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Optimal Control Problem for Bianchi Equation in Variable Exponent Sobolev Spaces

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Abstract In this paper, a necessary and sufficient condition, such as the Pontryagin's maximum principle for an optimal control problem with distributed parameters, is given by the third-order Bianchi equation with coefficients from variable exponent Lebesgue spaces. The statement of an optimal control problem is studied by using a new version of the increment method that essentially uses the concept of the adjoint equation of the integral form.

Keywords 3D optimal control · Pontryagin's maximum principle · Bianchi equation · Goursat problem · Variable exponent Sobolev spaces

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1 Introduction

It is well known that under different assumptions, for various optimal control problems described by Bianchi equations and the equations of mathematical physics, a number of necessary and sufficient conditions of optimality were obtained. Development of optimal control theory led to its application to various problems, such as mathematical modeling of a controlled objects, optimization of dynamical systems and others. Many of these optimal control problems are described by Bianchi equations, and there are numerous works devoted to solutions of these equations.

The Pontryagin's maximum principle is a fundamental result of the theory of necessary optimality conditions of the first order, which initially was proved (in the linear case by Gamkrelidze, in the nonlinear case by Boltyanskii [1]) for optimal control problems described by ordinary differential equations. Later works were dedicated for obtaining necessary conditions for optimality in more complex control problems with concentrated and distributed parameters. Optimal control problems described by hyperbolic equations under Goursat conditions, originate in [2]. Further various aspects of the problem of optimal control processes described by Goursat-Darboux systems, were investigated in [3–7] and others. Many of the processes occurring in the theory of filtration of fluids in fractured media were described by pseudoparabolic (hyperbolic) and parabolic equations with discontinuous coefficients. Note that some properties of the solutions of the Dirichlet problem for a parabolic equation with discontinuous coefficients in Sobolev-type spaces were investigated in [8] and, etc.

Well-defined solvability of the Goursat boundary value problem plays an important role in qualitative theory of optimal processes. The Goursat problem for hyperbolic equations with discontinuous coefficients under the nonclassical boundary conditions were studied in [9–12] and others. In the monograph of Mordukhovich [13], effective methods for solving complex optimization and control problems with a nonsmooth and nonconvex structure, based on simple problems of constructive approximations, were investigated. The present work is devoted for obtaining a necessary and sufficient condition such as the Pontryagin's maximum principle for an optimal control problem with distributed parameters described by a third-order Bianchi equation with coefficients in variable exponent Lebesgue spaces. Recently, the optimal control problem in the processes described by the Goursat problem for a hyperbolic equation in variable exponent Sobolev spaces with dominating mixed derivatives was studied in [14]. To study the optimal control problem, the Hardy operator appears in Lebesgue spaces. Therefore, investigation of the Hardy operator in different function spaces plays an important role, see, for example [15]. At the turn of the millennium, various developments lead to the start of a period of systematic intense study of variable exponent spaces. First, the connection was made between variable exponent spaces and variational integrals with nonstandard growth and coercivity conditions. It was also observed that these nonstandard variational problems are related to modeling of so-called electrorheological fluids. Moreover, progress in physics and engineering over the past ten year has made the study of fluid mechanical properties of these fluids an important issue, see [16].

In this paper, the optimal control problem for the third-order Bianchi equation with coefficients in variable exponent Lebesgue spaces with nonclassical Goursat boundary value problem is investigated. The statement of optimal control problem is studied by using a new version of the increment method that essentially uses the concept of an adjoint equation of the integral form. The method also includes the case where the coefficients of the equation are nonsmooth functions from variable exponent Lebesgue spaces. In this paper, it is shown that such an optimal control problem can be investigated with the help of a new concept of the adjoint equation, which can be regarded as an auxiliary equation for determination of Lagrange multipliers. In the future, we can consider a variety of classes of optimal control problem described by loaded integro-differential equations for various nonlocal boundary conditions. These optimal control problems actually describe more complex control processes that are very important in the theory of optimal processes. The results can be used in the theory of optimal processes for distribution Pontryagin's maximum principle for various controlled processes described by third-order Bianchi equation with discontinuous coefficients in variable exponent Sobolev spaces with dominant mixed derivatives.

The paper is organized as follows. Section 2 contains some preliminaries along with the standard ingredients used in the proofs. In Sect. 3, we give the problem statement, and in Sect. 4, we show the construction of an adjoint equation of the considered optimal control problem. In Sect. 5, we give the proof of the main result.

