{2}} \cos(b \ln|b_{1}s + b_{2}|) ds$$

$$\alpha_{4}(s) = \int (1 + b_{1}^{2})^{\frac{1}{2}} \sin(b \ln|b_{1}s + b_{2}|) ds$$
(21)

where a_1 , a_2 , a_3 $b_1 \neq 0$, b_2 , b_3 and $b = \frac{b_3}{b_1(1+b_1^2)^{\frac{1}{2}}}$ are real constants.

Proof. Let M be the flat rotation surface given by the parametrization (5). We suppose that M has pointwise 1-type Gauss map. By using (1) and (16), we have

$$f + f \langle C, e_1 \wedge e_2 \rangle = A(s)$$

$$f \langle C, e_2 \wedge e_3 \rangle = -B(s)$$

$$f \langle C, e_2 \wedge e_4 \rangle = D(s)$$
(22)

and

$$\langle C, e_1 \wedge e_3 \rangle = \langle C, e_1 \wedge e_4 \rangle = \langle C, e_3 \wedge e_4 \rangle = 0$$
 (23)

By differentiating (23) covariantly with respect to s, we have

$$-a(s)B(s) + A(s) - f = 0$$
$$a(s)D(s) = 0$$
$$D(s) = 0$$

In this case, firstly, we assume that a(s) = 0 and D(s) = 0. From (10), we obtain that $\alpha_1(s) = a_1$. Since the profile curve is unit speed spacelike curve, we can write $(\alpha_3'(s))^2 + (\alpha_4'(s))^2 = 1 + a_1^2$. Also we can put

$$\alpha_3'(s) = (1 + a_1^2)^{\frac{1}{2}} \cos \theta (s)$$

$$\alpha_4'(s) = (1 + a_1^2)^{\frac{1}{2}} \sin \theta (s)$$
(24)

where θ is smooth angle function. On the other hand, since D(s) = 0, from (19) we obtain as

$$d(s) = a_2, \quad a_2 \text{ is non zero constant.}$$
 (25)

By using (13), (24) and (25) we get

$$\theta\left(s\right) = a_2 s + a_3 \tag{26}$$

So from (24) and (26) we have

$$\alpha_3(s) = \frac{1}{a_2} \left(1 + a_1^2 \right)^{\frac{1}{2}} \cos \left(a_2 s + a_3 \right)$$

$$\alpha_4(s) = -\frac{1}{a_2} \left(1 + a_1^2 \right)^{\frac{1}{2}} \sin \left(a_2 s + a_3 \right)$$

Now we assume that $a(s) \neq 0$ and D(s) = 0. Since M is flat, (12) and (14) imply that

$$\alpha_1(s) = b_1 s + b_2 \tag{27}$$

for some constants $b_1 \neq 0$ and $b_2 = 0$. Since the profile curve is unit speed spacelike curve, we can write $(\alpha_3'(s))^2 + (\alpha_4'(s))^2 = 1 + b_1^2$. Also we can put

$$\alpha_3'(s) = (1 + b_1^2)^{\frac{1}{2}} \cos \theta (s)$$

$$\alpha_4'(s) = (1 + b_1^2)^{\frac{1}{2}} \sin \theta (s)$$
(28)

where θ is smooth angle function. By using (10), (11) and (19), we get

$$d(s) = \frac{b_3}{b_1 s + b_2} \tag{29}$$

On the other hand, by using (13), (27) and (28) we have

$$d(s) = (1 + b_1^2)^{\frac{1}{2}} \theta'(s) \tag{30}$$

By combining (29) and (30) we obtain

$$\theta\left(s\right) = b\ln\left|b_1 s + b_2\right| \tag{31}$$

where $b = \frac{b_3}{b_1(1+b_1^2)^{\frac{1}{2}}}$. So by substituting (31) into (28) we can write

$$\alpha_3(s) = \int (1 + b_1^2)^{\frac{1}{2}} \cos(b \ln|b_1 s + b_2|) ds$$

$$\alpha_4(s) = \int (1 + b_1^2)^{\frac{1}{2}} \sin(b \ln|b_1 s + b_2|) ds$$

Conversely, the surface M which is parametrized by (20) and (21) is pointwise 1-type Gauss map for

$$f(s) = -a(s)b'(s) - a^{2}(s)d^{2}(s) + d^{2}(s) - b^{2}(s)$$

and

$$C(s) = \frac{a(s)b'(s) + a^2(s)d^2(s)}{f(s)} (e_1 \wedge e_2) + \frac{b'(s) + a(s)d^2(s)}{f(s)} (e_2 \wedge e_3)$$

where it can be easily seen that $e_1(C(s)) = 0$ and $e_2(C(s)) = 0$. This completes the proof.

