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Helices in the n-dimensional Minkowski spacetime

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ABSTRACT

In this work, we find position vectors of non-null helices in the n-dimensional Minkowski spacetime. We present methods to generate helices from polynomial curves. We also present methods to generate helices from other helices which lie in different dimensional Minkowski spacetime.

1. Introduction

Helices are very interesting curves that attracted attention in a wide range of disciplines such as mathematics, physics, architecture, engineering and biology. Therefore, a rich literature exists [2,3,5–7,16].

In 3-dimensional Euclidean space, a helix is a curve whose tangent vector field makes a constant angle with a fixed direction called the axis of the helix. This definition suggests that a curve is a helix if and only if κ/τ is constant, where κ is the curvature and τ is the torsion of the curve [9,14]. The notion of helix can be extended to higher dimensional spaces using the same definition [7,12].

The notion of helix in 3-dimensional Minkowski spacetime is developed similarly and it can be extended to higher dimensions. Many different characterizations of helices in Minkowski spacetimes have been built, based on this definition, researchers gave many characterizations of these curves in 3-dimensional Minkowski spacetime.

Pythagorean-hodograph (PH) curve notion is introduced by Farouki and Sakkalis in [4]. Helical polynomial curves in Euclidean spaces are studied in [4,13]. PH curves in 3-dimensional Minkowski spacetime are studied in [10].

In this paper, we study position vectors of non-null helices similar to [1] in which Altunkaya and Kula studied polynomial helices in n-dimensional Euclidean space by using the PH curve notion. Although, there is a rich literature about helices in 3-dimensional Minkowski spacetime has been created, there are only a few papers discussing helices in n-dimensional Minkowski spacetime when $n > 3$. To the best of our knowledge, no example or application of position vectors of helices in n-dimensional Minkowski spacetime has been studied in the literature when n is even. The methods presented here can also be further used for finding different families of curves e.g. timelike polynomial helices with spacelike axis, etc.

2. Preliminaries

Let $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$ be nonzero vectors in the n-dimensional real vector space $\mathbb{R}^n$ and $\{e_1, e_2, ..., e_n\}$ be the standard orthonormal basis of this vector space. For $X, Y \in \mathbb{R}^n$

$$g\left(X, Y\right) = \sum_{i=1}^{n-1} x_i y_i - x_n y_n$$

is called Minkowski inner product. The couple $\{\mathbb{R}^n, g(,)\}$ is called Minkowski space(time) and denoted by $\mathbb{R}_1^n$ [15]. The vector X of $\mathbb{R}_1^n$ is called (see [11])

- timelike if $g(X, X) < 0$,
- spacelike if $g(X, X) > 0$ or $X = 0$,
- lightlike or null vector if $g(X, X) = 0, X \neq 0$.

For a given curve $\beta: I \subset \mathbb{R} \to \mathbb{R}_1^n$, we call the curve β is spacelike (resp. timelike, lightlike) if β' is spacelike (resp. timelike, lightlike) at any $t \in I$, where $\beta' = d\beta/dt$ [15].

The Frenet curvatures and Frenet equations of the curve β can be defined as follows.

Let $\beta: I \to \mathbb{R}_1^n$ be a non-null curve. The curve β is called Frenet curve of asculating order d if its higher order derivatives $\beta', \beta'', ..., \beta^d, \beta^{d+1}$ are no longer linearly independent for all $t \in I$. For each Frenet curve of order d, one can associate an orthonormal d-frame $V_1, V_2, ...V_d$ along β (such that $V_1 = \frac{\beta'}{\|\beta'\|}$) called the Frenet frame and $d-1$ functions

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$k_1, k_2, \ldots, k_{d-1}: I \to R$ called the Frenet curvatures. For $n = d$, the Frenet formulae are defined in the usual way:

$$\begin{cases} V_1' = \nu\varepsilon_2 k_1 V_2, \\ V_2' = -\nu\varepsilon_1 k_1 V_1 + \nu\varepsilon_3 k_2 V_3, \\ \vdots \\ V_i' = -\nu\varepsilon_{i-1} k_{i-1} V_{i-1} + \nu\varepsilon_{i+1} k_i V_{i+1}, \\ V_n' = -\nu\varepsilon_{n-1} k_{n-1} V_{n-1} \end{cases} \tag{1}$$

where $\nu = \|\beta'\| = |g(\beta', \beta')|^{1/2}$ and $\varepsilon_i = g(V_i, V_i)$ for $1 \leqslant i \leqslant n$ [8].