2 Preliminaries and Bianchi Equation in Variable Exponent Sobolev Spaces

Now we reduce an illustrative example demonstrating appearance of the variable exponent Sobolev space. Let $G_0 = G_1^0 \times G_2^0 \times G_3^0 = (0,\ h_1) \times (0, h_2) \times (0, h_3)$ be a rectangle in $\mathbb{R}^3$, and h_i $(i = 1, 2, 3)$ be fixed real numbers. We consider the following linear 3D Goursat boundary value problem

$$
\begin{aligned}
&D_1 D_2 D_3 u(x) + a_{1,1,0}(x) D_1 D_2 u(x)\\
&+ a_{0,1,1}(x) D_2 D_3 u(x) + a_{1,0,1}(x) D_1 D_3 u(x) = f(x),\ x \in G_0\\
&u|_{x_1=0} = F_1\big(x_2, x_3\big) \in SW^{(1,1)}_{(p_2(x_2), p_3(x_3))}\left(G_2^0 \times G_3^0\right),\\
&u|_{x_2=0} = F_2\big(x_1, x_3\big) \in SW^{(1,1)}_{(p_1(x_1), p_3(x_3))}\left(G_1^0 \times G_3^0\right),\\
&u|_{x_3=0} = F_3\big(x_1, x_2\big) \in SW^{(1,1)}_{(p_1(x_1), p_2(x_2))}\left(G_1^0 \times G_2^0\right),
\end{aligned}
$$

with agreement conditions

$$
\begin{aligned}
F_1\big(x_2, x_3\big)\big|_{x_3=0} &= F_3\big(x_1, x_2\big)\big|_{x_1=0},\\
F_1\big(x_2, x_3\big)\big|_{x_2=0} &= F_2\big(x_1, x_3\big)\big|_{x_1=0},\\
F_2\big(x_1, x_3\big)\big|_{x_3=0} &= F_3\big(x_1, x_2\big)\big|_{x_2=0}.
\end{aligned}
$$

Suppose $f \in L_{p(x)}(G_0)$. For example, let $p(x) = \sum_{i=1}^{3} p_i \chi_{G_i^0}(x)$, where $\chi_{G_i^0}(x)$ is the characteristic function of the set G_i^0. Then, the function $f(x) = x_1^{-\frac{1}{2}} x_2^{\frac{p_2-3}{p_2}} x_3^{-\frac{1}{3}}$ belongs to $L_{(p_1, p_2, p_3)}(G_0)$ if $p_1 < 2,\ p_2 > 2$ and $p_3 < 3$. This example shows that there is no a constant that characterizes the belonging of this function to the space $L_p(G_0)$. A more complicated situation appears, when p is not a simple function on G_0. In such cases, a variable exponent Sobolev spaces appear.

Let $\mathbb{R}^3$ be the three-dimensional Euclidean space of points $x = (x_1, x_2, x_3)$, $|x| = \left(\sum_{i=1}^3 x_i^2\right)^{1/2}$ and let $G = G_1 \times G_2 \times G_3 = (x_1^0, h_1) \times (x_2^0, h_2) \times (x_3^0, h_3)$ be a rectangle in $\mathbb{R}^3$, $x^0 = (x_1^0, x_2^0, x_3^0)$ and h_i $(i = 1, 2, 3)$ be fixed real numbers.

By $\mathcal{P}(G)$, we denote the set of Lebesgue measurable functions such that $p : G \mapsto [1, \infty)$. The functions $p \in \mathcal{P}(G)$ are called variable exponents on G. We define $\underline{p} = \operatorname*{essinf}_{x\in G} p(x)$ and $\overline{p} = \operatorname*{esssup}_{x\in G} p(x)$. We denote

$$r_1(x_1) = \lim_{\substack{x_3\to x_3^0+0\\ x_2\to x_2^0+0}} p(x_1, x_2, x_3), \quad r_2(x_2) = \lim_{\substack{x_3\to x_3^0+0\\ x_1\to x_1^0+0}} p(x_1, x_2, x_3)$$

and

$$r_3(x_3) = \lim_{\substack{x_2\to x_2^0+0\\ x_1\to x_1^0+0}} p(x_1, x_2, x_3).$$

Let $q(x)$ be the conjugate variable exponent function of p defined by $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. Assume $\frac{1}{r_1(x_1)} + \frac{1}{s_1(x_1)} = 1$ and $\frac{1}{r_2(x_2)} + \frac{1}{s_2(x_2)} = 1$, $\frac{1}{r_3(x_3)} + \frac{1}{s_3(x_3)} = 1$, where $x \in G$. Obviously, $\operatorname*{esssup}_{x\in G} q(x) = \overline{q} = \frac{\underline{p}}{\underline{p} - 1}$ and $\operatorname*{essinf}_{x\in G} q(x) = \underline{q} = \frac{\overline{p}}{\overline{p} - 1}$.

