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New Result on Matrix Summability Factors of Infinite Series and Fourier Series

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Abstract. In the present paper, we have extended the result of Sulaiman [10] dealing with $\left|\bar{N}, p_n\right|_k$ summability factors of Fourier series to the $|A, p_n|_k$ summability method.
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Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). By u_n^α and t_n^α we denote the nth Cesàro means of order α, with $\alpha > -1$, of the sequence (s_n) and (na_n), respectively, that is (see [2])

$$u_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \quad \text{and} \quad t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)...(\alpha+n)}{n!} = O(n^\alpha), \qquad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0. \tag{2}$$

The series $\sum a_n$ is said to be summable $|C,\alpha|_k$, $k \geq 1$, if (see [5],[8])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty. \tag{3}$$

If we take $\alpha = 1$, then $|C,\alpha|_k$ summability reduces to $|C,1|_k$ summability.
Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \tag{4}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v \tag{5}$$

defines the sequence (t_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [6]).

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The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1} \mid t_n - t_{n-1} \mid^k < \infty. \tag{6}$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), $\left|\bar{N}, p_n\right|_k$ summability is the same as $|C,1|_k$ (resp. $\mid \bar{N}, p_n \mid$) summability.

Let the formal expansion of a function f, periodic with period 2π, and integrable in the sense of Lebesgue over $[-\pi, \pi]$, in a Fourier trigonometric series be given by

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx) = \sum_{n=0}^{\infty} C_n(x),$$

We write

$$\phi(u) = f(x+u) + f(x-u) - 2f(x),$$
$$\varphi(t) = \int_t^{\delta} \frac{|\phi(u)|}{u} du, \quad \Phi(t) = \int_0^t |\phi(u)| du, \quad 0 < \delta \leq \pi,$$
$$\mu_n = \left(\prod_{v=1}^{\ell-1} log^v n\right)(log^{\ell} n)^{1+\epsilon}, \quad log^{\ell} n_0 > 0, \quad \epsilon > 0,$$

where

$$log^{\ell} n = log(log^{\ell-1} n), ..., log^2 n = loglogn.$$

The Known Results

Theorem 2.1 (Chow [4], 1941) If $\{\lambda_n\}$ is a convex sequence and the series $\sum n^{-1}\lambda_n$ convergent, then the series $\sum C_n(x)\lambda_n$ is $|C,1|$ summable for almost all values of x.
Theorem 2.2 (Cheng [3], 1948) If

$$\Phi(t) = 0(t), \quad as \quad t \to 0,$$

then the series

$$\sum_{n=2}^{\infty} C_n(x)/(logn)^{1+\epsilon}, \quad \epsilon > 0,$$

is summable $|C, \alpha|$, $\alpha > 1$.
Theorem 2.3 (Hsiang [7], 1970) If

$$\Phi(t) = 0(t), \quad as \quad t \to +0,$$

then the series

$$\sum_{n=0}^{\infty} C_n(x)/n^{\alpha}$$

is summable $|C,1|$, $\alpha > 0$.
Theorem 2.4 (Pandey [9], 1978) If

$$\varphi(t) = O\left\{(log^{\ell}(1/t))^{\eta}\right\} \quad as \quad t \to +0,$$

then the series

$$\sum_{n=0}^{\infty} C_n(x)/\mu_n$$

is summable $|C, 1|$ for $0 < \eta < \epsilon$.
Sulaiman has proved the more general theorem dealing with $|\bar{N}, p_n|_k$ summability in the following form, which includes the theorems of Chow and Pandey as special cases, and hence all the previous results.
Theorem 2.5[10] Let $\{|\lambda_n|\}$ be non-increasing sequence of constants such that $|\Delta\lambda_n| = O\left(\frac{|\lambda_n|}{n}\right)$. Let $P_n = O(np_n)$ and $|\Delta(\frac{p_n}{P_n})| = O(\frac{p_n}{nP_n})$.
(A) If

$$\sum p_n P_n^{-1} |\lambda_n|^k < \infty, \tag{7}$$

then the series $\sum C_n(x)\lambda_n$ is summable $|\bar{N}, p_n|_k$, $1 \leq k < 2$, for almost all values of x.
(B) If $\{\eta_n\}$ is a sequence of positive constants such that $n^{-\gamma}\eta_n \to 0$ as $n \to \infty$, for some γ, $0 < \gamma < 1$, and if

$$\varphi(t) = O\left\{\eta_{(1/t)}\right\}, \quad t \to 0 \tag{8}$$

$$\sum p_n P_n^{-1} |\lambda_n|^k \eta_n^k < \infty, \tag{9}$$

then the series $\sum C_n(x)\lambda_n$ is summable $|\bar{N}, p_n|_k$, $1 \leq k < \infty$.

The Main Results

Given a normal matrix $A = (a_{nv})$, i.e., a lower triangular matrix of nonzero diagonal entries. We associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \quad \bar{\Delta} a_{nv} = a_{nv} - a_{n-1}, v \quad a_{-1,0} = 0 \tag{10}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{\Delta}\bar{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{11}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) \quad = \textstyle\sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{12}$$

and

$$\bar{\Delta} A_n(s) \quad = \textstyle\sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{13}$$

Let $A = (a_{nv})$ be a normal matrix. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v \quad n = 0, 1, ... \tag{14}$$

The series $\sum a_n$ is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [11])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty, \tag{15}$$

where

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} s_v. \tag{16}$$

Note that in the special case when A is the matrix of weighted mean, i.e.,

$$a_{nv} = \begin{cases} \frac{p_v}{P_n}, & 0 \leq v \leq n \\ 0, & n > v, \end{cases}$$

then the summability $|A, p_n|_k$ reduces to the summability $|\bar{N}, p_n|_k$, and if we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n reduces to the summability $|C, 1|_k$. Also, if we take $p_n = 1$ for all values of n reduces to the summability $|A|_k$ (see [12]). For any sequence (λ_n) we write that $\Delta^2 \lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$. A sequence (λ_n) is said to be convex if $\Delta^2 \lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1} \geq 0$.

The aim of this paper is to generalize Theorem 2.5 for $|A, p_n|_k$ summability method. Moreover some new results will be obtained and extended by the author. Many papers have been made about this method (see [13]-[21]).

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