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Fractional maximal function and its commutators on Orlicz spaces

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Abstract In this paper, we find necessary and sufficient conditions for the boundedness of fractional maximal operator M_α on Orlicz spaces. As an application of this results we consider the boundedness of fractional maximal commutator $M_{b,\alpha}$ and nonlinear commutator of fractional maximal operator $[b, M_\alpha]$ on Orlicz spaces, when b belongs to the Lipschitz space, by which some new characterizations of the Lipschitz spaces are given.

Keywords Fractional maximal function · Commutator · Lipschitz space · Orlicz space

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1 Introduction

Norm inequalities for several classical operators of harmonic analysis have been widely studied in the context of Orlicz spaces. It is well known that many of such operators fail to have continuity properties when they act between certain Lebesgue spaces and, in some situations, the Orlicz spaces appear as adequate substitutes. For example, the Hardy-Littlewood maximal operator is bounded on L^p for $1 < p < \infty$, but not on L^1, but using Orlicz spaces, we can investigate the boundedness of the maximal operator near $p = 1$, see [3,5,6,9] for more precise statements.

Let T be the classical singular integral operator, the commutator $[b, T]$ generated by T and a suitable function b is given by

$$[b, T]f = bT(f) - T(bf). \tag{1.1}$$

A well known result due to Coifman, Rochberg and Weiss [2] (see e.g. [8]) states that $b \in BMO(\mathbb{R}^n)$ if and only if the commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In 1978, Janson [8] gave some characterizations of the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ (see Definition 4.1 below) via commutator $[b, T]$ and proved that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)(0 < \beta < 1)$ if and only if $[b, T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ where $1 < p < n/\beta$ and $1/p - 1/q = \beta/n$ (see also Paluszyński [11]).

Let $0 < \alpha < n$. The fractional maximal operator M_α is given by

$$M_\alpha f(x) = \sup_{B \ni x} |B|^{-1+\frac{\alpha}{n}} \int_B |f(y)|dy$$

and the fractional maximal commutator of M_α with a locally integrable function b is defined by

$$M_{b,\alpha} f(x) = \sup_{B \ni x} |B|^{-1+\frac{\alpha}{n}} \int_B |b(x) - b(y)||f(y)|dy,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x. If $\alpha = 0$, then $M \equiv M_0$ is the Hardy-Littlewood maximal operator and $M_b \equiv M_{b,0}$ is the maximal commutator of M.

On the other hand, similar to (1.1), we can define the (nonlinear) commutator of the fractional maximal operator M_α with a locally integrable function b by

$$[b, M_\alpha](f)(x)f = b(x)M_\alpha(f)(x) - M_\alpha(bf)(x).$$

For more details about the operators $M_{b,\alpha}$ and $[b, M_\alpha]$, where $0 \le \alpha < n$, we refer to [1,13] and references therein.

Our main aim is to characterize the functions involved in the boundedness on Orlicz spaces of the fractional maximal operator M_α. Actually, such a characterization was done in [3, Theorem 1]. But our technique of the proof and characterization different from the ones in [3]. As an application of this result we consider the boundedness of

$M_{b,\alpha}$ and $[b, M_\alpha]$ on Orlicz spaces when b belongs to the Lipschitz space, by which some new characterizations of the Lipschitz spaces are given.

Throughout the whole paper, the notation $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \leq CB$, where C is independent of appropriate quantities. If $C_1 B \leq A \leq C_2 B$ for some positive constants C_1 and C_2, we shall write $A \approx B$.

2 Preliminaries

Before we proceed with the proofs of the main results, we shall introduce some preliminary denitions and properties concerning Orlicz spaces.

Definition 2.1 A function $\Phi : [0, \infty) \to [0, \infty]$ is called a Young function if Φ is convex, left-continuous, $\lim_{r \to +0} \Phi(r) = \Phi(0) = 0$ and $\lim_{r \to \infty} \Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. The set of Young functions such that

$$0 < \Phi(r) < \infty \quad \text{for} \quad 0 < r < \infty$$

will be denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0, \infty)$ and bijective from $[0, \infty)$ to itself.

For a Young function Φ and $0 \leq s \leq \infty$, let

$$\Phi^{-1}(s) = \inf\{r \geq 0 : \Phi(r) > s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function' of Φ. It is well known that

$$r \leq \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \leq 2r, \quad r \geq 0, \tag{2.1}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0, \infty)\}, & r \in [0, \infty) \\ \infty, & r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \leq C\Phi(r), \quad r \geq 0$$

for some $C \geq 2$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2C}\Phi(Cr), \quad r \geq 0$$

for some $C > 1$. We can verify the following examples: The function $\Phi(r) = r$ satisfies the Δ_2-condition but does not satisfy the ∇_2-condition. If $1 < p < \infty$, then

$\Phi(r) = r^p$ satisfies both the conditions. The function $\Phi(r) = e^r - r - 1$ satisfies the ∇_2-condition but does not satisfy the Δ_2-condition.