Definition 2.1. A curve $\beta: I \subset \mathbb{R} \to \mathbb{R}_1^n$ is called helix if and only if there exists a constant vector $U \in \mathbb{R}_1^n$ with $g(V_1, U) \neq 0$ is constant [5]. U is called the axis of the curve β.

3. Spacelike helices in R_1^n

Now, we study spacelike helices with a spacelike axis.

3.1. Spacelike helices in $\mathbb{R}_1^n$ when n is even

Theorem 3.1. *Let*

$\beta: I \subset \mathbb{R} \to \mathbb{R}_1^4,$

be a curve defined by

$$\beta(t) = \left(c_1 t, \frac{c_2}{3}t^3, \frac{c_3}{3}t^3 + \frac{c_4}{5}t^5, \frac{c_5}{2}t^2\right).$$

If

$c_1 = -d_1,\ c_2^2 = 2d_1d_3,\ c_3 = -d_2,\ c_4 = d_3,\ c_5^2 = 2d_1d_2$

with $1 \leqslant j \leqslant 3,\ d_j \in \mathbb{R}^+,\ d_2^2 < 4d_1d_3$*, then* β *is a spacelike helix with the axis*

$$\frac{U}{\|U\|}$$

and the tangent vector

$$V_1(t) = \frac{1}{d_3t^4 - d_2t^2 + d_1}(c_1, c_2t^2, c_3t^2 + c_4t^4, c_5t)$$

where

$U = (-1, 0, 1, 0).$

Proof. If necessary calculations are executed, we get

$$V_1(t) = \frac{1}{d_3t^4 - d_2t^2 + d_1}(c_1, c_2t^2, c_3t^2 + c_4t^4, c_5t).$$

Additionally, since

$$g\left(V_1(t), \frac{U}{\|U\|}\right) = \frac{1}{\sqrt{2}}$$

and $g(V_1, V_1) = 1$, we conclude that β is a spacelike helix where

$U = (-1, 0, 1, 0).$ □

Example 3.1. If we choose $d_1 = 1,\ d_2 = d_3 = 2$ in Theorem 3.1, then it is $c_1 = -1; c_2 = 2; c_3 = -2; c_4 = 2; c_5 = 2$ and we have

$$\beta(t) = \left(-t, \frac{2}{3}t^3, -\frac{2}{3}t^3 + \frac{2}{5}t^5, t^2\right),$$

$$V_1(t) = \frac{1}{2t^4 - 2t^2 + 1}(-1, 2t^2, 2t^4 - 2t^2, 2\,t),$$

$$g\left(V_1(t), \frac{U}{\|U\|}\right) = \frac{1}{\sqrt{2}}.$$

Given the necessary calculations are made, we obtain

$g(\beta'(t), \beta'(t)) = (2t^4 - 2t^2 + 1)^2 > 0,\ \forall\ t \in \mathbb{R},$

then the curve β is a spacelike helix.

In $\mathbb{R}_1^6$, we can obtain a spacelike helix as follows.

Theorem 3.2. *Let*

$\beta: (a, b) \subset \mathbb{R} \to \mathbb{R}_1^6, \quad b > a > 1,$

be a curve defined by

$$\beta(t) = \left(c_1 t, \frac{c_2}{3}t^3, \frac{c_3}{4}t^4, \frac{c_4}{5}t^5, \frac{c_5}{5}t^5 + \frac{c_6}{7}t^7, \frac{c_7}{2}t^2\right).$$

For $2 \leqslant i \leqslant 3,\ c_i \in \mathbb{R} - \{0\}$*, if*

$$c_1 = d_1, \quad c_2^2 = d_2^2 - 2d_1d_3, \quad c_3^2 = 2d_2d_3 - 2d_1d_4,$$
$$c_4^2 = 2d_2d_4, \quad c_5 = d_3, \quad c_6 = d_4, \quad c_7^2 = 2d_1d_2,$$

with $1 \leqslant j \leqslant 4,\ d_j \in \mathbb{R}^+,\ d_1 < \sum_{j=2}^{4} d_j = d_2 + d_3 + d_4; d_2^2 = 2d_1d_3;\ d_2d_3 > d_1d_4$*, then* β *is a spacelike helix with the spacelike axis*

$$\frac{U}{\|U\|}$$

and the tangent vector

$$V_1(t) = \frac{1}{d_4t^6 + d_3t^4 + d_2t^2 - d_1}(c_1, c_2t^2, c_3t^3, c_4t^4, c_5t^4 + c_6t^6, c_7t)$$

where

$$U = \left(-1, \frac{d_2}{c_2}, 0, 0, 1, 0\right).$$

Proof. We omit the proof since it is quite similar to the proof of Theorem 3.1. □