Corollary 1. Let M be the flat rotation surface given by the parametrization (5). If M has pointwise 1-type Gauss map then the profile curve of M is a helix curve.

We will also use the following theorems and corollary.

Theorem 2. [17] Let M be an oriented maximal surface in the Minkowski space E_1^4 . Then M has pointwise 1-type Gauss map of the first kind if and only if M has flat normal bundle. Hence the Gauss map G satisfies (1.1) for $f = ||h||^2$ and C = 0.

Theorem 3. [18] Let M be a spacelike rotational surface in Minkowski 4-space given by the parametrization (5). If M marginally trapped surface then

$$\alpha_3(s) = \int \left(1 + (\alpha_1')^2\right)^{\frac{1}{2}} \cos\theta(s) ds$$

$$\alpha_4(s) = \int \left(1 + (\alpha_1')^2\right)^{\frac{1}{2}} \sin\theta(s) ds$$
(32)

and

$$\theta(s) = -\epsilon \int \frac{1 + (\alpha_1')^2 + \alpha_1' \alpha_1''}{\alpha_1 \left(1 + (\alpha_1')^2\right)^{\frac{1}{2}}}$$
(33)

where $\epsilon = \pm$.

Corollary 2. [18] Let M be a spacelike rotational surface in Minkowski 4-space given by the parametrization (5). If M is a extremal surface then a unit profile curve is given by

$$\alpha(s) = \left(f(s), 0, \cos \zeta_0 \sqrt{a_1} \arctan \left(\frac{s+a_2}{f(s)}\right), \sin \zeta_0 \sqrt{a_1} \arctan \left(\frac{s+a_2}{f(s)}\right)\right),$$

where $f(s) = \sqrt{a_1 - (s + a_2)^2}$ and $a_1, a_2, \zeta_0 \in \mathbb{R}$, $a_1 > 0$, being integration constants. In particular, the surface M is immersed in a totally geodesic Lorentzian 3-space.

Theorem 4. Let M be the marginally trapped surface given by the parametrization (5) in Minkowski 4-space. Then M has pointwise 1-type Gauss map if and only if the profile curve is given by or

$$\alpha_{1}(s) = (\lambda_{1} - 1)^{\frac{1}{2}} \left(u^{2}(s) + \lambda^{2}\right)^{\frac{1}{2}}$$

$$\alpha_{3}(s) = \int \left(\frac{\lambda_{1}u^{2} + \lambda^{2}}{u^{2} + \lambda^{2}}\right)^{\frac{1}{2}} \cos\theta(s) ds$$

$$\alpha_{4}(s) = \int \left(\frac{\lambda_{1}u^{2} + \lambda^{2}}{u^{2} + \lambda^{2}}\right)^{\frac{1}{2}} \sin\theta(s) ds$$
(34)

and

$$\theta(s) = -\epsilon \frac{\lambda_1}{(\lambda_1 - 1)^{\frac{1}{2}}} \int \frac{(u^2 + \lambda^2)^{\frac{1}{2}}}{\lambda_1 u^2 + \lambda^2} ds$$

where $u(s) = \delta s + \lambda_3$, $\lambda = \frac{\lambda_2}{\lambda_1 - 1}$, λ_1 , λ_2 , λ_3 , λ_3 , λ_4 and λ_2 are real constants.