Definition 2.2 (Orlicz Space). For a Young function Φ, the set

$$L^\Phi(\mathbb{R}^n) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < \infty \text{ for some } k > 0 \right\}$$

is called Orlicz space. If $\Phi(r) = r^p$, $1 \le p < \infty$, then $L^\Phi(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\Phi(r) = 0$, $(0 \le r \le 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L^\Phi(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. The space $L^\Phi_{\mathrm{loc}}(\mathbb{R}^n)$ is defined as the set of all functions f such that $f\chi_B \in L^\Phi(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$.

$L^\Phi(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L^\Phi} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right)dx \le 1 \right\}.$$

For a measurable set $\Omega \subset \mathbb{R}^n$, a measurable function f and $t > 0$, let $m(\Omega, f, t) = |\{x \in \Omega : |f(x)| > t\}|$. In the case $\Omega = \mathbb{R}^n$, we shortly denote it by $m(f, t)$.

Definition 2.3 The weak Orlicz space

$$WL^\Phi(\mathbb{R}^n) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{WL^\Phi} < \infty\}$$

is defined by the norm

$$\|f\|_{WL^\Phi} = \inf \left\{ \lambda > 0 : \sup_{t>0} \Phi(t)m\left(\frac{f}{\lambda}, t\right) \le 1 \right\}.$$

We note that $\|f\|_{WL^\Phi} \le \|f\|_{L^\Phi}$,

$$\sup_{t>0} \Phi(t)m(\Omega, f, t) = \sup_{t>0} t\, m(\Omega, f, \Phi^{-1}(t)) = \sup_{t>0} t\, m(\Omega, \Phi(|f|), t)$$

and

$$\int_\Omega \Phi\left(\frac{|f(x)|}{\|f\|_{L^\Phi(\Omega)}}\right)dx \le 1, \quad \sup_{t>0} \Phi(t)m\left(\Omega, \frac{f}{\|f\|_{WL^\Phi(\Omega)}}, t\right) \le 1, \qquad (2.2)$$

where $\|f\|_{L^\Phi(\Omega)} = \|f\chi_\Omega\|_{L^\Phi}$ and $\|f\|_{WL^\Phi(\Omega)} = \|f\chi_\Omega\|_{WL^\Phi}$.

The following analogue of the Hölder's inequality is well known (see, for example, [12]).

Theorem 2.4 *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and functions f and g measurable on Ω. For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid*

$$\int_\Omega |f(x)g(x)|dx \le 2\|f\|_{L^\Phi(\Omega)}\|g\|_{L^{\widetilde{\Phi}}(\Omega)}.$$

By elementary calculations we have the following property.

Lemma 2.5 *Let Φ be a Young function and B be a set in $\mathbb{R}^n$ with finite Lebesgue measure. Then*

$$\|\chi_B\|_{L^\Phi} = \|\chi_B\|_{WL^\Phi} = \frac{1}{\Phi^{-1}\left(|B|^{-1}\right)}.$$

By Theorem 2.4, Lemma 2.5 and (2.1) we get the following estimate.

Lemma 2.6 *For a Young function Φ and $B = B(x, r)$, the following inequality is valid:*

$$\int_B |f(y)|dy \leq 2|B|\Phi^{-1}\left(|B|^{-1}\right)\|f\|_{L^\Phi(B)}.$$

3 The boundedness of fractional maximal operator

In this section, we shall give a necessary and sufficient condition for the boundedness of M_α on Orlicz spaces and weak Orlicz spaces. We begin with the boundedness of the maximal operator on Orlicz spaces.

Theorem 3.1 *[10] Let Φ be a Young function.*

(i) The operator M is bounded from $L^\Phi(\mathbb{R}^n)$ to $WL^\Phi(\mathbb{R}^n)$, and the inequality

$$\|Mf\|_{WL^\Phi} \leq C_0\|f\|_{L^\Phi} \tag{3.1}$$

holds with constant C_0 independent of f.
(ii) The operator M is bounded on $L^\Phi(\mathbb{R}^n)$, and the inequality

$$\|Mf\|_{L^\Phi} \leq C_0\|f\|_{L^\Phi} \tag{3.2}$$

holds with constant C_0 independent of f if and only if $\Phi \in \nabla_2$.

We recall that, for functions Φ and Ψ from $[0, \infty)$ into $[0, \infty]$, the function Ψ is said to dominate Φ globally if there exists a positive constant c such that $\Phi(s) \leq \Psi(cs)$ for all $s \geq 0$.

In the theorem below we also use the notation

$$\widetilde{\Psi_P}(s) = \int_0^s r^{P'-1}(\mathcal{B}_P^{-1}(r^{P'}))^{P'}dr, \tag{3.3}$$

where $1 < P \leq \infty$ and $\widetilde{\Psi_P}(s)$ is the Young conjugate funtion to $\Psi_P(s)$, where $\mathcal{B}_P^{-1}(s)$ is inverses to

$$\mathcal{B}_P(s) = \int_0^s \frac{\Psi(t)}{t^{1+P'}}dt.$$

In [3], Cianchi found the necessary and sufficient conditions for the boundedness of M_α on Orlicz spaces.