Theorem 3.3. *Let* $n \geqslant 8$ *be an even number and*

$\beta: (a, b) \subset \mathbb{R} \to \mathbb{R}_1^n, \quad b > a > 1,$

be a curve defined by

$$\beta(t) = \left(c_1 t, \frac{c_2}{3}t^3, \ldots, \frac{c_{n-2}}{n-1}t^{n-1}, \frac{c_{n-1}}{n-1}t^{n-1} + \frac{c_n}{n+1}t^{n+1}, \frac{c_{n+1}}{2}t^2\right).$$

For $2 \leqslant i \leqslant n-4,\ c_i \in \mathbb{R} - \{0\}$*, if*

$$c_1 = d_1,\ c_2^2 = d_2^2 - 2d_1d_3,\ c_{n-2}^2 = 2d_{\frac{n-2}{2}}d_{\frac{n+2}{2}},$$
$$c_{n-1} = d_{\frac{n}{2}},\ c_n = d_{\frac{n+2}{2}},\ c_{n+1}^2 = 2d_1d_2,$$
$$c_{2k}^2 = d_{k+1}^2 - 2d_1d_{2k+1} + 2\sum_{j=2}^{k} d_jd_{2k-j+2}, \quad 2 \leqslant k \leqslant \frac{n-4}{2},$$
$$c_{2l-1}^2 = -2d_1d_{2l} + 2\sum_{j=2}^{l} d_jd_{2l-j+1}, \quad 2 \leqslant l \leqslant \frac{n-2}{2}$$

with $1 \leqslant j \leqslant \frac{n+2}{2},\ d_j \in \mathbb{R}^+,\ d_{\frac{n+4}{2}} = d_{\frac{n+6}{2}} = \ldots = d_{n-2} = 0$*, and* $\sum_{j=2}^{\frac{n+2}{2}} d_j > d_1$*, then* β *is a spacelike helix with the spacelike axis*

$$\frac{U_n}{\|U_n\|}$$

and the tangent vector

$$V_1(t) = \frac{1}{-d_1 + \sum_{j=2}^{\frac{n+2}{2}} d_jt^{2(j-1)}}\left(c_1,\ c_2t^2,\ \ldots,\ c_{n-1}t^{n-2} + c_nt^n,\ c_{n+1}t\right)$$

where

$$U_n = -e_1 + \sum_{m=2}^{\frac{n-2}{2}} \frac{d_m}{c_{2m-2}}e_{2m-2} + \frac{d_{\frac{n}{2}}}{c_{n-1}}e_{n-1}.$$

Proof. We can write

$$V_1(t) = \frac{\beta'(t)}{\|\beta'(t)\|} = \frac{1}{\|\beta'(t)\|}(c_1,\ c_2 t^2, \ldots, c_{n-2} t^{n-2},\ c_{n-1} t^{n-2} + c_n t^n,\ c_{n+1} t) = \frac{1}{\|\beta'(t)\|}\left(d_1,\ c_2 t^2, \ldots, c_{n-2} t^{n-2},\ d_{\frac{n}{2}} t^{n-2} + d_{\frac{n+2}{2}} t^n,\ c_{n+1} t\right)$$

Therefore, we have

$$g\left(\beta'(t), \beta'(t)\right) = \left(-d_1 + \sum_{j=2}^{\frac{n+2}{2}} d_j t^{2(j-1)}\right)^2$$

and since $g(\beta', \beta') > 0$, β is a spacelike curve, then

$$g\left(V_1(t), \frac{U_n}{\|U_n\|}\right) = \frac{1}{\|U_n\|} = constant.$$

Consequently, β is a spacelike helix. □

Example 3.2. If we choose $n = 8; d_1 = d_3 = d_4 = d_5 = 1,\ d_2 = 2$ in Theorem 3.3, then it is $c_1 = 1; c_2 = \sqrt{2}; c_3 = \sqrt{2}; c_4 = \sqrt{3}; c_5 = \sqrt{6}; c_6 = \sqrt{2}; c_7 = 1; c_8 = 1; c_9 = 2$ and we have

