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On Medium *-Clean Rings

Huanyin Chen, Marjan Sheibani Abdolyousefi and Handan Kose

Abstract. A *-ring R is called a medium *-clean ring if every element in R is the sum or difference of an element in its Jacobson radical and a projection that commute. We prove that a ring R is medium *-clean if and only if R is strongly *-clean and $R/J(R)$ is a Boolean ring, $\mathbb{Z}_3$ or the product of such rings, if and only if R weakly J-*-clean and $a^2 \in R$ is uniquely *-clean for all $a \in R$, if and only if every idempotent lifts modulo $J(R)$, R is abelian and $R/J(R)$ weakly *-Boolean. A subclass of medium *-clean rings with many nilpotents is thereby characterized.

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1. Introduction

Throughout, all rings are associative with an identity. An involution of a ring R is an operation $* : R \to R$ such that $(x + y)^* = x^* + y^*$, $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$. A ring R with involution * is called a *-ring. For general *-ring theory, we refer the reader [2]. An element e in a *-ring R is called a projection if $e = e^* = e^2$. Recently, the concepts of clean rings are considered for any *-ring. A *-ring R is strongly *-clean if every element in R is the sum of a unit and a projection [6,9] and [12]. A *-ring R is weakly J-*-clean if every element in R is the sum or difference of an element in its Jacobson radical and a projection. Such rings are the natural generalizations of weakly nil-clean rings (see [1,11]). The motivation of this paper is to explore the structure of certain weakly J-*-clean rings and obtain the relations to other closed classes.

A *-ring R is called a medium *-clean ring if every element of R is the sum or difference of an element in its Jacobson radical and a projection that commute. Clearly, {strongly J-*-clean rings} $\subset$ {medium *-clean rings} $\subset$ {weakly J-*-clean rings}. Here, a *-ring R is strongly J-*-clean if every element is the sum of a projection and a unit that commute. We shall prove that medium *-clean rings and abelian weakly J-*-clean rings coincide with each other. We show that a *-ring R is medium *-clean if and only if R

is strongly *-clean and $R/J(R)$ is a Boolean ring, $\mathbb{Z}_3$ or the product of such rings, if and only if R weakly J-*-clean and $a^2 \in R$ is uniquely *-clean for all $a \in R$, if and only if every idempotent lifts modulo $J(R)$, R is abelian and $R/J(R)$ weakly *-Boolean. A subclass of medium *-clean rings with many nilpotents is characterized in terms of medium *-cleanness. These completely determine the structure of *-clean rings involving their Jacobson radicals.

We use $N(R)$ to denote the set of all nilpotent elements in R and $J(R)$ the Jacobson radical of R. $\mathbb{N}$ stands for the set of all natural numbers.

2. Medium *-Clean Rings

The main purpose of this section is to explore some elementary properties of medium *-clean rings. Our starting point is the following.

Lemma 2.1. *Every medium *-clean rings is abelian.*

Proof. Let R be a medium *-clean ring, and let $e \in R$ be an idempotent. Then, we can find a projection f and a $w \in J(R)$ such that $e = f + w$ or $e = -f + w$ with $fw = wf$. If $e = f + w$, then $e - f \in J(R)$. As $(e-f)^3 = e - f$, we see that $(e-f)(1-(e-f)^2) = 0$, and so $e = f$. If $e = -f + w$, then $e + f \in J(R)$. As $(e-f)(e+f) = e - f$, we see that $(e-f)(1-(e+f)) = 0$. This implies that $e = f$. Therefore, $e \in R$ is a projection. Therefore, R is abelian, in terms of [9, Lemma 2.1]. □

Theorem 2.2. *Let R be a *-ring. Then, the following are equivalent:*

(1) *R is medium *-clean.*
(2) *R is abelian weakly J-*-clean.*
(3) *R is strongly *-clean and weakly J-clean.*

Proof. (1) ⇒ (3) Clearly, R is weakly J-clean. Let $a \in R$. Then, there exists a projection $e \in R$ such that $a = e + w$ or $-e + w$, $w \in J(R)$ and $ew = we$. If $a = -e + w$, then $a = (1-e) + (w-1)$, $w - 1 \in U(R)$, $(1-e)^2 = 1 - e = (1-e)^*$, $(1-e)(w-1) = (w-1)(1-e)$. So $a \in R$ is strongly *-clean. If $a = e + w$, then $a = (1-e) + (2e-1)[1 + (2e-1)w]$. Since $w \in J(R)$, we see that $1 + (2e-1)w \in U(R)$ and $(1-e)^2 = 1 - e = (1-e)^*$. So $a \in R$ is strongly *-clean, as desired.

(3) ⇒ (2) In light of [9, Theorem 2.2], R is abelian and every idempotent of R is a projection. Thus, R is weakly J-*-clean.