Theorem 3.2 *Let $0 < \alpha < n$.*

(i) M_α is bounded from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$ if and only if

$$\Phi \text{ dominates globally the function } Q, \tag{3.4}$$

whose inverse is given by

$$Q^{-1}(r) = r^{\alpha/n}\Psi^{-1}(r).$$

(ii) M_α is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$ if and only if

$$\int_0^1 \frac{\Psi(t)}{t^{1+n/(n-\alpha)}}dt < \infty \text{ and } \Phi \text{ dominates globally the function } \Psi_{n/\alpha}. \tag{3.5}$$

Here, $\Psi_{n/\alpha}$ is the Young function defined as in (3.3).

In order to prove our main theorem, we also need the following lemma.

Lemma 3.3 *If $B_0 := B(x_0, r_0)$, then $|B_0|^{\frac{\alpha}{n}} \leq M_\alpha \chi_{B_0}(x)$ for every $x \in B_0$.*

Proof For $x \in B_0$, we get

$$M_\alpha \chi_{B_0}(x) = \sup_{B \ni x} |B|^{-1+\frac{\alpha}{n}}|B \cap B_0| \geq |B_0|^{-1+\frac{\alpha}{n}}|B_0 \cap B_0| = |B_0|^{\frac{\alpha}{n}}.$$

$\square$

The following result completely characterizes the boundedness of M_α on Orlicz spaces.

Theorem 3.4 *Let $0 < \alpha < n$, Φ, Ψ be Young functions and $\Phi \in \mathcal{Y}$. The condition*

$$r^{-\frac{\alpha}{n}}\Phi^{-1}(r) \leq C\Psi^{-1}(r) \tag{3.6}$$

for all $r > 0$, where $C > 0$ does not depend on r, is necessary and sufficient for the boundedness of M_α from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, the condition (3.6) is necessary and sufficient for the boundedness of M_α from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Proof For arbitrary ball $B = B(x, r)$ we represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2B)}}(y), \quad r > 0,$$

and have

$$M_\alpha f(x) = M_\alpha f_1(x) + M_\alpha f_2(x).$$

Let y be an arbitrary point in B. If $B(y, t) \cap {}^\complement(B(x, 2r)) \neq \emptyset$, then $t > r$. Indeed, if $z \in B(y, t) \cap {}^\complement(B(x, 2r))$, then $t > |y - z| \geq |x - z| - |x - y| > 2r - r = r$.

On the other hand, $B(y,t) \cap {}^{\complement}(B(x,2r)) \subset B(x,2t)$. Indeed, if $z \in B(y,t) \cap {}^{\complement}(B(x,2r))$, then we get $|x - z| \leq |y - z| + |x - y| < t + r < 2t$.

Hence by Lemma 2.6

$$
\begin{aligned}
M_\alpha f_2(y) &\lesssim \sup_{t>0} \frac{1}{|B(y,t)|^{1-\frac{\alpha}{n}}} \int_{B(y,t) \cap {}^{\complement}(B(x,2r))} |f(z)| dz \\
&\lesssim \sup_{t>2r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |f(z)| dz \\
&\lesssim \|f\|_{L^\Phi} \sup_{r<t<\infty} t^\alpha \, \Phi^{-1}(|B(x,t)|^{-1}).
\end{aligned}
$$

Consequently from Hedberg's trick, see [7], and the last inequality, we have

$$
M_\alpha f(y) \lesssim r^\alpha M f(y) + \|f\|_{L^\Phi} \sup_{r<t<\infty} t^\alpha \, \Phi^{-1}(t^{-n}).
$$

Thus, by (3.6) we obtain

$$
|M_\alpha f(x)| \lesssim M f(x) \frac{\Psi^{-1}(r^{-n})}{\Phi^{-1}(r^{-n})} + \|f\|_{L^\Phi} \Psi^{-1}(r^{-n}).
$$

Choose $r > 0$ so that $\Phi^{-1}(r^{-n}) = \frac{Mf(x)}{C_0 \|f\|_{L^\Phi}}$. Then

$$
\frac{\Psi^{-1}(r^{-n})}{\Phi^{-1}(r^{-n})} = \frac{(\Psi^{-1} \circ \Phi)\left(\frac{Mf(x)}{C_0 \|f\|_{L^\Phi}}\right)}{\frac{Mf(x)}{C_0 \|f\|_{L^\Phi}}}.
$$

Therefore, we get

$$
|M_\alpha f(x)| \leq C_1 \|f\|_{L^\Phi} (\Psi^{-1} \circ \Phi)\left(\frac{Mf(x)}{C_0 \|f\|_{L^\Phi}}\right).
$$

