The Finite-Size Scaling Study of Five-Dimensional Ising Model

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The five-dimensional ferromagnetic Ising model is simulated on the Creutz cellular automaton algorithm using finite-size lattices with linear dimension $4 \leq L \leq 8$. The critical temperature value of infinite lattice is found to be $T^{\chi}(\infty) = 8.7811\,(1)$ using $4 \leq L \leq 8$ which is also in very good agreement with the precise result. The value of the field critical exponent ($\delta = 3.0067(2)$) is good agreement with $\delta = 3$ which is obtained from scaling law of Widom. The exponents in the finite-size scaling relations for the magnetic susceptibility and the order parameter at the infinite-lattice critical temperature are computed to be 2.5080 (1), 2.5005 (3) and 1.2501 (1) using $4 \leq L \leq 8$, respectively, which are in very good agreement with the theoretical predictions of $\frac{5}{2}$ and $\frac{5}{4}$. The finite-size scaling plots of magnetic susceptibility and the order parameter verify the finite-size scaling relations about the infinite-lattice temperature.

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1. Introduction

While the five-dimensional ferromagnetic Ising model is not directly applicable to real magnetic systems, it is useful to investigate the influence of dimensionality on phase transitions [1]. In fact, in Euclidean quantum field theory, the 4-d ferromagnetic Ising model describes the physical dimension. As the dimensionality and/or the lattice size increases, the simulation of the ferromagnetic Ising model by the conventional Monte Carlo method becomes impractical and faster algorithms are needed. The Creutz cellular automaton algorithm [2] does not require high-quality random numbers, it is an order of magnitude faster than the conventional Monte Carlo method and compared to the Q2R cellular automaton [3], it has the advantage of fluctuating internal energy from which the specific heat can be computed.

The four-dimensional Ising model in the presence of external magnetic field is simulated [3]. In two dimensions, the solution of ferromagnetic Ising model is investigated [4–8]. By considering different approximate methods, the approximations of the solutions of two-dimensional ferromagnetic Ising model are presented [9–13]. In addition, the four-dimensional ferromagnetic Ising model solution is approximated by using Creutz cellular automaton algorithm with nearest neighbor interactions and near the critical region [14–23]. The algorithm of approximating finite size behavior of ferromagnetic Ising model is extended to higher dimensions [14–32]. It is established that the algorithm has been powerful in terms of providing the values of static critical exponents near the critical region in four and higher dimensions with nearest neighbor interactions [14–32].

In this paper, we present the result of the simulation of the five-dimensional ferromagnetic Ising model with the Creutz cellular automaton algorithm. The value of the field critical exponent (δ) is obtained. The exponents $\left(\frac{d}{2}, \frac{d}{4}\right)$ for the magnetic susceptibility and the order parameter at T_c are obtained for $4 \leq L \leq 8$ which is also in very good agreement with the theoretical results.

The model is described in Sect. 2, the results are discussed in Sect. 3 and a conclusion is given in Sect. 4.

2. Model

In general, $n \geq 5$ binary bits are associated with each site of the lattice. The value for each site is determined from its value and from those of its nearest neighbors at the previous time step. The updating rule, which defines a deterministic cellular automaton, is as follows: of the n binary bits on each site, the first one is the Ising spin B_i. Its value may be "0" or "1". The Ising spin energy in the presence of an external magnetic field, H_{I}, is described by the Hamiltonian of the form

$$H_{\mathrm{I}} = -J \sum_{\langle ij \rangle} S_i S_j - h \sum_{\langle i \rangle} S_i, \tag{1}$$

by taking into account of the interaction between the nearest neighbors and also the interaction of the spins S_i with external magnetic field h, directed "up" ($S_i = +1$). The spins affected by the field are directed "up" and not changed during the simulation. Therefore, these spins play the role of the magnetic field. In the Hamiltonian, $S_i = 2B_i - 1$ and h is the ratio of the number of "up" spins to the number of all spins. The next $n-2$ bits are for the momentum variable conjugate to the spin (the demon). These $n-2$ bits form an integer which can take values within the interval $(0, \sum_{i=1}^{n-2} 2^{i-1})$. The kinetic energy (in units of J) associated with the demon can take on four times these integer values. The total energy