$$\beta(t) = \left(t,\ \frac{\sqrt{2}t^3}{3},\ \frac{\sqrt{2}t^4}{4},\ \frac{\sqrt{3}t^5}{5},\ \frac{\sqrt{2}t^6}{6},\ \frac{\sqrt{2}t^7}{7},\ \frac{t^9}{9} + \frac{t^7}{7},\ t^2\right),$$

$$V_1(t) = \frac{1}{t^8 + t^6 + t^4 + 2t^2 - 1}(1,\ \sqrt{2}t^2,\ \sqrt{2}t^3,\ \sqrt{3}t^4,\ \sqrt{6}t^5,\ \sqrt{2}t^6,\ t^8 + t^6,\ 2t),$$

$$g\left(V_1(t), \frac{U}{\|U\|}\right) = \sqrt{\frac{3}{13}}$$

where

$$U = \left(-1,\ \sqrt{2},\ 0,\ \frac{1}{\sqrt{3}},\ 0,\ 0,\ 1,\ 0\right).$$

Given the necessary calculations are made, we obtain

$$g(\beta'(t), \beta'(t)) = (t^8 + t^6 + t^4 + 2t^2 - 1)^2 > 0,\ \forall\ t \in \mathbb{R},$$

then the curve β is a spacelike helix.

In the rest of the paper, the proofs are similar to the proofs of the Theorem 3.1 or Theorem 3.3, therefore we will omit the proofs.

3.2. Spacelike helices in $\mathbb{R}_1^n$ when n is odd

Theorem 3.4. *Let*

$$\beta: (a, b) \subset \mathbb{R} \to \mathbb{R}_1^3, \quad b > a > 1,$$

be a curve defined by

$$\beta(t) = \left(c_1 t,\ \frac{c_2}{3} t^3,\ \frac{c_3}{2} t^2\right).$$

If

$$c_1 = d_1,\ c_2 = d_2,\ c_3^2 = 2d_1 d_2$$

with $d_1, d_2 \in \mathbb{R}^+$, $d_1 < d_2$, then β is a spacelike helix with the axis

$$\frac{U}{\|U\|}$$

and the tangent vector

$$V_1(t) = \frac{1}{d_2 t^2 - d_1}(c_1,\ c_2 t^2,\ c_3 t)$$

where

$$U = (-1, 1, 0).$$

Theorem 3.5. *Let*

$$\beta: (a, b) \subset \mathbb{R} \to \mathbb{R}_1^5, \quad b > a > 1,$$

be a curve defined by

$$\beta(t) = \left(c_1 t,\ \frac{c_2}{3} t^3,\ \frac{c_3}{4} t^4,\ \frac{c_4}{5} t^5,\ \frac{c_5}{2} t^2\right).$$

For $c_2 \in \mathbb{R} - \{0\}$, if

$$c_1 = d_1,\ c_2^2 = d_2^2 - 2d_1 d_3,\ c_3^2 = 2d_2 d_3,\ c_4 = d_3,\ c_5^2 = 2d_1 d_2$$

with $1 \leqslant j \leqslant 3$, $d_j \in \mathbb{R}^+$, $d_1 < d_2 + d_3$, then β is a spacelike helix with the spacelike axis

$$\frac{U}{\|U\|}$$

and the tangent vector

$$V_1(t) = \frac{1}{d_3 t^4 + d_2 t^2 - d_1}(c_1,\ c_2 t^2,\ c_3 t^3,\ c_4 t^4,\ c_5 t)$$

where

$$U = \left(-1,\ \frac{d_2}{c_2},\ 0,\ 1,\ 0\right).$$

Example 3.3. If we take $d_1 = d_3 = 1$, $d_2 = 2$ in Theorem 3.5, then it is $c_1 = 1; c_2 = \sqrt{2}; c_3 = 2; c_4 = 1; c_5 = 2$ and we have

$$\beta(t) = \left(t,\ \frac{\sqrt{2}}{3} t^3,\ \frac{1}{2} t^4,\ \frac{1}{5} t^5,\ t^2\right).$$

The curve β is a spacelike helix with the spacelike axis

$$\frac{U}{\|U\|} = \left(-\frac{1}{2},\ \frac{1}{\sqrt{2}},\ 0,\ \frac{1}{2},\ 0\right)$$

and the tangent vector

$$V_1(t) = \frac{1}{t^4 + 2t^2 - 1}\left(1,\ \sqrt{2}\, t^2,\ 2t^3,\ t^4,\ 2t\right).$$