Definition 2.1 [16,17] Let $p \in \mathcal{P}(G)$. By $L_{p(x)}(G)$, we denote the space of Lebesgue measurable functions f on G such that for some $\lambda_0 > 0$

$$\int_G \left(\frac{|f(x)|}{\lambda_0}\right)^{p(x)} \mathrm{d}x < \infty.$$

Note that the functional

$$\|f\|_{L_{p(x)}(G)} = \|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_G \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}x \le 1\right\}$$

defines the norm in $L_{p(x)}(G)$ and $L_{p(x)}(G)$ is a Banach function space (see, [16,17]).

Definition 2.2 Let $p \in \mathcal{P}(G)$. By $SW_{p(x)}^{(1,1,1)}(G)$, we define the variable exponent Sobolev spaces of function with dominating mixed derivatives as

$$SW^{(1,1,1)}_{p(x)}(G) := \left\{u \in L^{\text{loc}}_1(G) : \ D^{i_1}_1 D^{i_2}_2 D^{i_3}_3 u(x) \in L_{p(x)}(G),\ i_k = 0, 1,\ k = 1, 2, 3\right\}.$$

Obviously, the expression

$$\|u\|_{SW^{(1,1,1)}_{p(\cdot)}(G)} = \sum_{i_1,i_2,i_3=0}^{1} \left\| D^{i_1}_1 D^{i_2}_2 D^{i_3}_3 u \right\|_{L_{p(\cdot)}(G)} < \infty$$

defines the norm in $SW^{(1,1,1)}_{p(x)}(G)$.

Lemma 2.1 *Let $p \in \mathcal{P}(G)$ and $1 < \underline{p} \le \overline{p} < \infty$. Then, the space $SW^{(1,1,1)}_{p(x)}(G)$ is complete.*

Proof Let $\{u_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $SW^{(1,1,1)}_{p(x)}(G)$. Then, $\left\{D^{i_1}_1 D^{i_2}_2 D^{i_3}_3 u_n\right\}$ is a Cauchy sequence in $L_{p(x)}(G)$ for all $0 \le i_1, i_2, i_3 \le 1$. By the completeness of $L_{p(x)}(G)$ (see [17]), there exists $g_{i_1,i_2,i_3} \in L_{p(x)}(G)$ such that

$$\left\| D^{i_1}_1 D^{i_2}_2 D^{i_3}_3 u_n - g_{i_1,i_2,i_3} \right\|_{L_{p(x)}(G)} \to 0 \ \text{as} \ n \to \infty$$

and for all $0 \le i_1, i_2, i_3 \le 1$. Applying the Hölder inequality in variable exponent Lebesgue spaces (see [16,17]), for $\varphi \in C^{\infty}_c(G)$ we get

$$\int_G \left(D^{i_1}_1 D^{i_2}_2 D^{i_3}_3 u_n(x) - g_{i_1,i_2,i_3}(x)\right) D^{i_1}_1 D^{i_2}_2 D^{i_3}_3 \varphi(x)\, \mathrm{d}x$$
$$\le \left(\frac{1}{\underline{p}} + \frac{1}{\overline{p}}\right) \left\| D^{i_1}_1 D^{i_2}_2 D^{i_3}_3 u_n - g_{i_1,i_2,i_3} \right\|_{L_{p(x)}(G)} \left\| D^{i_1}_1 D^{i_2}_2 D^{i_3}_3 \varphi \right\|_{L_{q(x)}(G)}.$$

Since $\left\| D^{i_1}_1 D^{i_2}_2 D^{i_3}_3 u_n - g_{i_1,i_2,i_3} \right\|_{L_{p(x)}(G)} \to 0$ and $D^{i_1}_1 D^{i_2}_2 D^{i_3}_3 \varphi(x)$ are bounded for any $\varphi \in C^{\infty}_c(G)$, by the Lebesgue dominated convergence theorem in variable Lebesgue spaces (see [17]), we have

$$\lim_{n\to\infty} \int_G u_n(x)\, D^{i_1}_1 D^{i_2}_2 D^{i_3}_3 \varphi(x)\, \mathrm{d}x = \int_G g_{i_1,i_2,i_3}(x)\, D^{i_1}_1 D^{i_2}_2 D^{i_3}_3 \varphi(x)\, \mathrm{d}x.$$

Therefore, for all $\varphi \in C^{\infty}_c(G)$, we have

$$\begin{aligned}\int_G u(x) D^{i_1}_1 D^{i_2}_2 D^{i_3}_3 \varphi(x)\, \mathrm{d}x &= \lim_{n\to\infty} \int_G u_n(x) D^{i_1}_1 D^{i_2}_2 D^{i_3}_3 \varphi(x)\, \mathrm{d}x \\ &= (-1)^{i_1+i_2+i_3} \lim_{n\to\infty} \int_G D^{i_1}_1 D^{i_2}_2 D^{i_3}_3 u_n(x)\varphi(x)\, \mathrm{d}x \\ &= (-1)^{i_1+i_2+i_3} \int_G g_{i_1,i_2,i_3}(x)\varphi(x)\, \mathrm{d}x.\end{aligned}$$