(2) ⇒ (1) This is obvious. □

Example. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$. Define $\sigma : R \to R$ by $\sigma(x, y) = (y, x)$. Consider the ring $T_2(R, \sigma) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in R \right\}$ with the following operations: $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} + \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ 0 & a+c \end{pmatrix}$, $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \cdot \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = \begin{pmatrix} ac & ad + b\sigma(c) \\ 0 & ac \end{pmatrix}$. Define $* : T_2(R, \sigma) \to T_2(R, \sigma)$ by $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^* = \begin{pmatrix} a & \sigma(b) \\ 0 & a \end{pmatrix}$. Then, $T_2(R, \sigma)$ is weakly J-*-clean, but it is not medium *-clean.

Proof. Let $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in T_2(R, \sigma)$. Then, $E = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ is a projection. Further, $A - E = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in J(T_2(R, \sigma))$. Therefore, $T_2(R, \sigma)$ is weakly J-*-clean.

Let $A = \begin{pmatrix} (0,1) & (0,0) \\ (0,0) & (0,1) \end{pmatrix}$. We check that $A^2 = A \in T_2(R, \sigma)$ is not central, and so $T_2(R, \sigma)$ is not abelian. Therefore, by Theorem 2.2 the ring $T_2(R, \sigma)$ is not medium *-clean. □

Theorem 2.3. *Let $L = \prod_{i \in I} R_i$ be the direct product of *-rings R_i and $|I| \geq 2$. Then, the following are equivalent:*

(1) *L is medium *-clean;*
(2) *Each R_i is medium *-clean and at most one is not strongly J-*-clean.*

Proof. $\Longrightarrow$ Obviously, each R_i is medium *-clean. Suppose R_{i_1} and $R_{i_2} (i_1 \neq i_2)$ are not strongly J-*-clean. Then, there exist some $x_{i_j} \in R_{i_j} (j = 1, 2)$ such that $x_{i_1} \in R_{i_1}$ and $-x_{i_2} \in R_{i_2}$ are not strongly J-*-clean. Choose $x = (x_i)$ where $x_i = 0$ whenever $i \neq i_j (j = 1, 2)$. Then, x and $-x$ are both not strongly J-*-clean. This gives a contradiction. Therefore, each R_i is a medium *-clean and at most one is not strongly J-*-clean.

$\Longleftarrow$ Suppose that R_{i_0} is medium *-clean and all the others R_i are strongly J-*-clean. Then, $\prod_{i \neq i_0} R_i$ is strongly J-*-clean (see [4]). We directly check that R is medium *-clean. □

Corollary 2.4. *Let $L = \prod_{i \in I} R_i$ be the direct product of *-rings $R_i \cong R$ and $|I| \geq 2$. Then, the following are equivalent:*

(1) *L is medium *-clean;*
(2) *L is strongly J-*-clean.*
(3) *R is strongly J-*-clean.*

Proof. (1) $\Rightarrow$ (3) Since L is medium *-clean, it follows by Theorem 2.4 that R is strongly J-*-clean.

(3) $\Rightarrow$ (2) Straightforward.

(2) $\Rightarrow$ (1) This is trivial. □

We come now to record the strongly weak J-*-cleanness for some related rings.

Proposition 2.5. *Let R be medium *-clean, and let $e \in R$ be an idempotent. Then, eRe is medium *-clean.*

Proof. Let R be medium *-clean ring, and let $e \in R$ be an idempotent. In view of Theorem 2.2, R is strongly *-clean. Thus, R is abelian and every idempotent of R is a projection from [9, Theorem 2.2]. Let $eae \in eRe$. Then, there exists a projection $f \in R$ such that $a = f + w$ or $-f + w$ where $w \in J(R)$ and $fw = wf$. Hence, $eae = efe + ewe$ or $-efe + ewe$ and $ewe \in eJ(R)e = J(eRe)$. Hence, $(efe)^2 = efe = (efe)^*$. This completes the proof. □

Proposition 2.6. *Let R be a *-ring. Then, R is medium *-clean if and only if so is $R[[x]]$.*

Proof. $\Longrightarrow$ In light of Theorem 2.2, R is strongly $*$-clean. It follows by [9, Corollary 2.10] that $R[[x]]$ is strongly *-clean. Let $f(x) \in R[[x]]$. Then, there exists an idempotent $e \in R$ such that $f(0) - e$ or $f(0) + e$ in $J(R)$. Hence, $f(x) - e$ or $f(x) + e$ in $J(R[[x]])$. This implies that $R[[x]]$ is weakly J-clean. Therefore, $R[[x]]$ is medium *-clean, by Theorem 2.2.

$\Longleftarrow$ Let $a \in R$. There exists an idempotent $f(x) \in R[[x]]$ such that $a - f(x)$ or $a + f(x)$ in $J(R[[x]])$ and $af(x) = f(x)a$. Set $e = f(0)$. Then, $a - f(0) \in J(R), af(0) = f(0)a$ and $f(0) \in R$ is an idempotent and, hence, the result. $\square$

Let R be a *-ring, and let $T(R, R)$ be the trivial extension of R by R, i.e., $T(R, R) = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in R\}$. Define $* : T(R, R) \to T(R, R)$ given by $(x, y) \to (x^*, y^*)$. Then, $T(R, R)$ is a *-ring.