Let C_0 be as in (3.1). Then by Theorem 3.1, we have

$$
\begin{aligned}
\sup_{r>0} \Psi(r)\, m\left(B, \frac{|M_\alpha f(x)|}{C_1 \|f\|_{L^\Phi}}, r\right) &= \sup_{r>0} r\, m\left(B, \Psi\left(\frac{|M_\alpha f(x)|}{C_1 \|f\|_{L^\Phi}}\right), r\right) \\
&\leq \sup_{r>0} r\, m\left(B, \Phi\left(\frac{Mf(x)}{C_0 \|f\|_{L^\Phi}}\right), r\right) \leq \sup_{r>0} \Phi(r)\, m\left(\frac{Mf(x)}{\|Mf\|_{WL^\Phi}}, r\right) \leq 1,
\end{aligned}
$$

i.e.

$$
\|M_\alpha f\|_{WL^\Psi(B)} \lesssim \|f\|_{L^\Phi}. \tag{3.7}
$$

By taking supremum over B in (3.7), we get

$$
\|M_\alpha f\|_{WL^\Psi} \lesssim \|f\|_{L^\Phi},
$$

since the constants in (3.7) don't depend on x and r.

Let C_0 be as in (3.2). Since $\Phi \in \nabla_2$, by Theorem 3.1, we have

$$\int_B \Psi \left(\frac{|M_\alpha f(x)|}{C_1 \|f\|_{L^\Phi}} \right) dx \leq \int_B \Phi \left(\frac{Mf(x)}{C_0 \|f\|_{L^\Phi}} \right) dx \leq \int_{\mathbb{R}^n} \Phi \left(\frac{Mf(x)}{\|Mf\|_{L^\Phi}} \right) dx \leq 1,$$

i.e.

$$\|M_\alpha f\|_{L^\Psi(B)} \lesssim \|f\|_{L^\Phi}. \tag{3.8}$$

By taking supremum over B in (3.8), we get

$$\|M_\alpha f\|_{L^\Psi} \lesssim \|f\|_{L^\Phi},$$

since the constants in (3.8) don't depend on x and r.

We shall now prove the necessity. Let $B_0 = B(x_0, r_0)$ and $x \in B_0$. By Lemma 3.3, we have $r_0^\alpha \leq C M_\alpha \chi_{B_0}(x)$. Therefore, by Lemma 2.5, we have

$$r_0^\alpha \lesssim \Psi^{-1}(|B_0|^{-1}) \|M_\alpha \chi_{B_0}\|_{WL^\Psi(B_0)} \lesssim \Psi^{-1}(|B_0|^{-1}) \|M_\alpha \chi_{B_0}\|_{WL^\Psi}$$
$$\lesssim \Psi^{-1}(|B_0|^{-1}) \|\chi_{B_0}\|_{L^\Phi} \lesssim \frac{\Psi^{-1}(r_0^{-n})}{\Phi^{-1}(r_0^{-n})}$$

and

$$r_0^\alpha \lesssim \Psi^{-1}(|B_0|^{-1}) \|M_\alpha \chi_{B_0}\|_{L^\Psi(B_0)} \lesssim \Psi^{-1}(|B_0|^{-1}) \|M_\alpha \chi_{B_0}\|_{L^\Psi}$$
$$\lesssim \Psi^{-1}(|B_0|^{-1}) \|\chi_{B_0}\|_{L^\Phi} \lesssim \frac{\Psi^{-1}(r_0^{-n})}{\Phi^{-1}(r_0^{-n})}.$$

Since this is true for every $r_0 > 0$, we are done. □

We recover the following well known result by taking $\Phi(t) = t^p$ at Theorem 3.4.

Corollary 3.5 *Let $0 < \alpha < n$ and $1 \leq p \leq n/\alpha$. Then the condition $1/q = 1/p - \alpha/n$ is necessary and sufficient for the boundedness of M_α from $L^p(\mathbb{R}^n)$ to $WL^q(\mathbb{R}^n)$ and for $p > 1$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

From Theorems 3.2 and 3.4 we have the following corollary.

Corollary 3.6 *Let $0 < \alpha < n$, Φ, Ψ be Young functions and $\Phi \in \mathcal{Y}$, then:*

(1) Condition (3.4) holds if and only if condition (3.6) holds.
(2) Moreover if $\Phi \in \nabla_2$, then condition (3.5) holds if and only if (3.6) holds.

4 Characterization of Lipschitz spaces via commutators

In this section, as an application of Theorem 3.4 we consider the boundedness of $M_{b,\alpha}$ and $[b, M_\alpha]$ on Orlicz spaces when b belongs to the Lipschitz space, by which some new characterizations of the Lipschitz spaces are given.

Definition 4.1 Let $0 < \beta < 1$, we say a function b belongs to the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ if there exists a constant C such that for all $x, y \in \mathbb{R}^n$,

$$|b(x) - b(y)| \leq C|x - y|^\beta.$$

The smallest such constant C is called the $\dot{\Lambda}_\beta(\mathbb{R}^n)$ norm of b and is denoted by $\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}$.

To prove the theorems, we need auxiliary results. The first one is the following characterizations of Lipschitz space, which is due to DeVore and Sharply [4].