Besides,

$$g\left(V_1(t), \frac{U}{\|U\|}\right) = \frac{1}{2}.$$

Theorem 3.6. *Let $n \geqslant 7$be an odd number and*

$$\beta: (a, b) \subset \mathbb{R} \to \mathbb{R}_1^n, \quad b > a > 1,$$

be a curve defined by

$$\beta(t) = \left(c_1 t,\ \frac{c_2}{3} t^3,\ \ldots,\ \frac{c_{n-1}}{n} t^n,\ \frac{c_n}{2} t^2\right).$$

For $2 \leqslant i \leqslant \frac{n-1}{2}$, $c_i \in \mathbb{R} - \{0\}$, if

$$c_1 = d_1,\ c_2^2 = d_2^2 - 2d_1 d_3,\ c_{n-1} = d_{\frac{n+1}{2}},\ c_n^2 = 2d_1 d_2,$$

$$c_{2k}^2 = d_{k+1}^2 - 2d_1 d_{2k+1} + 2\sum_{j=2}^{k} d_j d_{2k-j+2}, \quad 2 \leqslant k \leqslant \frac{n-3}{2},$$

$$c_{2l-1}^2 = -2d_1 d_{2l} + 2\sum_{j=2}^{l} d_j d_{2l-j+1}, \quad 2 \leqslant l \leqslant \frac{n-1}{2}$$

with $1 \leqslant j \leqslant \frac{n+1}{2}$, $d_j \in \mathbb{R}^+$, $d_{\frac{n+3}{2}} = d_{\frac{n+5}{2}} = \ldots = d_{n-1} = 0$, $\sum_{j=2}^{\frac{n}{2}} d_j > d_1$, then the curve β is a spacelike helix with the spacelike axis

$$\frac{U_n}{\|U_n\|}$$

and the tangent vector

$$V_1(t) = \frac{1}{-d_1 + \sum_{j=2}^{\frac{n+1}{2}} d_j t^{2(j-1)}}\left(c_1,\ c_2 t^2,\ ...,\ c_{n-1} t^{n-1},\ c_n t\right)$$

where

$$U_n = -e_1 + \sum_{m=2}^{\frac{n+1}{2}} \frac{d_m}{c_{2m-2}} e_{2m-2}.$$

Example 3.4. If we take $n = 7; d_1 = d_3 = d_4 = 1, d_2 = 2$ in Theorem 3.6, then it is $c_1 = 1; c_2 = \sqrt{2}; c_3 = \sqrt{2}; c_4 = \sqrt{5}; c_5 = \sqrt{2}; c_6 = 1; c_7 = 2$ and we have

$$\beta(t) = \left(t,\ \frac{\sqrt{2}\,t^3}{3},\ \frac{\sqrt{2}\,t^4}{4},\ \frac{\sqrt{5}\,t^5}{5},\ \frac{\sqrt{2}\,t^6}{6},\ \frac{t^7}{7},\ t^2\right).$$

The curve β is a spacelike helix with the spacelike axis

$$\frac{U}{\|U\|} = \left(-\sqrt{\frac{5}{21}},\ \sqrt{\frac{10}{21}},\ 0,\ \frac{1}{\sqrt{21}},\ 0,\ \sqrt{\frac{5}{21}},\ 0\right)$$

and the tangent vector

$$V_1(t) = \frac{1}{t^6 + t^4 + 2t^2 - 1}(1,\ \sqrt{2}\,t^2,\ \sqrt{2}\,t^3,\ \sqrt{5}\,t^4,\ \sqrt{2}\,t^5,\ t^6,\ 2t).$$

Besides,

$$g\left(V_1(t),\ \frac{U}{\|U\|}\right) = \sqrt{\frac{5}{21}}.$$

As a result of Theorem 3.3, we have the following corollary.