Proposition 2.7. *Let R be a *-ring. Then, $T(R, R)$ is medium *-clean if and only if so is R.*

Proof. $\Longrightarrow$ Straightforward.

$\Longleftarrow$ In view of Theorem 2.2, R is strongly *-clean and weakly J-clean. Hence, $T(R, R)$ is strongly *-clean, by [9, Example 2.4]. As $J\big(T(R, R)\big) = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a \in J(R), b \in R\}$, one easily checks that $T(R, R)$ is weakly J-clean. This completes the proof by Theorem 2.2. $\square$

Let R be a *-ring. Define $* : T(R, R) \to T(R, R)$ given by $(x, y) \to (x^*, -y^*)$. Analogously, we prove that $T(R, R)$ is medium *-clean if and only if so is R.

3. Homomorphic Images

We say that an ideal I of a *-ring R is a *-ideal in case $I^* \subseteq I$. If I is a *-ideal of a *-ring, it is easy to check that R/I is also a *-ring.

Lemma 3.1. *Let R be a $*$-ring, let $I \subseteq J(R)$, and let $e \in R$ be an idempotent. If $e - e^* \in I$, then there exists a projection $f \in R$ such that $eR = fR$ and $e - f \in I$.*

Proof. Let $z = 1 + (e^* - e)^*(e^* - e)$. Then $z \in U(R)$ and $z^* = z$. Let $t = z^{-1}$. Then $t^* = t$. We check that $ez = e(1 - e - e^* + ee^* + e^*e) = ee^*e = (1 - e - e^* + ee^* + e^*e)e = ze$; whence, $et = te$ and $e^*t = te^*$. Let $f = ee^*t$. Then $f^* = f$, and $f^2 = ee^*tee^*t = tee^*ee^*t = (ee^*e)te^*t = (ez)te^*t = ee^*t = f$. Hence, $f \in R$ is a projection. Obviously, $fR \subseteq eR$, and from $fe = ee^*te = ee^*et = ezt = e$ one has $eR \subseteq fR$. Therefore, $eR = fR$. Further, $e - f = e(ez - ee^*)t = e(ee^*e - ee^*)t = ee^*(e - e^*)t \in I$, as asserted. $\square$

Theorem 3.2. *Let I be a *-ideal of a *-ring R. If $I \subseteq J(R)$, then R is medium *-clean if and only if*

(1) R is strongly *-clean;
(2) R/I is medium *-clean.

Proof. One direction is obvious from Theorem 2.2. It will suffice to prove the converse. For any idempotent $e \in R$, $\overline{e} \in R/I$ is an idempotent. By (1), R/I is strongly *-clean. In light of [9, Theorem 2.2], $\overline{e} \in R/I$ is a projection. Thus, $e - e^* \in I \subseteq J(R)$. In view of Lemma 4, there exists a projection $f \in R$ such that $eR = fR$. We infer that $e = fe$ and $f = ef$. By (2), we get $e = f$. Therefore, every idempotent of R is a projection, and R is abelian. There we easily complete the proof. □

As a consequence, we can derive

Corollary 3.3. *A *-ring R is medium *-clean if and only if*

(1) R is strongly *-clean;
(2) $R/J(R)$ is medium *-clean.

Proof. This is obvious, by Theorem 3.2. □

Corollary 3.4. *A *-ring R is medium *-clean if and only if*

(1) R is strongly *-clean;
(2) $R/6R$ is medium *-clean and $6 \in J(R)$.

Proof. One direction is obvious by Theorem 3.2. Conversely, assume that R is medium *-clean. Then, there exists a projection $e \in R$ such that $2 = e + w$ or $2 = -e + w$ for a $w \in J(R)$. If $2 = e + w$, then $1 - e = w - 1 \in U(R)$; hence, $e = 0$. We infer that $2 = w \in J(R)$. If $2 = -e + w$, then $4 = e + w'$ for some $w' \in J(R)$. This implies that $6 = w + w' \in J(R)$. In any case, $6 \in J(R)$. By virtue of Theorem 3.2, we complete the proof. □

Recall that a ring R is weakly Boolean if for any $a \in R$, either a or $-a$ is an idempotent.

Lemma 3.5. *Let R be a *-ring. Then, R is medium *-clean if and only if*

(1) *R is strongly *-clean;*
(2) *$R/J(R)$ is weakly Boolean.*

Proof. $\Longrightarrow$ This is clear.