Lemma 4.2 *Let* $0 < \beta < 1$, *we have*

$$\|f\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \approx \sup_B \frac{1}{|B|^{1+\beta/n}} \int_B |f(x) - f_B| dx,$$

where $f_B = \frac{1}{|B|} \int_B f(y) dy$.

Lemma 4.3 *Let* $0 < \beta < 1$, $0 \leq \alpha < n$, $0 < \alpha + \beta < n$ *and* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, *then the following pointwise estimate holds:*

$$M_{b,\alpha} f(x) \leq C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} M_{\alpha+\beta} f(x).$$

Proof If $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, then

$$M_{b,\alpha}(f)(x) = \sup_{B \ni x} |B|^{-1+\frac{\alpha}{n}} \int_B |b(x) - b(y)||f(y)| dy$$

$$\leq C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \sup_{B \ni x} |B|^{-1+\frac{\alpha+\beta}{n}} \int_B |f(y)| dy$$

$$= C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} M_{\alpha+\beta} f(x).$$

$\square$

Lemma 4.4 *If* $b \in L^1_{loc}(\mathbb{R}^n)$ *and* $B_0 := B(x_0, r_0)$, *then*

$$|B_0|^{\frac{\alpha}{n}} |b(x) - b_{B_0}| \leq M_{b,\alpha} \chi_{B_0}(x) \text{ for every } x \in B_0.$$

Proof For $x \in B_0$, we get

$$M_{b,\alpha} \chi_{B_0}(x) = \sup_{B \ni x} |B|^{-1+\frac{\alpha}{n}} \int_B |b(x) - b(y)| \chi_{B_0}(y) dy$$

$$= \sup_{B \ni x} |B|^{-1+\frac{\alpha}{n}} \int_{B \cap B_0} |b(x) - b(y)| dy \geq |B_0|^{-1+\frac{\alpha}{n}} \int_{B_0 \cap B_0} |b(x) - b(y)| dy$$

$$\geq \left||B_0|^{-1+\frac{\alpha}{n}} \int_{B_0} (b(x) - b(y)) dy\right| = |B_0|^{\frac{\alpha}{n}} |b(x) - b_{B_0}|.$$

$\square$

The following theorem is valid.

Theorem 4.5 *Let* $0 < \beta < 1$, $0 \leq \alpha < n$, $0 < \alpha + \beta < n$, $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, Φ, Ψ *be Young functions and* $\Phi \in \mathcal{Y}$.

1. *If* $\Phi \in \nabla_2$ *and the condition*

$$t^{-\frac{\alpha+\beta}{n}} \Phi^{-1}(t) \leq C \Psi^{-1}(t), \tag{4.1}$$

   holds for all $t > 0$, *where* $C > 0$ *does not depend on* t, *then the condition* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *is sufficient for the boundedness of* $M_{b,\alpha}$ *from* $L^\Phi(\mathbb{R}^n)$ *to* $L^\Psi(\mathbb{R}^n)$.
2. *If the condition*

$$\Psi^{-1}(t) \leq C \Phi^{-1}(t) t^{-\frac{\alpha+\beta}{n}}, \tag{4.2}$$

   holds for all $t > 0$, *where* $C > 0$ *does not depend on* t, *then the condition* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *is necessary for the boundedness of* $M_{b,\alpha}$ *from* $L^\Phi(\mathbb{R}^n)$ *to* $L^\Psi(\mathbb{R}^n)$.
3. *If* $\Phi \in \nabla_2$ *and* $\Psi^{-1}(t) \approx \Phi^{-1}(t) t^{-\frac{\alpha+\beta}{n}}$, *then the condition* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *is necessary and sufficient for the boundedness of* $M_{b,\alpha}$ *from* $L^\Phi(\mathbb{R}^n)$ *to* $L^\Psi(\mathbb{R}^n)$.

Proof (1) The first statement of the theorem follows from Theorem 3.4 and Lemma 4.3.

(2) We shall now prove the second part. Suppose that $\Psi^{-1}(t) \lesssim \Phi^{-1}(t) t^{-(\alpha+\beta)/n}$ and $M_{b,\alpha}$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$. Choose any ball B in $\mathbb{R}^n$, by Lemmas 2.5 and 2.6

$$\frac{1}{|B|^{1+\frac{\beta}{n}}} \int_B |b(y) - b_B| dy = \frac{1}{|B|^{1+\frac{\alpha+\beta}{n}}} \int_B \left| \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B (b(y) - b(z)) dz \right| dy$$

$$\leq \frac{1}{|B|^{1+\frac{\alpha+\beta}{n}}} \int_B M_{b,a}(\chi_B)(y) dy$$

$$\leq \frac{2\Psi^{-1}(|B|^{-1})}{|B|^{\frac{\alpha+\beta}{n}}} \|M_{b,\alpha}(\chi_B)\|_{L^\Psi(B)}$$

$$\leq \frac{C}{|B|^{\frac{\alpha+\beta}{n}}} \frac{\Psi^{-1}(|B|^{-1})}{\Phi^{-1}(|B|^{-1})} \leq C.$$

Thus by Lemma 4.2 we get $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

(3) The third statement of the theorem follows from the first and second parts of the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

If we take $\alpha = 0$ at Theorem 4.5, we have the following result.