Corollary 3.1. *If $d_{\frac{n+2}{2}} = 0$ in Theorem 3.3, then the curve β lies in the (n-1)-dimensional hyperplane of $\mathbb{R}^n_1$. So it can be considered as a spacelike helix γin (n-1)-dimensional Minkowski spacetime.*

4. Generating spacelike helices from spacelike helices that lie in different dimensional Minkowski spacetime

4.1. Generating spacelike helices in $\mathbb{R}^4_1$ from spacelike helices in $\mathbb{R}^3_1$

Let $d_1, d_2 \in \mathbb{R}^+$,

$\gamma_1(t) = d_1 t,$

$\gamma_2(t) = \frac{d_2}{3} t^3,$

$\gamma_3(t) = \frac{\sqrt{2d_1 d_2}}{2} t^2.$

Then, $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ (see Fig. 1) is a spacelike helix in $\mathbb{R}^3_1$ and the Minkowski scalar product of tangent vector of γ with the spacelike vector $\frac{U}{\|U\|} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ is $\frac{1}{\sqrt{2}}$, where $U = (-1, 1, 0)$.

Let $\beta(t) = (\beta_1(t), \beta_2(t), \beta_3(t), \beta_4(t))$ be a curve in $\mathbb{R}^4_1$ denoted by

$\beta_1(t) = \lambda(t),$

$\beta_2(t) = \lambda(t)\gamma_1(t),$

$\beta_3(t) = \lambda(t)\gamma_2(t),$

$\beta_4(t) = \lambda(t)\gamma_3(t).$

with the Minkowski scalar product of tangent vector V of β with the spacelike vector $\frac{\hat{U}}{\|\hat{U}\|} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ is constant (equal to the corresponding of γ), where $\lambda(t)$ is a real valued function. Thus, we have

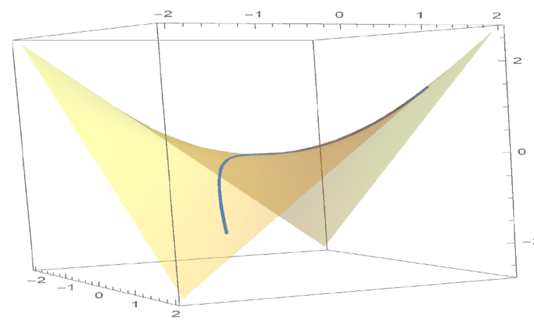

Fig. 1. For $d_1 = 1$, $d_2 = 2$, the spacelike helix γ lies on $y = \frac{2}{3}xz$.

$$g(V_1(t), v) = \frac{1}{\sqrt{2}}.$$

By solving this equation, we find

$$\lambda(t) = \frac{c}{d_1 d_2 t^4 + 6}$$

where $c \in \mathbb{R} - \{0\}$. Then, we have

$$\beta(t) = \frac{c}{d_1 d_2\, t^4 + 6}\left(1,\ d_1 t,\ \frac{d_2}{3} t^3,\ \frac{\sqrt{2d_1 d_2}}{2} t^2\right).$$

The curve β is a spacelike helix and the Minkowski scalar product of tangent vector of β with the spacelike vector

$$\frac{\hat{U}}{\|\hat{U}\|}$$

is constant where

$$\hat{U} = (0, -1, 1, 0).$$

4.2. Generating spacelike helices in $\mathbb{R}^5_1$ from spacelike helices in $\mathbb{R}^4_1$

Let $d_1, d_2, d_3 \in \mathbb{R}^+$,

$\gamma_1(t) = -d_1 t,$

$\gamma_2(t) = \frac{\sqrt{2d_1 d_3}}{3} t^3,$

$\gamma_3(t) = -\frac{d_2}{3} t^3 + \frac{d_3}{5} t^5,$

$\gamma_4(t) = \frac{\sqrt{2d_1 d_2}}{2} t^2.$

From Theorem 3.1, $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t), \gamma_4(t))$ is a spacelike helix in $\mathbb{R}^4_1$ the Minkowski scalar product of tangent vector of γ with the spacelike vector $\frac{U}{\|U\|} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right)$ is $\frac{1}{\sqrt{2}}$, where $U = (-1, 0, 1, 0)$.

Let $\beta(t) = (\beta_1(t), \beta_2(t), \beta_3(t), \beta_4(t), \beta_5(t))$ be a curve in $\mathbb{R}^5_1$ denoted by