$\Longleftarrow$ Since R is strongly *-clean, it follows by [9, Theorem 2.2] that R is an abelian ring in which every idempotent in R is a projection. In light of [9, Corollary 2.11], every idempotent lifts modulo $J(R)$. So the lemma is true. □

A ring R is a Yaqub ring if it is the subdirect product of $\mathbb{Z}_3$'s. We record

Lemma 3.6 (see [5, Lemma 4.1]). *Let R be a ring in which $x = x^3$ for all $x \in R$. Then, R is a Boolean ring, a Yaqub ring or the product of such rings.*

Theorem 3.7. *Let R be a *-ring. Then, R is medium *-clean if and only if*

(1) *R is strongly *-clean;*
(2) *$R/J(R)$ is a Boolean ring, $\mathbb{Z}_3$ or the product of such rings.*

Proof. $\Longrightarrow$ Clearly, R is strongly *-clean. In view of Lemma 3.6, $R/J(R)$ is a Boolean ring R_1, a Yaqub ring R_2 or the product of such rings. As $3 \in J(R)$, we see that $3 = 0$ in R_1, a contradiction. This implies that $R/J(R) \cong R_2$ is a Yaqub ring. In light of Lemma 3.5, $R/J(R)$ weakly Boolean. This forces $R_2 \cong \mathbb{Z}_3$, as desired.

$\Longleftarrow$ By hypothesis, $R/J(R)$ is weakly Boolean. Therefore, the result follows by Lemma 3.5. □

Corollary 3.8. *Let R be a *-ring. Then, R is strongly J-*-clean if and only if*

(1) *$2 \in J(R)$;*
(2) *R is medium *-clean.*

Proof. $\Longrightarrow$ This is obvious.

$\Longleftarrow$ In light of Theorem 3.7, $R/J(R)$ is Boolean and R is strongly *-clean. Therefore, the result follows, by [4, Theorem 2.6]. □

Corollary 3.9. *Let R be a *-ring. Then, $R/J(R) \cong \mathbb{Z}_3$ if and only if*

(1) *$3 \in J(R)$;*
(2) *R is medium *-clean.*

Proof. $\Longrightarrow$ Clearly, $3 \in J(R)$. Let $a \in R$. Then, $\overline{a} = \overline{0}, \overline{1}$ or $-\overline{1}$ in $R/J(R)$. Hence, $a - 0, a - 1$ or $a + 1$ in $J(R)$. Therefore, R is medium *-clean.

$\Longleftarrow$ In view of Theorem 3.7, $R/J(R)$ is a Boolean ring $R_1, \mathbb{Z}_3$ or the product of such rings. As $3 \in J(R)$, we see that $3 = 0$ in R_1, a contradiction. This implies that $R/J(R) \cong \mathbb{Z}_3$, as desired. □

Example. Let $R = \mathbb{Z}_{(3)}$ be the localization of the ring $\mathbb{Z}$ of integers at (3), and $* = 1_R$, the identical automorphism of R. Then, R is medium *-clean, but it is not a strongly J-*-clean.

Proof. It is obvious that R is a local ring with $J(R) = 3R$. Then, $\frac{2}{1} - (\frac{2}{1})^2$ is not in $J(R)$. Hence, $R/J(R)$ is not a Boolean ring. By [4, Theorem 2.6], R is not strongly J-*-clean. Since $R/J(R) \cong \mathbb{Z}_3$, it follows by Theorem 3.7 that R is medium *-clean. □

Corollary 3.10. *Let R be a local *-ring. Then, the following are equivalent:*

(1) *R is medium *-clean.*
(2) *$R/J(R) \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Proof. (1) $\Rightarrow$ (2) By virtue of Theorem 3.7, $R/J(R)$ is a Boolean ring, $\mathbb{Z}_3$ or the product of such rings. But every idempotent in R is trivial; hence, $R/J(R) \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$.

(2) $\Rightarrow$ (1) Since R is a local *-ring, it is strongly *-clean. Thus, we complete the proof, by Theorem 3.7. □

The Brown–McCoy radical of R can be defined as the intersection of the maximal two-sided ideals and denote it by $BM(R)$.

Theorem 3.11. *Let R be a *-ring. Then, R is medium *-clean if and only if*

(1) *R is strongly *-clean;*
(2) *For all maximal ideals M of R, $R/M \cong \mathbb{Z}_2$ or at most one $\mathbb{Z}_3$.*

Proof. $\Longrightarrow$ By virtue of Lemma 3.5, R is strongly *-clean and $R/J(R)$ is weakly Boolean. Let M be a maximal ideal of R. If $J(R) \not\subseteq M$, then $J(R) + M = R$; hence, $x + y = 1$ for some $x \in J(R), y \in M$. This shows that $y = 1-x \in U(R)$, a contradiction. Thus, $J(R) \subseteq M$. Clearly, $R/M \cong \frac{R/J(R)}{M/J(R)}$; hence, R/M is weakly Boolean. Since R/M is simple and every idempotent in R/M is central, we see that $R/M \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$.

Assume that M_1, M_2 are distinct maximal ideals of R such that R/M_1, $R/M_2 \cong \mathbb{Z}_3$. Since $R/(M_1 \bigcap M_2) \cong \frac{R/J(R)}{(M_1 \bigcap M_2)/J(R)}$, we see that $R/(M_1 \bigcap M_2)$ is weakly Boolean. As $M_1 + M_2 = R$, By Chinese Reminder Theorem, we have $R/(M_1 \bigcap M_2) \cong R/M_1 \times R/M_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, which is not weakly Boolean. Hence, there is at most one maximal ideal M such that $R/M \cong \mathbb{Z}_3$, as desired.