Corollary 4.6 *Let* $0 < \beta < 1$, $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, Φ, Ψ *be Young functions and* $\Phi \in \mathcal{Y}$.

1. *If* $\Phi \in \nabla_2$ *and the condition* $\Phi^{-1}(t) t^{-\beta/n} \lesssim \Psi^{-1}(t)$ *holds, then the condition* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *is sufficient for the boundedness of* M_b *from* $L^\Phi(\mathbb{R}^n)$ *to* $L^\Psi(\mathbb{R}^n)$.
2. *If* $\Psi^{-1}(t) \lesssim \Phi^{-1}(t) t^{-\beta/n}$, *then the condition* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *is necessary for the boundedness of* M_b *from* $L^\Phi(\mathbb{R}^n)$ *to* $L^\Psi(\mathbb{R}^n)$.

3. If $\Phi \in \nabla_2$ and $\Psi^{-1}(t) \approx \Phi^{-1}(t)t^{-\beta/n}$, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of M_b from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

If we take $\Phi(t) = t^p$ and $\Psi(t) = t^q$ with $1 \leq p < \infty$ and $1 \leq q \leq \infty$ at Theorem 4.5, we have the following result.

Corollary 4.7 Let $0 < \beta < 1$, $0 \leq \alpha < n$, $0 < \alpha + \beta < n$, $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, $1 < p < q \leq \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha+\beta}{n}$. Then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of $M_{b,\alpha}$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

Remark 4.8 For $\alpha = 0$, Corollary 4.7 was proved in [15].

The following theorem is valid.

Theorem 4.9 Let $0 < \beta < 1$, $0 \leq \alpha < n$, $0 < \alpha + \beta < n$, $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, Φ, Ψ be Young functions and $\Phi \in \mathcal{Y}$.

1. If condition (4.1) holds, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is sufficient for the boundedness of $M_{b,\alpha}$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$.
2. If condition (4.2) holds and $\frac{t^{1+\varepsilon}}{\Psi(t)}$ is almost decreasing for some $\varepsilon > 0$, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary for the boundedness of $M_{b,\alpha}$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$.
3. If $\Psi^{-1}(t) \approx \Phi^{-1}(t)t^{-(\alpha+\beta)/n}$ and $\frac{t^{1+\varepsilon}}{\Psi(t)}$ is almost decreasing for some $\varepsilon > 0$, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of $M_{b,\alpha}$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$.

Proof (1) The first statement of the theorem follows from Theorem 3.4 and Lemma 4.3.

(2) For any fixed ball B_0 such that $x \in B_0$ by Lemma 4.4 we have $|B_0|^{\alpha/n}|b(x) - b_{B_0}| \leq M_{b,\alpha}\chi_{B_0}(x)$. This together with the boundedness of $M_{b,\alpha}$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$ and Lemma 2.5

$$|\{x \in B_0 : |B_0|^{\alpha/n}|b(x) - b_{B_0}| > \lambda\}| \leq |\{x \in B_0 : M_{b,\alpha}\chi_{B_0}(x) > \lambda\}|$$

$$\leq \frac{1}{\Psi\left(\frac{\lambda}{C\|\chi_{B_0}\|_{L^\Phi}}\right)} = \frac{1}{\Psi\left(\frac{\lambda\Phi^{-1}(|B_0|^{-1})}{C}\right)}.$$

Let $t > 0$ be a constant to be determined later, then

$$\int_{B_0} |b(x) - b_{B_0}|dx = |B_0|^{-\alpha/n}\int_0^\infty |\{x \in B_0 : |b(x) - b_{B_0}| > |B_0|^{-\alpha/n}\lambda\}|d\lambda$$

$$= |B_0|^{-\alpha/n}\int_0^t |\{x \in B_0 : |b(x) - b_{B_0}| > |B_0|^{-\alpha/n}\lambda\}|d\lambda$$

$$+ |B_0|^{-\alpha/n}\int_t^\infty |\{x \in B_0 : |b(x) - b_{B_0}| > |B_0|^{-\alpha/n}\lambda\}|d\lambda$$

$$\leq t|B_0|^{1-\alpha/n} + |B_0|^{-\alpha/n}\int_t^\infty \frac{1}{\Psi\left(\frac{\lambda\Phi^{-1}(|B_0|^{-1})}{C}\right)}d\lambda$$

$$\lesssim t|B_0|^{1-\alpha/n} + \frac{|B_0|^{-\alpha/n}t}{\Psi\left(\frac{t\Phi^{-1}(|B_0|^{-1})}{C}\right)},$$

where we use almost decreasingness of $\frac{t^{1+\varepsilon}}{\Psi(t)}$ in the last step. Set $t = C|B_0|^{\frac{\alpha+\beta}{n}}$ in the above estimate, we have

$$\int_{B_0} |b(x) - b_{B_0}|dx \lesssim |B_0|^{1+\beta/n}.$$

Thus by Lemma 4.2 we get $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ since B_0 is an arbitrary ball in $\mathbb{R}^n$.