This shows that $D_1^{i_1} D_2^{i_2} D_3^{i_3} u$ exists weakly and $g_{i_1,i_2,i_3} = D_1^{i_1} D_2^{i_2} D_3^{i_3} u$. Thus, $u \in SW_{p(x)}^{(1,1,1)}(G)$ and $u_n \to u$ as $n \to \infty$, that completes the proof. □

3 Problem Statement

By $L_{(p_1(x_1),p_2(x_2),p_3(x_3))}(G)$, we denote the variable Lebesgue spaces of the mixed norm defined as

$$\|f\|_{L_{(p_1(x_1),p_2(x_2),p_3(x_3))}(G)} = \left\| \left\| \|f\|_{L_{p_1(x_1)}(G_1)} \right\|_{L_{p_2(x_2)}(G_2)} \right\|_{L_{p_3(x_3)}(G_3)} < \infty.$$

Let the controlled object be described by the Bianchi equation

$$\begin{aligned} \big(V_{1,1,1}u\big)(x) \equiv D_1 D_2 D_3 u(x) + \sum_{\substack{i_1,i_2,i_3=0 \\ 0 \le i_1+i_2+i_3 \le 2}}^{1} a_{i_1,i_2,i_3}(x) D_1^{i_1} D_2^{i_2} D_3^{i_3} u(x) \\ = \varphi(x, \nu(x)), \end{aligned} \tag{1}$$

and the following nonclassical Goursat conditions (see [9])

$$\begin{cases} V_{0,0,0}u \equiv u\big(x^0\big) = \varphi_{0,0,0}, \\ \big(V_{1,0,0}u\big)\big(x_1\big) \equiv D_1 u\left(x_1, x_2^0, x_3^0\right) = \varphi_{1,0,0}\big(x_1\big), \\ \big(V_{0,1,0}u\big)\big(x_2\big) \equiv D_2 u\left(x_1^0, x_2, x_3^0\right) = \varphi_{0,1,0}\big(x_2\big), \\ \big(V_{0,0,1}u\big)\big(x_3\big) \equiv D_3 u\left(x_1^0, x_2^0, x_3\right) = \varphi_{0,0,1}\big(x_3\big), \\ \big(V_{1,1,0}u\big)\big(x_1, x_2\big) \equiv D_1 D_2 u\left(x_1, x_2, x_3^0\right) = \varphi_{1,1,0}\big(x_1, x_2\big), \\ \big(V_{0,1,1}u\big)\big(x_2, x_3\big) \equiv D_2 D_3 u\left(x_1^0, x_2, x_3\right) = \varphi_{0,1,1}\big(x_2, x_3\big), \\ \big(V_{1,0,1}u\big)\big(x_1, x_3\big) \equiv D_1 D_3 u\left(x_1, x_2^0, x_3\right) = \varphi_{1,0,1}\big(x_1, x_3\big), \end{cases} \tag{2}$$

where $a_{0,0,0}(x) \in L_{p(x)}(G)$, $a_{1,0,0}(x) \in L_{(\infty,r_2(x_2),r_3(x_3))}(G)$,
$a_{0,1,0}(x) \in L_{(r_1(x_1),\infty,r_3(x_3))}(G)$, $a_{0,0,1}(x) \in L_{(r_1(x_1),r_2(x_2),\infty)}(G)$,
$a_{1,1,0}(x) \in L_{(\infty,\infty,r_3(x_3))}(G)$, $a_{0,1,1}(x) \in L_{(r_1(x_1),\infty,\infty)}(G)$,
$a_{1,0,1}(x) \in L_{(\infty,r_2(x_2),\infty)}(G)$, $\varphi_{0,0,0} \in R$, $\varphi_{1,0,0}(x_1) \in L_{r_1(x_1)}(G_1)$,
$\varphi_{0,1,0}\big(x_2\big) \in L_{r_2\left(x_2\right)}(G_2)$, $\varphi_{0,0,1}(x_3) \in L_{r_3(x_3)}(G_3)$,
$\varphi_{1,1,0}(x_1, x_2) \in L_{(r_1(x_1),r_2(x_2))}(G_1 \times G_2)$,
$\varphi_{0,1,1}\,(x_2, x_3) \in L_{(r_2(x_2),r_3(x_3))}(G_2 \times G_3)$,
$\varphi_{1,0,1}(x_1, x_3) \in L_{(r_1(x_1),r_3(x_3))}(G_1 \times G_3)$ and $D_k = \dfrac{\partial}{\partial x_k}$ $(k = 1, 2, 3)$ is the generalized differential operator in the weak sense. Let $\nu(x) = (\nu_1(x), \ldots, \nu_m(x))$ be a m-dimensional control vector function and $\varphi\,(x, \nu(x))$ be a given function defined on $G \times \mathbb{R}^m$ and satisfying Caratheodory condition on $G \times \mathbb{R}^m$:

(1) $\varphi(x, \nu)$ is measurable by x in G for all $\nu \in \mathbb{R}^m$;
(2) $\varphi(x, \nu)$ is continuous by ν in $\mathbb{R}^m$ for almost all $x \in G$;
(3) for any $\delta > 0$ there exists $\varphi_\delta^0(x) \in L_{p(x)}(G)$ such that $|\varphi(x, \nu)| \le \varphi_\delta^0(x)$ for almost all $x \in G$ and $\|\nu\| = \sum_{i=1}^{m} |\nu_i| \le \delta$.

Since the coefficients of the Bianchi equation (1) are nonsmooth, we mean the solution of the problem (1)–(2) in the weak sense. Let the vector function $\nu(x)$ be measurable and bounded on G and for almost every $x \in G$ it takes its value from the given set $\Omega \subset \mathbb{R}^m$. Then, the vector function is called an admissible control. The set of all admissible controls is denoted by Ω_∂.

Now consider the following 3D optimal control problem: Find an admissible control $\nu(x)$ from Ω_∂, for which the solution of the problem (1)–(2) $u \in SW_{p(x)}^{(1,1,1)}(G)$ that minimize of the multi-point functional

$$F(\nu)=\sum_{k=1}^{N}\Big[\alpha_k^{(1,0,0)}u\big(x_1^{(k)},h_2,h_3\big)+\alpha_k^{(0,1,0)}u\big(h_1,x_2^{(k)},h_3\big) + \alpha_k^{(0,0,1)}u\big(h_1,h_2,x_3^{(k)}\big)+\alpha_k^{(1,1,0)}u\big(x_1^{(k)},x_2^{(k)},h_3\big)+\alpha_k^{(0,1,1)}u\big(h_1,x_2^{(k)},x_3^{(k)}\big) + \alpha_k^{(1,0,1)}u\big(x_1^{(k)},h_2,x_3^{(k)}\big)+\alpha_k^{(1,1,1)}u\big(x_1^{(k)},x_2^{(k)},x_3^{(k)}\big)\Big]\to \min, \tag{3}$$

where $\big(x_1^{(k)},x_2^{(k)},x_3^{(k)}\big)\in\overline{G}$ are the given fixed points, $\alpha_k^{(i_1,i_2,i_3)}$ are the given real numbers and N is a positive integer, $i_j=0,1,\ j=1,2,3$ and $1\le i_1+i_2+i_3\le 3$.

4 The Construction of Adjoint Equation for the Optimal Control Problem (1)–(3)

To obtain the necessary and sufficient conditions for optimality first we find the increment of the functional (3). Let $\nu(x)$ and $\nu(x)+\Delta\nu(x)$ be different admissible controls, and $u(x)$ and $u(x)+\Delta u(x)$ solutions of the problem (1)–(2) in the space $SW_{p(x)}^{(1,1,1)}(G)$. Then, the increment of the functional (3) is of the form

$$\Delta F(\nu)=\sum_{k=1}^{N}\Big[\alpha_k^{(1,0,0)}\Delta u\big(x_1^{(k)},h_2,h_3\big)+\alpha_k^{(0,1,0)}\Delta u\big(h_1,x_2^{(k)},h_3\big) + \alpha_k^{(0,0,1)}\Delta u\big(h_1,h_2,x_3^{(k)}\big)+\alpha_k^{(1,1,0)}\Delta u\big(x_1^{(k)},x_2^{(k)},h_3\big) + \alpha_k^{(0,1,1)}\Delta u\big(h_1,x_2^{(k)},x_3^{(k)}\big)+\alpha_k^{(1,0,1)}\Delta u\left(x_1^{(k)},h_2,x_3^{(k)}\right) + \alpha_k^{(1,1,1)}\Delta u\left(x_1^{(k)},x_2^{(k)},x_3^{(k)}\right)\Big]. \tag{4}$$