$\Longleftarrow$ It is easy to check that $R/BM(R)$ is isomorphic to a subring of $\prod_{M \in Max(R)} R/M$ which is weakly Boolean. Hence, $R/BM(R)$ is weakly Boolean.

For all maximal ideals M of R, as in the preceding discussion, $J(R) \subseteq M$, and so $J(R) \subseteq BM(R)$. Since R is strongly *-clean, R is an abelian clean ring. In view of [13, Proposition 4.1], R is a right quasi-duo ring, i.e., every maximal right ideal is an ideal. Hence, $BM(R) \subseteq J(R)$, and so $BM(R) = J(R)$. Thus, $R/J(R)$ is weakly Boolean. This completes the proof, by Lemma 3.5. □

4. Uniqueness for Projections

The aim of this section is to determine medium *-clean rings by means of the unique property of projections. The following observation is crucial.

Theorem 4.1. *Let R be a *-ring. Then, R is medium *-clean if and only if*

(1) *R weakly J-*-clean;*
(2) *For any projections $e, f \in R$, $e - f \in J(R)$ implies $e = f$.*

Proof. $\Longrightarrow$ Clearly, R weakly J-*-clean. Let $e, f \in R$ be projections. In light of Theorem 2.2, R is abelian. Thus, $ef = fe$; hence, $(e-f)^3 = e - f$, and so $(e-f)(1-(e-f)^2) = 0$. Hence, $e = f$, as desired.

$\Longleftarrow$ Let $e \in R$ be an idempotent. Then, there exists a projection $g \in R$ such that $e - g \in J(R)$ or $e + g \in J(R)$. If $e - g \in J(R)$, then $e^* - g \in J(R)$, and so $e - e^* = (e-g) - (e^* - g) \in J(R)$. If $e + g \in J(R)$, similarly, we have $e - e^* \in J(R)$.

Set $z = 1 + (e-e^*)^*(e-e^*)$. Write $t = z^{-1}$. Since $z^* = z$, $t^* = t$. Also $e^*z = e^*ee^* = ze^*$, and so $e^*t = te^*$, and $et = te$. Set $f = e^*et = te^*e$. Then, $f^* = f, f^2 = e^*ete^*et = e^*ee^*(tet) = e^*ztet = e^*et = f, fe = f$ and $ef = ee^*et = ezt = e$. Now $e = f + (e - f)$ and $e - f = e - e^*et = ee^*et - e^*et = (e-e^*)e^*et \in J(R)$. Here, $f = f^* = f^2$. In addition, $f = e^*e\big(1 + (e^*-e)(e-e^*)\big)^{-1}$.

Set $z' = 1 + (e^* - e)^*(e^* - e)$. Write $t' = (z')^{-1}$. Since $(z')^* = z'$, $(t')^* = t'$. Also $ez' = ee^*e = z'e$. Set $f' = ee^*t' = t'ee^*$. As in the preceding proof, we see that $f' = (f')^2 = (f')^*$ and $ef' = f'$, $f'e = e$. In addition,

$$e - f' = f'e - f' = t'ee^*(e - e^*) \in J(R),$$

where $f' = \big(1 + (e - e^*)(e^* - e)\big)^{-1} ee^*$.

Thus, $e - f, e - f' \in J(R)$, f and f' are projections. Hence, $f - f' = (e - f') - (e - f) \in J(R)$. By hypothesis, $f = f'$, and so

$$e^*e\big(1 + (e^* - e)(e - e^*)\big)^{-1} = \big(1 + (e - e^*)(e^* - e)\big)^{-1} ee^*.$$

It follows that

$$\big(1 + (e - e^*)(e^* - e)\big)e^*e = ee^*\big(1 + (e^* - e)(e - e^*)\big).$$

Obviously, $(e - e^*)(e^* - e)e^*e = -e^*e + e^*ee^*e$ and $ee^*(e^* - e)(e - e^*) = -ee^* + ee^*ee^*$. Consequently, $e^*ee^*e = ee^*ee^*$. One easily checks that

$$(e - e^*)^3 - (e - e^*) = -ee^*e + e^*ee^*;$$
$$\big((e - e^*)^3 - (e - e^*)\big)(e + e^*) = (e - e^*)^3 - (e - e^*).$$

Thus $(e - e^*)\big((e - e^*)^2 - 1\big)\big((e + e^*) - 1\big) = 0$.

As $e - f \in J(R)$, we see that $e^* - f \in J(R)$. Thus, $(e + e^*) - 2f \in J(R)$. This implies that $(e + e^*) - 1 = (2f - 1) + \big((e + e^*) - 2f\big) \in U(R)$, as $(2f - 1)^2 = 1$. Since $(e - e^*)^2 - 1, (e + e^*) - 1 \in U(R)$, we get $e = e^*$. Thus, every idempotent in R is a projection. In light of [9, Lemma 2.1], R is abelian. Therefore, R is medium *-clean. □

Projections e, f in R are said to be equivalent, write $e \sim f$, in case there exists $w \in R$ such that $w^*w = e$ and $ww^* = f$ (see [2]).