(3) The third statement of the theorem follows from the first and second parts of the theorem.

If we take $\alpha = 0$ at Theorem 4.9, we have the following result.

Corollary 4.10 *Let $0 < \beta < 1$, $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, Φ, Ψ be Young functions and $\Phi \in \mathcal{Y}$.*

1. *If the condition $\Phi^{-1}(t)t^{-\beta/n} \lesssim \Psi^{-1}(t)$ holds, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is sufficient for the boundedness of M_b from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$.*
2. *If $\Psi^{-1}(t) \lesssim \Phi^{-1}(t)t^{-\beta/n}$ and $\frac{t^{1+\varepsilon}}{\Psi(t)}$ is almost decreasing for some $\varepsilon > 0$, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary for the boundedness of M_b from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$.*
3. *If $\Psi^{-1}(t) \approx \Phi^{-1}(t)t^{-\beta/n}$ and $\frac{t^{1+\varepsilon}}{\Psi(t)}$ is almost decreasing for some $\varepsilon > 0$, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of M_b from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$.*

If we take $\Phi(t) = t^p$ and $\Psi(t) = t^q$ with $1 \le p < \infty$ and $1 \le q \le \infty$ at Theorem 4.9, we have the following result.

Corollary 4.11 *Let $0 < \beta < 1$, $0 \le \alpha < n$, $0 < \alpha + \beta < n$, $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, $1 \le p < q \le \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha+\beta}{n}$. Then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of $M_{b,\alpha}$ from $L^p(\mathbb{R}^n)$ to $WL^q(\mathbb{R}^n)$.*

Remark 4.12 For $\alpha = 0$, Corollary 4.11 was proved in [15].

To state our results, we recall the definition of the maximal operator with respect to a ball. For a fixed ball B_0, the fractional maximal function with respect to B_0 of a function f is given by

$$M_{\alpha,B_0}(f)(x) = \sup_{B_0 \supseteq B \ni x} \frac{1}{|B_0|^{1-\frac{\alpha}{n}}} \int_B |f(y)|dy, \quad 0 \le \alpha < n,$$

where the supremum is taken over all the balls B with $B \subseteq B_0$ and $x \in B$.

Theorem 4.13 *Let $0 < \beta < 1$, $0 \le \alpha < n$, $0 < \alpha + \beta < n$ and b be a locally integrable non-negative function. Suppose that Φ, Ψ be Young functions, $\Phi \in \mathcal{Y} \cap \nabla_2$ and $\Psi^{-1}(t) \approx \Phi^{-1}(t)t^{-\frac{\alpha+\beta}{n}}$. Then the following statements are equivalent:*

1. $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.
2. $[b, M_\alpha]$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.
3. There exists a constant $C > 0$ such that

$$\sup_B |B|^{-\beta/n} \Psi^{-1}(|B|^{-1}) \|b(\cdot) - |B|^{-\alpha/n} M_{\alpha,B}(b)(\cdot)\|_{L^\Psi(B)} \leq C. \qquad (4.3)$$

Proof (1) $\Rightarrow$ (2): The following estimate was proved in [14]. Let b be any non-negative locally integrable function. Then

$$|[b, M_\alpha](f)(x)| \leq M_{b,\alpha}(f)(x), \quad x \in \mathbb{R}^n \qquad (4.4)$$

holds for all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

It follows from (4.4) and Theorem 4.5 that $[b, M_\alpha]$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$ since $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

(2) $\Rightarrow$ (3): For any fixed ball $B \subset \mathbb{R}^n$ and all $x \in B$, we have (see (2.4) in [13]).

$$M_\alpha(\chi_B)(x) = |B|^{\alpha/n} \quad \text{and} \quad M_\alpha(b\chi_B)(x) = M_{\alpha,B}(b)(x).$$

Then,

$$
\begin{aligned}
&|B|^{-\beta/n} \Psi^{-1}(|B|^{-1}) \|b(\cdot) - |B|^{-\alpha/n} M_{\alpha,B}(b)(\cdot)\|_{L^\Psi(B)} \\
&= |B|^{-\frac{\alpha+\beta}{n}} \Psi^{-1}(|B|^{-1}) \|b(\cdot) M_\alpha(\chi_B)(\cdot) - M_\alpha(b\chi_B)(\cdot)\|_{L^\Psi(B)} \\
&= |B|^{-\frac{\alpha+\beta}{n}} \Psi^{-1}(|B|^{-1}) \|[b, M_\alpha](\chi_B)\|_{L^\Psi(B)} \qquad (4.5) \\
&\leq C|B|^{-\frac{\alpha+\beta}{n}} \Psi^{-1}(|B|^{-1}) \|\chi_B\|_{L^\Phi} \\
&\leq C
\end{aligned}
$$

which implies (3) since the ball $B \subset \mathbb{R}^n$ is arbitrary.