$$H = H_{\mathrm{I}} + H_{\mathrm{K}}, \tag{2}$$

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is conserved; here H_K is the kinetic energy of the lattice. For a given total energy, the system temperature T (in units of J/k_B where k_B is the Boltzmann constant) is obtained from the average value of the kinetic energy of a demon. The n-th bit provides a checkerboard style updating, therefore it allows the simulation of the Ising model on a cellular automaton. The black sites of the checkerboard are updated and then their colour is changed into white; the white sites are changed into black without being updated. The updating rules for the spin and the momentum variables are as follows: For a site to be updated its spin is flipped and the change in the Ising energy (internal energy), H_I, is calculated. If this energy change is transferable to or from the momentum variable associated with this site, such that the total energy H is conserved, then this change is made and the momentum is appropriately changed. Otherwise, the spin and the momentum are not changed. As the initial configuration all the spins are taken ordered (up or down). The initial kinetic energy is given to the lattice via the second and the third bits of the momentum variables in the white sites randomly, such that the value of the initial kinetic energy for such a demon is 24 (in units of J).

Simulations are carried out on simple hypercubic lattices L^5 of linear dimensions $4 \le L \le 8$ with periodic boundary conditions. The cellular automaton develops 9.6×10^5 ($L = 4, 6, 8$) sweeps for each run, with 7 runs for each total energy.

3. Results and discussion

The temperature dependences of the order parameter and the magnetic susceptibility are illustrated in Fig. 1 for $4 \le L \le 8$. The critical temperatures of the finite-size lattices obtained from the magnetic susceptibility maxima $T_c^{\chi}(L)$ for $4 \le L \le 8$ are listed in Table I. The computed values of χ_c and M_c are listed in Table II.

TABLE I

The maximum values and the critical temperatures of the magnetic susceptibility for $4 \le L \le 8$.

L	$T_c^{\chi}(L)$	χ_{max}
4	8.576(2)	1541(2)
6	8707(3)	4.256(4)
8	8745(1)	8767(3)

The dependence of the critical temperatures $T_c^{\chi}(L)$ obtained from the magnetic susceptibility maxima of the finite-size lattices on linear dimension L is given by the following expression [14, 26, 33–37]:

$$T_c^{\chi}(\infty) - T_c^{\chi}(L) \propto L^{-d/2}. \quad (3)$$

The value of the infinite-lattice critical temperature for the five-dimensional ferromagnetic Ising model, 8.7811 (1) is obtained from the straight line fit of the magnetic susceptibility maxima for $4 \le L \le 8$ (Fig. 2, Table III). The value obtained from infinite lattice critical temperature $T^{\chi}(\infty) = 8.7811(1)$ for $4 \le L \le 8$ agrees with the results obtained previously using different methods [14, 26, 37–46].

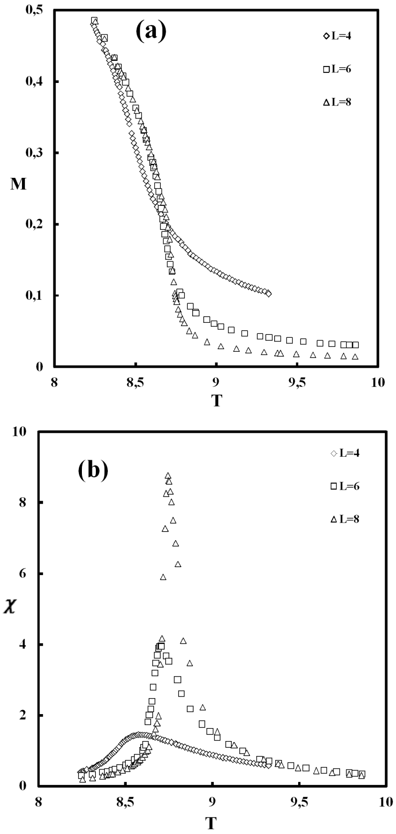

Fig. 1. The temperature dependence of (a) the order parameter (M) and (b) the magnetic susceptibility (χ) for $4 \le L \le 8$.

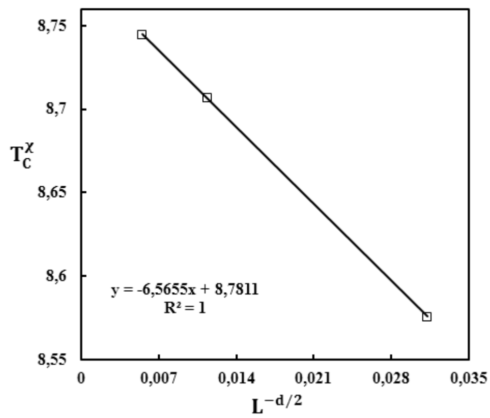

Fig. 2. The value of the infinite-lattice critical temperature for the five-dimensional ferromagnetic Ising model, $T_c^{\chi}(\infty) = 8.7811\,(1)$, obtained by extrapolating temperatures of the lattice with the linear dimension $4 \le L \le 8$ as $L \to \infty$.