Corollary 4.2. *Let R be a *-ring. Then, R is medium *-clean if and only if*

1. *R weakly J-*-clean;*
2. *For any projections $e, f \in R$, $e \sim f$ implies $e = f$.*

Proof. $\Longrightarrow$ (1) is clear. In light of Lemma 3.5, $R/J(R)$ is weakly Boolean; hence, it is strongly π-regular. By virtue of [3, Theorem 13.1.7], R has stable range 1. If $e \sim f$ with projections $e, f \in R$, then $eR \cong fR$. It follows by [3, Lemma 1.4.6] that $u^{-1}eu = f$. In light of Theorem 2.2, R is abelian; hence, $e = f$. This proves (2).

$\Longleftarrow$ Let $e, f \in R$ be projections such that $e - f \in J(R)$. Set $u = 1 - e - f$. Then, $eu = -ef = uf$. Clearly, $u = u^* = u^{-1} \in U(R)$. Set $w = fu^{-1}e$. Then, $f = u^{-1}eu = ww^*$ and $e = ufu^{-1} = w^*w$. Hence, $e \sim f$. By hypothesis, $e = f$. According to Theorem 4.1, R is medium *-clean. □

Let R be a $*$-ring. An element $a \in R$ is called a partial isometry provided that $a = aa^*a$. An element $u \in R$ is called a unitary element provided that $uu^* = u^*u = 1$ (see [2]).

Corollary 4.3. *Let R be a *-ring. Then, R is medium *-clean if and only if*

(1) *R weakly J-*-clean;*
(2) *For any partial isometry $a \in R$, there exist a projection e and a unitary u such that $a = eu = ue$.*

Proof. $\Longrightarrow$ Clearly, R weakly J-*-clean. Let $w \in R$ be a partial isometry. Then, $w = ww^*w$. Hence, $w^* = w^*ww^*$, ww^* and w^*w are projections with $ww^*R \cong w^*wR$. In light of Corollary 4.2, $ww^* = w^*w$. Let $u = 1 - w^*w + w$. Then, $u^* = 1 - w^*w + w^*$ and $uu^* = u^*u = 1$, i.e., $u \in R$ is a unitary element. Let $e = ww^*$. Then, $e \in R$ is a projection. Furthermore, $w = ww^*(1 - ww^* + w) = eu$. In virtue of Theorem 2.2, R is ableian. Hence, $w = ue$, as desired.

$\Longleftarrow$ Suppose $e \sim f$ for projections $e, f \in R$. Write $e = w^*w$ and $f = ww^*$. We may assume that $w \in fRe$ and $w^* \in eRf$. Hence, $ww^*w = we = w$, i.e., $w \in R$ is a partial isometry. By hypothesis, there exist a projection g and a unitary u such that $w = gu = ug$. Accordingly, $e = w^*w = (u^*g)(gu) = u^*gu = (u^*u)g = g$ and $f = ww^* = (gu)(u^*g) = g(uu^*)g = g$, and then $e = f$. In light of Corollary 4.2, the result follows. □

Recall that an element a in a *-ring R is uniquely *-clean provided that there exists a unique projection e such that $a - e$ is invertible. We have

Theorem 4.4. *Let R be a *-ring. Then, R is medium *-clean if and only if*

(1) *R weakly J-*-clean;*
(2) *$a^2 \in R$ is uniquely *-clean for all $a \in R$.*

Proof. $\Longrightarrow$ Let $a \in R$. Then, there exist a projection $e \in R$ and a $w \in J(R)$ such that $a = e + w$ or $a = -e + w$ with $ae = ea$. Hence, $a^2 = e + w'$ where $w' \in J(R)$. This implies that $a^2 = (1 - e) + \big((2e - 1) + w'\big)$. Clearly, $(2e-1)+w' = (2e-1)\big(1+(2e-1)w'\big) \in U(R)$. Thus, $a^2 \in R$ is *-clean. Assume that $a^2 = f+v$ where $f \in R$ is a projection and $v \in U(R)$. Then, $e-f \in U(R)$. As R is abelian, it follows from $(e-f)^3 = e-f$ that $(e-f)(1-(e-f)^2) = 0$; hence, $1 - e + 2ef - f = 0$. Thus, $f = (1 - 2e)^{-1}(1 - e) = 1 - e$. Therefore, $a^2 \in R$ is uniquely clean.

$\Longleftarrow$ Clearly, R is weakly J-*-clean. Let $e, f \in R$ be projections with $e - f \in J(R)$. By hypothesis, e^2 is uniquely *-clean. Obviously, $e^2 = (1-e) + (2e-1) = (1-f) + \big((2f-1)+(e-f)\big)$. We that $1-e, 1-f$ are both projections, $(2e - 1)^2 = 1$ and $(2f - 1) + (e - f) = (2f - 1)\big(1 + (2f - 1)(e - f)\big) \in U(R)$. Thus, $1 - e = 1 - f$; hence, $e = f$. According to Theorem 4.1, R is medium *-clean. □

Example. Let $\mathbb{Z}_3$ be the $*$-ring with $*$ the identical automorphism. Then, $\mathbb{Z}_3$ is medium *-clean, but $-1 \in \mathbb{Z}_3$ is not unique *-clean, as $-1 = 0 + (-1) = 1 + 1$.