$\beta_1(t) = \lambda(t),$

$\beta_2(t) = \lambda(t)\gamma_1(t),$

$\beta_3(t) = \lambda(t)\gamma_2(t),$

$\beta_4(t) = \lambda(t)\gamma_3(t),$

$\beta_5(t) = \lambda(t)\gamma_4(t)$

with the Minkowski scalar product of tangent vector V_1 of β with the spacelike vector $\frac{\hat{U}}{\|\hat{U}\|} = \left(0, \frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right)$ is constant (equal to the corresponding of γ), where $\lambda(t)$ is a real valued function. Thus, we have

$$g\left(V_1(t),\ \frac{\hat{U}}{\|\hat{U}\|}\right) = \frac{1}{\sqrt{2}}.$$

By solving this equation, we find

$$\beta(t) = \frac{c}{-16d_1d_3t^6 + 15d_1d_2t^4 + 90}\left(1, -d_1t, \frac{\sqrt{2d_1d_3}t^3}{3}, \frac{d_2t^3}{3} + \frac{d_3t^5}{5}, \frac{\sqrt{2d_1d_2}t^2}{2}\right)$$

where $c \in \mathbb{R} - \{0\}$.

The curve β is a spacelike helix and the Minkowski scalar product of tangent vector of β with the spacelike vector

$$\frac{\widehat{U}}{\|\widehat{U}\|}$$

is constant where

$$\widehat{U} = (0, -1, 0, 1, 0).$$

We can similarly construct spacelike helices in upper dimensions.

5. Timelike helices in $\mathbb{R}_1^n$

In this section, we study timelike helices with null axis.

5.1. Timelike helices with null axis in $\mathbb{R}_1^n$

Theorem 5.1. *Let*

$$\beta: I \subset \mathbb{R} \to \mathbb{R}_1^3, \quad 0 \notin I$$

be a curve defined by

$$\beta(t) = \left(c_1t, \frac{c_2}{2}t^2, \frac{c_3}{3}t^3 + c_1t\right).$$

If

$$c_1 = \frac{c_2^2}{2c_3}, \qquad c_1, c_3 \in \mathbb{R}^+, \ c_2 \in \mathbb{R} - \{0\},$$

then β is a timelike helix with the null axis

$$U = (1, 0, 1)$$

and the tangent vector

$$V_1(t) = \frac{2}{c_3t^2}\left(c_1, c_2t, c_3t^2 + c_1\right).$$

Besides,

$$g(V_1(t), U) = -1.$$

Theorem 5.2. *Let $n \geqslant 4$ and*

$$\beta(t) = \left(c_1t, \frac{c_2}{2}t^2, \frac{c_3}{3}t^3, \ldots, \frac{c_{n-1}}{n-1}t^{n-1}, c_1t + \frac{c_n}{3}t^3 + \frac{c_{n+1}}{5}t^5 + \ldots + \frac{c_{2n-3}}{2n-3}t^{2n-3}\right)$$

be a curve in $\mathbb{R}_1^n$ with $c_1 \in \mathbb{R}^+$ and $c_i \in \mathbb{R} - \{0\}$, $2 \leqslant i \leqslant n-1$. If

$$c_n = \frac{c_2^2}{2c_1}, \ c_{n+1} = \frac{c_3^2}{2c_1}, \ \ldots, c_{2n-3} = \frac{c_{n-1}^2}{2c_1},$$

then β is a timelike helix with the null axis

$$U = e_1 + e_n$$

and the tangent vector

$$V_1(t) = \frac{2c_1}{\sum_{j=3}^{n-1} c_j^2t^{2n-4}}\left(c_1, c_2t, c_3t^2, \ldots, c_{n-1}t^{n-2}, c_1 + c_nt^2 + c_{n+1}t^4 + \ldots + c_{2n-3}t^{2n-4}\right).$$

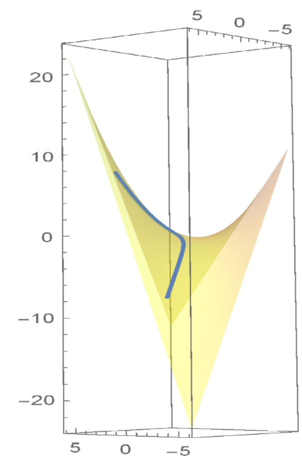

Fig. 2. The timelike helix γ lies on $z = x(\frac{\sqrt{2}}{3}y + 1)$.

Example 5.1. If we take $n = 4$, $c_1 = c_2 = c_3 = 1$ in Theorem 5.2, then we obtain

$$\beta(t) = \left(t, \frac{1}{2}t^2, \frac{1}{3}t^3, \frac{1}{10}t^5 + \frac{1}{6}t^3 + t\right).$$

The curve β is a timelike helix with the axis

$$U = (1, 0, 0, 1)$$

and the tangent vector

$$V_1(t) = \frac{2}{t^4 + t^2}\left(1, t, t^2, 1 + \frac{1}{2}t^2 + \frac{1}{2}t^4\right).$$

Besides,

$$g(V_1(t), U) = -1.$$

We can give the following corollary as a result of Theorem 5.2.