Obviously, in this case, the function $\Delta u \in SW_{p(x)}^{(1,1,1)}(G)$ is the solution of the equation

$$V_{1,1,1}\Delta u(x)=\Delta\varphi(x), \tag{5}$$

satisfying the trivial conditions

$$\begin{aligned} &V_{0,0,0}\Delta u=0,\\ &\left(V_{1,0,0}\Delta u\right)\left(x_1\right)=0,\\ &\left(V_{0,1,0}\Delta u\right)\left(x_2\right)=0,\\ &\left(V_{0,0,1}\Delta u\right)\left(x_3\right)=0,\\ &\left(V_{1,1,0}\Delta u\right)\left(x_1,x_2\right)=0,\\ &\left(V_{0,1,1}\Delta u\right)\left(x_2,x_3\right)=0,\\ &\left(V_{1,0,1}\Delta u\right)\left(x_1,x_3\right)=0, \end{aligned} \tag{6}$$

where $\Delta\varphi(x)=\varphi\left(x,\nu(x)+\Delta\nu(x)\right)-\varphi\left(x,\nu(x)\right)$. Let us denote

$$V=\left(V_{1,1,1},V_{0,0,0},V_{1,0,0},V_{0,1,0},V_{0,0,1},V_{1,1,0},V_{0,1,1},V_{1,0,1}\right)$$

and

$$\begin{aligned} E_{p(x)}(G)\equiv L_{p(x)}(G)&\times\mathbb{R}\times L_{r_1(x_1)}\left(G_1\right)\times L_{r_2(x_2)}\left(G_2\right)\\ &\times L_{r_3(x_3)}\left(G_3\right)\times L_{(r_1(x_1),r_2(x_2))}\left(G_1\times G_2\right)\\ &\times L_{(r_2(x_2),r_3(x_3))}\left(G_2\times G_3\right)\times L_{(r_1(x_1),r_3(x_3))}\left(G_1\times G_3\right). \end{aligned}$$

Let $B(G)$ denote the set of variable exponents $p(x)$ such that V is bounded from $SW^{(1,1,1)}_{p(x)}(G)$ to $E_{p(x)}(G)$. Then, the operator $V:SW^{(1,1,1)}_{p(x)}(G)\mapsto E_{p(x)}(G)$ generated by the problem (1)–(2) is bounded by the above-mentioned assumptions.

The integral representation of the functions in the space $SW^{(1,1,1)}_{p(x)}(G)$

$$\begin{aligned} u(x)=u\left(x_1^0,x_2^0,x_3^0\right)&+\int_{x_1^0}^{x_1}u_{\alpha_1}\left(\alpha_1,x_2^0,x_3^0\right)\mathrm{d}\alpha_1+\int_{x_2^0}^{x_2}u_{\alpha_2}\left(x_1^0,\alpha_2,x_3^0\right)\mathrm{d}\alpha_2\\ &+\int_{x_3^0}^{x_3}u_{\alpha_3}\left(x_1^0,x_2^0,\alpha_3\right)\mathrm{d}\alpha_3+\int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2}u_{\alpha_1\alpha_2}\left(\alpha_1,\alpha_2,x_3^0\right)\mathrm{d}\alpha_1\mathrm{d}\alpha_2\\ &+\int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}u_{\alpha_2\alpha_3}\left(x_1^0,\alpha_2,\alpha_3\right)\mathrm{d}\alpha_2\mathrm{d}\alpha_3+\int_{x_1^0}^{x_1}\int_{x_3^0}^{x_3}u_{\alpha_1\alpha_3}\left(\alpha_1,x_2^0,\alpha_3\right)\mathrm{d}\alpha_1\mathrm{d}\alpha_3\\ &+\int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}u_{\alpha_1\alpha_2\alpha_3}\left(\alpha_1,\alpha_2,\alpha_3\right)\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3 \end{aligned} \tag{7}$$

holds. It is obvious that the weak derivatives have the form

$$\begin{aligned} D_1u(x)&=u_{x_1}(x_1,x_2^0,x_3^0)+\int_{x_2^0}^{x_2}u_{x_1\alpha_2}(x_1,\alpha_2,x_3^0)\mathrm{d}\alpha_2\\ &\quad+\int_{x_3^0}^{x_3}u_{x_1\alpha_3}(x_1,x_2^0,\alpha_3)\mathrm{d}\alpha_3+\int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}u_{x_1\alpha_2\alpha_3}(x_1,\alpha_2,\alpha_3)\mathrm{d}\alpha_2\mathrm{d}\alpha_3,\\ D_2u(x)&=u_{x_2}(x_1^0,x_2,x_3^0)+\int_{x_1^0}^{x_1}u_{\alpha_1x_2}(\alpha_1,x_2,x_3^0)\mathrm{d}\alpha_1\\ &\quad+\int_{x_3^0}^{x_3}u_{x_2\alpha_3}(x_1^0,x_2,\alpha_3)\mathrm{d}\alpha_3+\int_{x_1^0}^{x_1}\int_{x_3^0}^{x_3}u_{\alpha_1x_2\alpha_3}(\alpha_1,x_2,\alpha_3)\mathrm{d}\alpha_1\mathrm{d}\alpha_3, \end{aligned}$$