For a lattice linear dimension L and very small h at $T = T_c(L)$, the order parameter is given by

$$M(L) \propto h^{1/\delta(L)}, \quad (4)$$

where $\delta(L)$ is the field critical exponent (Fig. 3). Scaling law of Widom is the following:

$$\gamma = \beta(\delta - 1), \tag{5}$$

where $\gamma = 1$, $\beta = 1/2$ and $\delta = 3$ for $d = 5$ [29, 30, 41, 42].

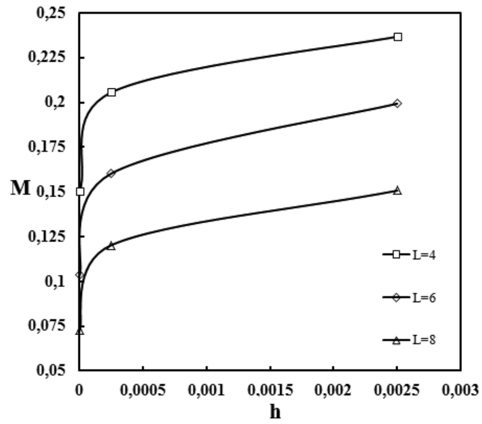

Fig. 3. The dependence of M against h for the lattices with the linear dimension $4 \leq L \leq 8$ ($T_\mathrm{c} = 8.7811\,(1)$).

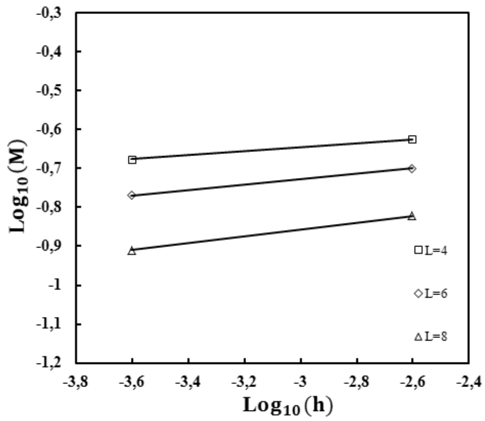

Fig. 4. The log–log plots of $M(L, T_\mathrm{c}(L,h))$ against h with the slope giving the value of $1/\delta = 0.0502$, $1/\delta = 0.0696$ and $1/\delta = 0.0875$ for $L = 4$, 6 and 8 at $T_\mathrm{c} = 8.7811(1)$, $8.7856(6)$, $8.8370(1)$, respectively.

The log–log plots of $M(L)$ at $T = T_\mathrm{c}(L,h)$ versus h for h in the interval $0 \leq h \leq 0.0025$ yield to $1/\delta(L)$ (Fig. 4). The straight line which fits to the plot of $\delta(L)$ against $1/L$ results in the infinite-lattice critical exponents $\delta = 3.0136(3)$ (Fig. 5). The result for the $\delta(\infty)$ is compared with $\delta = 3$ which is obtained from scaling law of Widom [35, 47, 48].

In $d \geq 5$ dimensions the finite-size scaling relation for the free-energy density is given as [33]:

$$f_L = L^{-d} F(tL^{y_t^*}, hL^{y_h^*}), \quad h \to 0, \quad L \to \infty, \tag{6}$$

where $y_t^* = d/(\gamma + 2\beta)$ and $y_h^* = d(\gamma + \beta)/(\gamma + 2\beta)$, with $\alpha = 2 - d/y_t^*$, $\beta = d/y_t^* - \Delta$ and $\gamma = 2\Delta - d/y_t^*$, $t = (T - T_\mathrm{C})/T_\mathrm{C}$ is the reduced temperature with $t > 0$ for $T > T_\mathrm{C}$ and $t < 0$ for $T < T_\mathrm{C}$, h is the reduced external magnetic field, α, β and γ are the critical exponents for the specific heat, order parameter and the magnetic susceptibility of the infinite lattice, respectively. Thus,

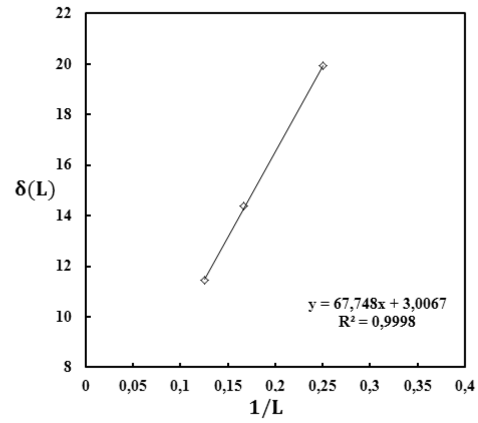

Fig. 5. The plot of $\delta(L)$ against $1/L$. The extrapolation of the fit lines to $1/L \to \infty$ gives $\delta = 3.0067(2)$.