Corollary 5.1. *If $c_{n-1} = 0$ in Theorem 5.2, then the curve β lies in the (n-1)-dimensional hyperplane of $\mathbb{R}_1^n$. So it can be considered as a timelike helix in (n-1)-dimensional Minkowski spacetime.*

6. Generating timelike helices from timelike helices that lie in different dimensional Minkowski spacetime

6.1. Generating timelike helices in $\mathbb{R}_1^4$ from timelike helices in $\mathbb{R}_1^3$

Let

$$\gamma_1(t) = t,$$
$$\gamma_2(t) = \frac{\sqrt{2}}{2}t^2,$$
$$\gamma_3(t) = \frac{1}{3}t^3 + t,$$

then $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ (see Fig. 2) is a timelike helix in $\mathbb{R}_1^3$ and the Minkowski scalar product of tangent vector of γ with the null vector $U = (1, 0, 1)$ is constant.

Let $\beta(t) = (\beta_1(t), \beta_2(t), \beta_3(t), \beta_4(t))$ be a curve in $\mathbb{R}_1^4$ denoted by

$$\beta_1(t) = \lambda(t),$$
$$\beta_2(t) = \lambda(t)\gamma_1(t),$$
$$\beta_3(t) = \lambda(t)\gamma_2(t),$$
$$\beta_4(t) = \lambda(t)\gamma_3(t)$$

with the Minkowski scalar product of tangent vector V_1 of β with the null vector $\widehat{U} = (0, 1, 0, 1)$ is constant (equal to the corresponding of γ), where $\lambda(t)$ is a real valued function. Therefore, we have the differential equation

$g(V_1(t), \hat{U}) = -1.$

By solving this differential equation, we find

$$\lambda(t) = \frac{c}{t^4 - 6}$$

where $c \in \mathbb{R} - \{0\}$. Then, we have

$$\beta(t) = \frac{c}{t^4 - 6}\left(1,\ t,\ \frac{\sqrt{2}}{2}t^2,\ \frac{1}{3}t^3 + t\right).$$

The curve β is a timelike helix with the null axis

$U = (0, 1, 0, 1).$

6.2. *Generating timelike helices in* $\mathbb{R}_1^5$ *from timelike helices in* $\mathbb{R}_1^4$

Let

$$\gamma_1(t) = t,$$
$$\gamma_2(t) = \tfrac{1}{2}t^2,$$
$$\gamma_3(t) = \tfrac{1}{3}t^3,$$
$$\gamma_4(t) = \tfrac{1}{10}t^5 + \tfrac{1}{6}t^3 + t.$$

From Theorem 5.2, $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t), \gamma_4(t))$ is a timelike helix in $\mathbb{R}_1^4$ and the Minkowski scalar product of tangent vector of γ with the null vector $U = (1, 0, 0, 1)$ is constant.

Let $\beta(t) = (\beta_1(t), \beta_2(t), \beta_3(t), \beta_4(t), \beta_5(t))$ be a curve in $\mathbb{R}_1^5$ denoted by

$$\beta_1(t) = \lambda(t),$$
$$\beta_2(t) = \lambda(t)\gamma_1(t),$$
$$\beta_3(t) = \lambda(t)\gamma_2(t),$$
$$\beta_4(t) = \lambda(t)\gamma_3(t),$$
$$\beta_5(t) = \lambda(t)\gamma_4(t),$$

with the Minkowski scalar product of tangent vector V_1 of β with the null vector $\hat{U} = (0, 1, 0, 0, 1)$ is constant (equal to the corresponding of γ), where $\lambda(t)$ is a real valued function. Therefore, we have the differential equation

$g(V_1(t), \hat{U}) = -1.$

By solving this differential equation, we find

$$\beta(t) = \frac{c}{16t^6 + 15t^4 - 180}\left(1,\ t,\ \frac{1}{2}t^2,\ \frac{1}{3}t^3,\ \frac{1}{10}t^5 + \frac{1}{6}t^3 + t\right)$$

where $c \in \mathbb{R} - \{0\}$.

The curve β is a timelike helix with null axis

$\hat{U} = (0, 1, 0, 0, 1).$

We can similarly construct timelike helices in upper dimensions.

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