Lemma 4.5. *Let R be a *-ring. Then, R is medium *-clean if and only if*

(1) *Every idempotent lifts modulo $J(R)$;*
(2) *For any projections $e, f \in R$, $e - f \in J(R)$ implies $e = f$;*
(3) *$R/J(R)$ weakly *-Boolean.*

Proof. $\Longrightarrow$ This is obvious.

$\Longleftarrow$ Let $a \in R$. By (3), we can find some $e \in R$ such that $e - e^2, e - e^* \in J(R)$ such that $a - e \in J(R)$ or $a + e \in J(R)$. By (1), there exists an idempotent $f \in R$ such that $e - f \in J(R)$. Hence, $a - f \in J(R)$ or $a + f \in J(R)$. Additionally, $f - f^* \in J(R)$. In light of Lemma , we have a projection $g \in R$ such that $f - g \in J(R)$. Therefore, $a - g \in J(R)$ or $a + g \in J(R)$. Hence, R is weakly J-*-clean. Therefore, R is medium *-clean, by Theorem 4.1. □

Theorem 4.6. *Let R be a *-ring. Then, R is medium *-clean if and only if*

(1) *Every idempotent lifts modulo $J(R)$;*
(2) *R is abelian;*
(3) *$R/J(R)$ weakly *-Boolean.*

Proof. $\Longrightarrow$ This is obvious, by Theorem 2.2 and Lemma 4.5.

$\Longleftarrow$ Suppose $e - f \in J(R)$ with projections $e, f \in R$. Since R is abelian, $(e - f)^3 = e - f$; hence, $(e - f)\big(1 - (e - f)\big) = 0$. We infer that $e = f$. Therefore, we complete the proof, by Lemma 4.5. □

5. Medium Nil-*-clean Rings

In this section, we are concern on a subclass of medium *-clean rings. A *-ring R is medium nil-*-clean if for any $a \in R$ there exists a projection $e \in R$ such that $a - e$ or $a + e$ is nilpotent and $ea = ae$. We derive

Theorem 5.1. *Let R be a *-ring. Then, R is medium nil-*-clean if and only if*

(1) *R is medium *-clean.*
(2) *$J(R)$ is nil.*

Proof. $\Longrightarrow$ Clearly, R is strongly 2-nil-clean. In light of [5, Theorem 3.3 and Theorem 3..6], $N(R)$ forms an ideal of R and $J(R)$ is nil. Hence, $N(R) \subseteq J(R)$. Therefore, R is medium *-clean.

$\Longleftarrow$ This is obvious. □

A $*$-ring R is called strongly nil $*$-clean if every element of R is the sum of a projection and a nilpotent element that commute with each other.

Corollary 5.2. *Let R be a *-ring. Then, R is medium nil-*-clean if and only if R is strongly nil-*-clean, or $R/J(R) \cong \mathbb{Z}_3$, with $J(R)$ is nil, or R is the direct product of two such rings.*

Proof. $\Longleftarrow$ By virtue of Theorems 5.1 and 3.7, $R/J(R)$ is a Boolean ring, $\mathbb{Z}_3$ or the product of such rings. Suppose that $R/J(R)$ is Boolean. Since $J(R)$ is nil and every idempotent in R is a projection, we easily check that R is strongly nil-*-clean, as desired. $\Longrightarrow$ This is obvious by Theorems 3.7 and 5.1. □

Corollary 5.3. *Let R be medium nil-*-clean, and let $e \in R$ be an idempotent. Then, eRe is medium nil-*-clean.*

Proof. In view of Theorem 2.2, R is abelian. As $J(eRe) = eJ(R)e$, we complete the proof, by Theorem 5.1 and Proposition 2.6. □

Theorem 5.4. *Let R be a *-ring. Then, R is medium nil-*-clean if and only if*

(1) *R is strongly *-clean;*
(2) *R is strongly weakly nil-clean.*

Proof. $\Longrightarrow$ This is obvious.

$\Longleftarrow$ In light of [9, Theorem 2.2], every idempotent in R is a projection, the result follows. □

Corollary 5.5. *Let R be a *-ring. Then, R is medium nil-*-clean if and only if*

(1) *R is strongly *-clean;*
(2) *R is R_1, R_2 or $R_1 \times R_2$, where R_1 is strongly nil-clean and $R_2/J(R_2) \cong \mathbb{Z}_3$ and $J(R_2)$ is nil.*

Proof. $\Longrightarrow$ In view of Theorem 2.2, R is strongly *-clean, and hence proving (1). Clearly, R is strongly weakly nil-clean. Therefore we prove (2) by [8, Theorem 1].