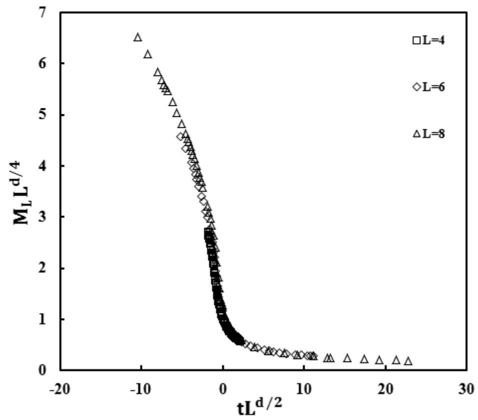

Fig. 6. Finite-size scaling plot of M_L at $T_\mathrm{c} = 8.7811\,(1)$ for $4 \leq L \leq 8$.

f_L takes the following form [16, 33, 35, 36, 49]:

$$f_L = L^{-d} F(tL^{\frac{1}{(2-\alpha)/d}}, hL^{\frac{(\gamma+\beta)}{(2-\alpha)/d}}). \tag{7}$$

According to Eq. (6), M_L and χ_L have the scaling form

$$M_L = \frac{\partial f_L}{\partial h} = L^{-d/4} X(tL^{d/2}, hL^{3d/4}), \tag{8}$$

$$\chi_L = \frac{\partial^2 f_L}{\partial h^2} = L^{d/2} Y(tL^{d/2}, hL^{3d/4}), \tag{9}$$

where the scaling functions X and Y are obtained from f_L. These finite-size scaling relations take the following form $T = T_\mathrm{c}$ [14, 16, 35]:

$$M_\mathrm{c} \propto L^{-d/4}, \tag{10}$$

$$\chi_\mathrm{c} \propto L^{d/2}. \tag{11}$$

The relations for $4 \leq L \leq 8$ can be tested by simulations directly. The finite-size scaling plots for $|M_L(t)|$ and $\chi_L(t)$ are given in Figs. 6 and 7, respectively. The overlap of the plots of the scaled quantities for different L verify the finite-size scaling relations given in Eqs. (8) and (9) at $T_\mathrm{c} = 8.7811(1)$.

The slope of straight lines in Fig. 8 (Eq. (10)) for $4 \leq L \leq 8$ give the results of $\frac{d}{4} = 1.2501\,(1)$ which is also in very good agreement with the theoretical re-

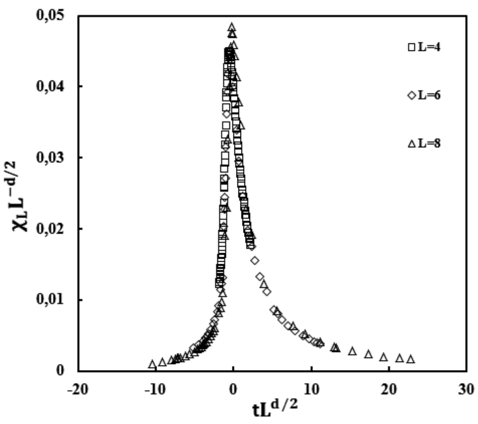

Fig. 7. Finite-size scaling plot of χ_L at $T_c = 8.7811\,(1)$ for $4 \le L \le 8$.

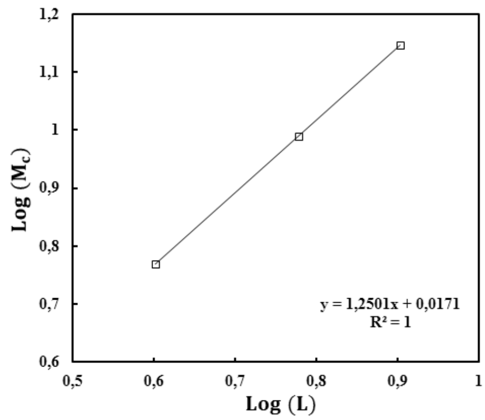

Fig. 8. The log–log plot of M_c against L at $T_c = 8.7811\,(1)$ for $4 \le L \le 8$. The slope gives $\frac{d}{4} = 1.2501\,(1)$.

sult $\left(\frac{d}{4} = 1.25\right)$. The slopes of straight lines in Figs. 9 and 10 (Eq. (11)) for $4 \le L \le 8$ give the results of $\frac{d}{2} = 2.5080\,(1)$ and $\frac{d}{2} = 2.5005\,(3)$, respectively, which are also in very good agreement with the theoretical result $\left(\frac{d}{2} = 2.5\right)$ are given in Table IV.