Proof. Let M be marginally trapped surface. This means ||H|| = 0 that is $\langle H, H \rangle = 0$. By using (15), we get

$$-(b(s) + c(s)) = \epsilon d(s) \tag{35}$$

where $\epsilon = \pm$. In this case, by using (35) we can rewrite the Laplacian ΔG of the Gauss map G as

$$\Delta G = A(s) (e_1 \wedge e_2) - \epsilon N(s) (e_2 \wedge e_3) + N(s) (e_2 \wedge e_4)$$
(36)

where

$$N(s) = d'(s) - \epsilon a(s)d^{2}(s)$$
(37)

We assume that M has pointwise 1-type Gauss map. Then we have

$$f + f \langle C, e_1 \wedge e_2 \rangle = A(s)$$

$$f \langle C, e_2 \wedge e_3 \rangle = \epsilon N(s)$$

$$f \langle C, e_2 \wedge e_4 \rangle = N(s)$$
(38)

and

$$\langle C, e_1 \wedge e_3 \rangle = \langle C, e_1 \wedge e_4 \rangle = \langle C, e_3 \wedge e_4 \rangle = 0$$
 (39)

By differentiating (39) covariantly with respect to s, we have

$$\epsilon a(s)N(s) + A(s) - f = 0$$

$$a(s)N(s) = 0$$

$$N(s) = 0$$

In this case, firstly, we assume that a(s) = 0 and N(s) = 0. From (10) and (12), we obtain that $\alpha_1(s) = a_1$ and c(s) = 0, respectively. Hence from (35) we get

$$-b(s) = \epsilon d(s) \tag{40}$$

By using (40) and (17) we obtain that A(s) = 0. So we have that f = 0. This is a contradiction.

Now we assume that $a(s) \neq 0$ and N(s) = 0. By combining (10), (11), (12), (13), (35) and (37), we obtain a differential equation as follows:

$$\left(1 + (\alpha_1'(s))^2 + \alpha_1'(s)\alpha_1''(s)\right)'\alpha_1(s)\left(1 + (\alpha_1'(s))^2\right) = 0$$

Since $\alpha_1 > 0$ and $1 + (\alpha'_1(s))^2 \neq 0$ we have

$$1 + (\alpha_1'(s))^2 + \alpha_1'(s)\alpha_1''(s) = \lambda_1$$

whose the solition

$$\alpha_1(s) = (\lambda_1 - 1)^{\frac{1}{2}} \left((\delta s + \lambda_3)^2 + \frac{\lambda_2}{(\lambda_1 - 1)^2} \right)^{\frac{1}{2}}$$
(41)

By using (33) and (41) we get

$$\theta(s) = -\epsilon \mu \int \frac{(u^2 + \lambda)^{\frac{1}{2}}}{\lambda_1 u^2 + \lambda} ds \tag{42}$$

where $u(s) = \delta s + \lambda_3$, $\lambda = \frac{\lambda_2}{\lambda_1 - 1}$ and $\mu = \frac{\lambda_1}{(\lambda_1 - 1)^{\frac{1}{2}}}$.

Conversely, the surface M which is parametrized by (34) has pointwise 1-type Gauss map with

$$f(s) = 2b(s)c(s)$$

and

$$C(s) = 0$$

This completes the proof.

Corollary 3. Let M be marginally trapped surface given by the parametrization (5) in Minkowski 4-space. Then M has pointwise 1-type Gauss map then M is pointwise 1-type Gauss map of the first kind.

Corollary 4. Let M be a spacelike rotational surface in Minkowski 4-space given by the parametrization (5). If M is extremal surface then M has pointwise 1-type Gauss map of the first kind.

Proof. We assume that M is a spacelike rotational surface given by the parametrization (5). In that case by using (3) and (7) we obtain that M has flat normal bundle. Hence from Theorem (2) If M is extremal surface then M has pointwise 1-type Gauss map of the first kind.