$$D_3u(x) = u_{x_3}(x_1^0, x_2^0, x_3) + \int_{x_2^0}^{x_2} u_{\alpha_2 x_3}(x_1^0, \alpha_2, x_3)\mathrm{d}\alpha_2 + \int_{x_1^0}^{x_1} u_{\alpha_1 x_3}(\alpha_1, x_2^0, x_3)\mathrm{d}\alpha_1 + \int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2} u_{\alpha_1\alpha_2 x_3}(\alpha_1, \alpha_2, x_3)\mathrm{d}\alpha_1\mathrm{d}\alpha_2,$$

$$D_1D_2u(x) = u_{x_1x_2}(x_1, x_2, x_3^0) + \int_{x_3^0}^{x_3} u_{x_1x_2\alpha_3}(x_1, x_2, \alpha_3)\mathrm{d}\alpha_3,$$

$$D_2D_3u(x) = u_{x_2x_3}(x_1^0, x_2, x_3) + \int_{x_1^0}^{x_1} u_{\alpha_1x_2x_3}(\alpha_1, x_2, x_3)\mathrm{d}\alpha_1$$

and

$$D_1D_3u(x) = u_{x_1x_3}(x_1, x_2^0, x_3) + \int_{x_2^0}^{x_2} u_{x_1\alpha_2x_3}(x_1, \alpha_2, x_3)\mathrm{d}\alpha_2.$$

Remark 4.1 Note that in the case $p(x) = p = const$ the integral representation (7) was obtained in [18]. The proof of integral representation (7) in the variable exponent case is similar to the constant exponent case.

Next, we show that the operator V has an adjoint operator $V^\star = \big(\omega_{1,1,1}, \omega_{0,0,0}, \omega_{1,0,0}, \omega_{0,1,0}, \omega_{0,0,1}, \omega_{1,1,0}, \omega_{0,1,1}, \omega_{1,0,1}\big)$, which boundedly acts in the spaces

$$E_{q(x)}(G) \equiv L_{q(x)}(G) \times \mathbb{R} \times L_{s_1(x_1)}(G_1) \times L_{s_2(x_2)}(G_2) \times L_{s_3(x_3)}(G_3) \times L_{(s_1(x_1),s_2(x_2))}(G_1 \times G_2) \times L_{(s_2(x_2),s_3(x_3))}(G_2 \times G_3) \times L_{(s_1(x_1),s_3(x_3))}(G_1 \times G_3)$$

and satisfies conditions (5) and (6). Let

$$f = \big(f_{1,1,1}(x), f_{0,0,0}, f_{1,0,0}(x_1), f_{0,1,0}(x_2), f_{0,0,1}(x_3), f_{1,1,0}(x_1, x_2), f_{0,1,1}(x_2, x_3), f_{1,0,1}(x_1, x_3)\big) \in E_{q(x)}(G)$$

be an arbitrary linear bounded functional on $E_{p(x)}(G)$, $u \in SW_{p(x)}^{(1,1,1)}(G)$ and $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. Then, by the general form of linear functional in $E_{p(x)}(G)$, we have

$$f(Vu) = \int_G f_{1,1,1}(x)(V_{1,1,1}u)(x)dx + f_{0,0,0}V_{0,0,0}u + \int_{G_1} f_{1,0,0}(x_1)(V_{1,0,0}u)(x_1)\mathrm{d}x_1 + \int_{G_2} f_{0,1,0}(x_2)(V_{0,1,0}u)(x_2)\mathrm{d}x_2 + \int_{G_3} f_{0,0,1}(x_3)(V_{0,0,1}u)(x_3)\mathrm{d}x_3 + \int_{G_1\times G_2} f_{1,1,0}(x_1, x_2)(V_{1,1,0}u)(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2$$

$$+\int_{G_2\times G_3} f_{0,1,1}(x_2,x_3)(V_{0,1,1}u)(x_2,x_3)\mathrm{d}x_2\mathrm{d}x_3$$
$$+\int_{G_1\times G_3} f_{1,0,1}(x_1,x_3)(V_{1,0,1}u)(x_1,x_3)\mathrm{d}x_1\mathrm{d}x_3.$$