(3) $\Rightarrow$ (1): From [14] we have,

$$\frac{1}{|B|^{1+\frac{\beta}{n}}} \int_B |b(x) - b_B| dx \leq \frac{2}{|B|^{1+\frac{\beta}{n}}} \int_B |b(x) - |B|^{-\alpha/n} M_{\alpha,B}(b)(x)| dx.$$

it follows from Lemma 2.6 and (4.3) that

$$\frac{1}{|B|^{1+\frac{\beta}{n}}} \int_B |b(x) - b_B| dx \leq \frac{4}{|B|^{\frac{\beta}{n}}} \Psi^{-1}(|B|^{-1}) \|b(\cdot) - |B|^{-\alpha/n} M_{\alpha,B}(b)(\cdot)\|_{L^\Psi(B)} \leq C.$$

Thus by Lemma 4.2 we get $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. $\qquad \square$

If we take $\alpha = 0$ at Theorem 4.13, we have the following result.

Corollary 4.14 *Let $0 < \beta < 1$ and b be a locally integrable non-negative function. Suppose that Φ, Ψ be Young functions, $\Phi \in \mathcal{Y} \cap \nabla_2$ and $\Psi^{-1}(t) \approx \Phi^{-1}(t) t^{-\frac{\beta}{n}}$. Then the following statements are equivalent:*

1. $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.
2. $[b, M]$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.
3.   There exists a constant $C > 0$ such that

$$\sup_B |B|^{-\beta/n} \Psi^{-1}(|B|^{-1}) \|b(\cdot) - M_B(b)(\cdot)\|_{L^\Psi(B)} \le C.$$

If we take $\Phi(t) = t^p$ and $\Psi(t) = t^q$ with $1 \le p < \infty$ and $1 \le q \le \infty$ at Theorem 4.13, we have the following result.

Corollary 4.15 *Let* $0 < \beta < 1$, $0 \le \alpha < n$, $0 < \alpha + \beta < n$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, b *be a locally integrable non-negative function,* $1 < p < q \le \infty$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha + \beta}{n}$. *Then the following statements are equivalent:*

1. $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.
2. $[b, M_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.
3.   There exists a constant $C > 0$ such that

$$\sup_B \frac{1}{|B|^{\beta/n}} \left(\frac{1}{|B|} \int_B |b(x) - |B|^{-\alpha/n} M_{\alpha,B}(b)(x)|^q dx \right)^{1/q} \le C.$$

Remark 4.16 For $\alpha = 0$, Corollary 4.15 was proved in [15].

Remark 4.17 From the proof of Theorem 4.13 one can see that the assumption $b \ge 0$ is not used in (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1). This means (2) and (3) are sufficient conditions for $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. But we don't know if (2) and (3) are necessary for $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

Indeed, we have obtained the following result.

Corollary 4.18 *Let* $0 < \beta < 1$, $0 \le \alpha < n$, $0 < \alpha + \beta < n$ *and* b *be a locally integrable function. Suppose that* Φ, Ψ *be Young functions,* $\Phi \in \mathcal{Y} \cap \nabla_2$ *and* $\Psi^{-1}(t) \approx \Phi^{-1}(t) t^{-\frac{\alpha+\beta}{n}}$. *If one of the following statements is true, then* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$:

1. $[b, M_\alpha]$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.
2.   There exists a constant $C > 0$ such that

$$\sup_B |B|^{-\beta/n} \Psi^{-1}(|B|^{-1}) \|b(\cdot) - |B|^{-\alpha/n} M_{\alpha,B}(b)(\cdot)\|_{L^\Psi(B)} \le C.$$

Theorem 4.19 *Let* $b \ge 0$ *be a locally integrable function,* $0 < \beta < 1$, $0 \le \alpha < n$, $0 < \alpha + \beta < n$ *and* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. *Suppose that* Φ, Ψ *be Young functions,* $\Phi \in \mathcal{Y}$ *and condition* (4.1) *holds. Then* $[b, M_\alpha]$ *is bounded from* $L^\Phi(\mathbb{R}^n)$ *to* $WL^\Psi(\mathbb{R}^n)$.

Proof Obviously, it follows from (4.4) and Theorem 4.9.   □

If we take $\Phi(t) = t^p$ and $\Psi(t) = t^q$ with $1 \le p < \infty$ and $1 \le q \le \infty$ at Theorem 4.19, we have the following result.

Corollary 4.20 *Let* $b \ge 0$ *be a locally integrable function,* $0 < \beta < 1$, $0 \le \alpha < n$, $0 < \alpha + \beta < n$, $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, $1 \le p < q \le \infty$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha+\beta}{n}$. *Then* $[b, M_\alpha]$ *is bounded from* $L^p(\mathbb{R}^n)$ *to* $WL^q(\mathbb{R}^n)$.

Remark 4.21 For $\alpha = 0$, Corollary 4.20 was proved in [15].

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