References

- [1] Aksoyak F. and Yaylı Y. Flat Rotational Surfaces with pointwise 1-type Gauss map in E^4 , arXiv:1302.2804 (under review).
- [2] Aksoyak F. and Yaylı Y. General rotational surfaces with pointwise 1-type Gauss map in pseudo-Euclidean space E_2^4 , arXiv:1302.2910 (under review).
- [3] Arslan K., Bayram, B.K., Bulca, B., Kim, Y.H., Murathan, C. and Öztürk, G. Rotational embeddings in E^4 with pointwise 1-type Gauss map, Turk. J. Math. 35, 493-499, 2011.
- [4] Arslan K., Bayram B.K., Kim, Y.H., Murathan, C. and Öztürk, G. Vranceanu surface in E^4 with pointwise 1-type Gauss map, Indian J. Pure. Appl. Math. 42, 41-51, 2011.
- [5] Arslan K., Bulca B., Kılıç B., Kim Y.H., Murathan C. and Öztürk G. Tensor Product Surfaces with Pointwise 1-Type Gauss Map. Bull. Korean Math. Soc. 48, 601-609, 2011.
- [6] Chen, B.Y. Choi, M. and Kim, Y.H. Surfaces of revolution with pointwise 1-type Gauss map, J. Korean Math. 42, 447-455, 2005.
- [7] Chen, B.Y. and Piccinni, P. Submanifolds with Finite Type-Gauss map, Bull. Austral. Math. Soc., 35, 161-186, 1987.
- [8] B.Y.Chen , Classification of marginally trapped Lorentzian flat surfaces in E_2^4 and its application to biharmonic surfaces, J. Math. Anal. Appl. 340, 861-875, 2008.
- [9] Choi, M. and Kim, Y.H. Characterization of the helicoid as ruled surfaces with pointwise 1-type Gauss map, Bull. Korean Math. Soc. 38, 753-761, 2001.
- [10] Choi, M., Kim, D.S., Kim Y.H, Helicoidal surfaces with pointwise 1-type Gauss map, J. Korean Math. Soc. 46, 215-223, 2009.
- [11] Choi, M. and Kim, Y.H. and Yoon, D.W. Classification of ruled surfaces with pointwise 1-type Gauss map in Minkowski 3-space, Taiwanese J. Math. 15, 1141-1161, 2011.
- [12] Dursun, U. and Turgay, N.C., General rotational surfaces in Euclidean space E^4 with pointwise 1-type Gauss map, Math. Commun. 17, 71-81, 2012.
- [13] Dursun, U., Hypersurfaces with pointwise 1-type Gauss map in Lorentz-Minkowski space, Proc. Est. Acad. Sci. 58, 146-161, 2009.

- [14] Dursun, U., Coşkun, E. Flat surfaces in the Minkowski space E_1^3 with pointwise 1-type Gauss map, Turk. J. Math. 35, 1-1, 2011.
- [15] Dursun, U. and Arsan, G.G. Surfaces in the Euclidean space E^4 with pointwise 1-type Gauss map, Hacet. J. Math. Stat. 40, 617-625, 2011.
- [16] Dursun, U. and Turgay, N.C., On spacelike surfaces in Minkowski 4-space with pointwise 1-type Gauss map of second type, Balkan J. Geom. App., 17, 34-45, 2012.
- [17] Dursun, U. and Turgay, N.C., Space-like surfaces in Minkowski space E_1^4 with pointwise 1-type Gauss map. arXiv:1305.5419
- [18] Haesen S. and Ortega M., Boost invariant marginally trapped surfaces in Minkowski 4-space. Class. Quantum Grav. 24, 5441-5452, 2007.
- [19] Haesen S. and Ortega M., Marginally trapped surfaces in Minkowski 4-space invariant under a rotation subgroup of the Lorentz group. Gen Relativ Gravit 41 (2009), 1819-1834.
- [20] Kim, Y.H. and Yoon, D.W. Ruled surfaces with pointwise 1-type Gauss map, J. Geom. Phys. 34, 191-205, 2000
- [21] Kim, Y.H. and Yoon, D.W. Classification of rotation surfaces in pseudo Euclidean space, J. Korean Math. 41, 379-396, 2004.
- [22] Milousheva, V. Marginally trapped surfaces with pointwise 1-type Gauss map in Minkowski 4-space, Int. Journal of Geom., 2, 1, 34-43, 2013.
- [23] Niang, A. Rotation surfaces with 1-type Gauss map, Bull. Korean Math. Soc. 42, 23-27, 2005
- [24] Niang, A. On rotation surfaces in the Minkowski 3- dimensional space with pointwise 1-type Gauss map, J. Korean Math. Soc. 41, 1007-1021, 2004.
- [25] Yoon, D.W. Rotation surfaces with finite type Gauss map in E^4 , Indian J. Pure. Appl. Math. 32, 1803-1808, 2001.
- [26] Yoon, D.W. Some properties of the Clifford torus as rotation surface, Indian J. Pure. Appl. Math. 34, 907-915, 2003.