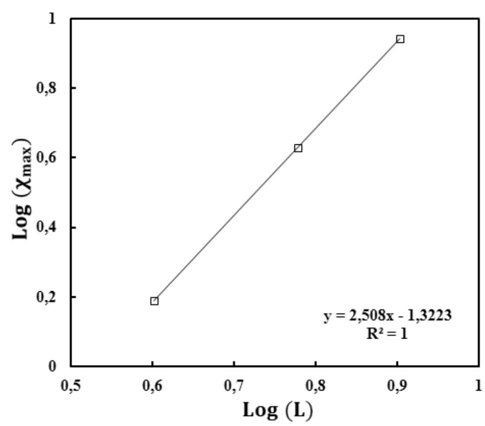

Fig. 9. The log–log plot of $\chi_{\max}(L)$ against L at $T_c = 8.7811\,(1)$ for $4 \le L \le 8$. The slope gives $\frac{d}{2} = 2.5080\,(1)$.

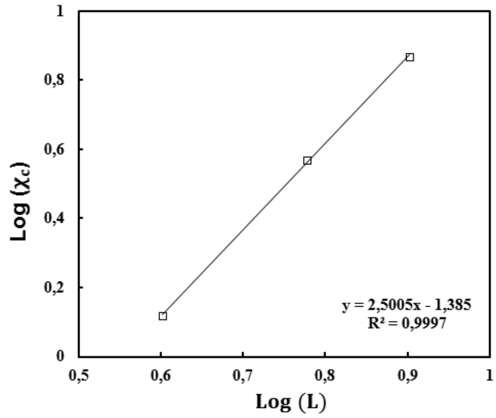

Fig. 10. The log–log plot of χ_c against L at $T_c = 8.7811\,(1)$ for $4 \le L \le 8$. The slope gives $\frac{d}{2} = 2.5005(3)$.

TABLE II

The values of χ_c and M_c obtained at $T_c = 8.7811\,(1)$ for $4 \le L \le 8$.

L	χ_c	M_c
4	1.310(3)	0.170(1)
6	3.701(1)	0.102(3)
8	7.392(4)	0.071(2)

4. Conclusions

The five-dimensional ferromagnetic Ising model is simulated on the Creutz cellular automaton algorithm by using the finite-size lattices with the linear dimensions $L = 4$, 6, and 8. In our work, the critical temperature value of infinite lattice for $4 \le L \le 8$ is in agreement with the other simulation results. In this study, the value of the field critical exponent ($\delta = 3.0067(2)$) is satisfied by scaling law of Widom. The finite-size scaling relations of $|M_L(t)|$ and $\chi_L(t)$ at the infinite-lattice critical temperature for $4 \le L \le 8$ are verified. The exponents in the finite-size scaling relations for the magnetic susceptibility and the order parameter at the infinite-lattice critical

TABLE III

The values of the infinite-lattice critical temperature for $4 \le L \le 8$.

T_c^χ	Method
8.7774(35) [31]	Monte Carlo
8.7812(23) [32, 33]	Monte Carlo
8.778475(31) [34]	Monte Carlo
8.780(10) [35]	Monte Carlo
8.78(1) [36]	Monte Carlo
8.77832(54) [37]	series expansion
8.7769(12) [38]	series expansion
8.7780(5) [39]	series expansion
8.77886(77) [38, 40]	dynamic Monte Carlo
8.779(8), 8.7572 [14, 26]	Creutz cellular automaton
8.7811(1), this work	Creutz cellular automaton

TABLE IV

The slopes of the log–log plots of $\chi_{\max}$, χ_c and M_c against L for $4 \le L \le 8$.

$\frac{d}{2}$	$\frac{d}{2}$	$\frac{d}{4}$
2.5080(1)	2.5005(3)	1.2501(1)

temperature, $\frac{d}{4}$ and $\frac{d}{2}$ for $4 \le L \le 8$ are in agreement with the theoretical predictions of $\frac{5}{4}$ and $\frac{5}{2}$.

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