$\Longleftarrow$ In light of [8, Theorem 1], R is strongly weakly nil-clean. Thus, we complete the proof, by Theorem 5.4. □

Example. Let $R = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \mid a, b, c \in \mathbb{Z}_3 \right\}$. Define

$$\begin{pmatrix} a & b \\ c & a \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & a' \end{pmatrix} = \begin{pmatrix} a+a' & b+b' \\ c+c' & a+a' \end{pmatrix},$$

$$\begin{pmatrix} a & b \\ c & a \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & a' \end{pmatrix} = \begin{pmatrix} aa' & ab'+ba' \\ ca'+ac' & aa' \end{pmatrix}$$

and $* : R \to R$, $\begin{pmatrix} a & b \\ c & a \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & a \end{pmatrix}$. Then, R is medium *-clean, but it is not strongly nil-*-clean.

Proof. In view of [4, Example 2.2], R is strongly J-*-clean, and so it is medium *-clean. The projections in R are the zero and the identity matrix. Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. It is obvious that A can not be written as the sum of a projection and a nilpotent. Thus, R is not strongly weakly nil-*-clean. □

Example. Let $R = \left\{ \begin{pmatrix} a & 2b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_4 \right\}$, and let

$$* : R \to R, \begin{pmatrix} a & 2b \\ 0 & c \end{pmatrix} \mapsto \begin{pmatrix} c & -2b \\ 0 & a \end{pmatrix}.$$

Then, R is strongly weakly nil clean, but it is not medium nil-*-clean.

Proof. Let $A = \begin{pmatrix} a & 2b \\ 0 & c \end{pmatrix}$ be a projection in R. Then, $\begin{pmatrix} a & 2b \\ 0 & c \end{pmatrix}^2 = \begin{pmatrix} a^2 & 2ab+2bc \\ 0 & c^2 \end{pmatrix} = \begin{pmatrix} c & -2b \\ 0 & a \end{pmatrix}$. This implies that $a = a^2, c = c^2, a = c$ and $2ab + 2bc = -2b$, and so $(2a-1)(2b) = 0$. As $(2a-1)^2 = 1$, we see that $2b = 0$, and so $A = 0$ or I_2. Now let $A = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} \in R$. It is obvious that A can not be written as the sum or difference of a projection and a nilpotent. Hence, R is not medium nil-*-clean. However for any$A \in R$, $A - A^2 \in N(R)$, then R is weakly strongly nil-clean ring, as desired. □

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References

[1] Breaz, S., Danchev, P., Zhou, Y.: Rings in which every element in either a sum or a difference of a nilpotent and an idempotent. J. Algebra Appl. **15**, 1650148 (2016). https://doi.org/10.1142/S0219498816501486

[2] Berberian, S.K.: Baer *-Rings. Springer, Heidelberg (2011)

[3] Chen, H.: Rings Related Stable Range Conditions, Series in Algebra 11. World Scientific, Hackensack (2011)

[4] Chen, H., Harmanci, A., Özcan, A.C.: Strongly J-clean rings with involutions. Contemp Math **609**, 33–44 (2014)

[5] Chen, H., Sheibani, M.: Strongly 2-nil-clean rings. J. Algebra Appl. **16**, 1750178 (2017). https://doi.org/10.1142/S021949881750178X

[6] Cui, J., Wang, Z.: A note on strongly *-clean rings. J. Korean Math. Soc. **52**, 839–851 (2015)

[7] Hirano, Y., Tominaga, H.: Rings in which every element is a sum of two idempotents. Bull. Austral. Math. Soc. **37**, 161–164 (1988)

[8] Kosan, M.T., Zhou, Y.: On weakly nil-clean rings. Front. Math. China (2016). https://doi.org/10.1007/s11464-016-0555-6

[9] Li, C., Zhou, Y.: On strongly *-clean rings. J. Algebra Appl. **10**, 1363–1370 (2011)

[10] Li, Y., Parmenter, M.M., Yuan, P.: On *-clean group rings. J. Algebra Appl. **14**, 1550004 (2015). https://doi.org/10.1142/S0219498815500048

[11] Stancu, A.: A note on commutative weakly nil clean rings. J. Algebra Appl. **15**, 1620001 (2016). https://doi.org/10.1142/S0219498816200012

[12] Vas, L.: *-Clean rings; some clean and almost clean Baer *-rings and von Neumann algebras. J. Algebra **324**, 3388–3400 (2010)

[13] Yu, H.P.: On quasi-duo rings. Glasg. Math. J. **37**, 21–31 (1995)

Huanyin Chen
Department of Mathematics
Hangzhou Normal University
Hangzhou
China
e-mail: huanyinchen@aliyun.com

Marjan Sheibani Abdolyousefi
Women's University of Semnan (Farzanegan)
Semnan
Iran
e-mail: sheibani@fgusem.ac.ir

Handan Kose
Department of Mathematics
Ahi Evran University
Kirsehir
Turkey
e-mail: handankose@gmail.com

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