By (2), we get

$$\begin{aligned}
f(Vu) = &\int_{G_3} f_{0,0,1}(x_3)D_3u(x_1^0,x_2^0,x_3)\mathrm{d}x_3 + f_{0,0,0}u(x^0)\\
&+\int_G f_{1,1,1}(x)\Bigg[D_1D_2D_3u(x) + \sum_{\substack{i_1,i_2,i_3=0\\0\le i_1+i_2+i_3\le 2}}^{1} a_{i_1,i_2,i_3}(x)D_1^{i_1}D_2^{i_2}D_3^{i_3}u(x)\Bigg]\mathrm{d}x\\
&+\int_{G_1} f_{1,0,0}(x_1)D_1u(x_1,x_2^0,x_3^0)\mathrm{d}x_1 + \int_{G_2} f_{0,1,0}(x_2)D_2u(x_1^0,x_2,x_3^0)\mathrm{d}x_2\\
&+\int_{G_1}\int_{G_2} f_{1,1,0}(x_1,x_2)D_1D_2u(x_1,x_2,x_3^0)\mathrm{d}x_1\mathrm{d}x_2\\
&+\int_{G_2}\int_{G_3} f_{0,1,1}(x_2,x_3)D_2D_3u(x_1^0,x_2,x_3)\mathrm{d}x_2\mathrm{d}x_3\\
&+\int_{G_1}\int_{G_3} f_{1,0,1}(x_1,x_3)D_1D_3u(x_1,x_2^0,x_3)\mathrm{d}x_1\mathrm{d}x_3.
\end{aligned}\tag{8}$$

By substituting expressions for the weak derivatives and (7) in (8), we get

$$\begin{aligned}
f(Vu) = \int_G f_{1,1,1}(x)\Bigg\{&D_1D_2D_3u(x) + a_{1,1,0}(x)\Bigg[u_{x_1x_2}u(x_1,x_2,x_3^0)\\
&+\int_{x_3^0}^{x_3} u_{x_1x_2\alpha_3}(x_1,x_2,\alpha_3)\mathrm{d}\alpha_3\Bigg]\\
&+a_{0,1,1}(x)\Bigg[u_{x_2x_3}u(x_1^0,x_2,x_3) + \int_{x_1^0}^{x_1} u_{\alpha_1x_2x_3}(\alpha_1,x_2,x_3)\mathrm{d}\alpha_1\Bigg]\\
&+a_{1,0,1}(x)\Bigg[u_{x_1x_3}u(x_1,x_2^0,x_3) + \int_{x_2^0}^{x_2} u_{x_1\alpha_2x_3}(x_1,\alpha_2,x_3)\mathrm{d}\alpha_2\Bigg]\\
&+a_{1,0,0}(x)\Bigg[u_{x_1}(x_1,x_2^0,x_3^0) + \int_{x_2^0}^{x_2} u_{x_1\alpha_2}(x_1,\alpha_2,x_3^0)\mathrm{d}\alpha_2\\
&+\int_{x_3^0}^{x_3} u_{x_1\alpha_3}(x_1,x_2^0,\alpha_3)\mathrm{d}\alpha_3 + \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3} u_{x_1\alpha_2\alpha_3}(x_1,\alpha_2,\alpha_3)\mathrm{d}\alpha_2\mathrm{d}\alpha_3\Bigg]\\
&+a_{0,1,0}(x)\Bigg[u_{x_2}u(x_1^0,x_2,x_3^0) + \int_{x_1^0}^{x_1} u_{\alpha_1x_2}(\alpha_1,x_2,x_3^0)\mathrm{d}\alpha_1\\
&+\int_{x_3^0}^{x_3} u_{x_2\alpha_3}(x_1^0,x_2,\alpha_3)\mathrm{d}\alpha_3 + \int_{x_1^0}^{x_1}\int_{x_3^0}^{x_3} u_{\alpha_1x_2\alpha_3}(\alpha_1,x_2,\alpha_3)\mathrm{d}\alpha_1\mathrm{d}\alpha_3\Bigg]
\end{aligned}$$

$$+ a_{0,0,1}(x)\Big[u_{x_3}u(x_1^0, x_2^0, x_3) + \int_{x_2^0}^{x_2} u_{\alpha_2 x_3}(x_1^0, \alpha_2, x_3)\mathrm{d}\alpha_2$$

$$+ \int_{x_1^0}^{x_1} u_{\alpha_1 x_3}(\alpha_1, x_2^0, x_3)\mathrm{d}\alpha_1 + \int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2} u_{\alpha_1\alpha_2 x_3}(\alpha_1, \alpha_2, x_3)\mathrm{d}\alpha_1\mathrm{d}\alpha_2\Big]$$

$$+ a_{0,0,0}(x)\Big[u(x_1^0, x_2^0, x_3^0) + \int_{x_1^0}^{x_1} u_{\alpha_1}(\alpha_1, x_2^0, x_3^0)\mathrm{d}\alpha_1$$

$$+ \int_{x_2^0}^{x_2} u_{\alpha_2}(x_1^0, \alpha_2, x_3^0)\mathrm{d}\alpha_2 + \int_{x_3^0}^{x_3} u_{\alpha_3}(x_1^0, x_2^0, \alpha_3)\mathrm{d}\alpha_3$$

$$+ \int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2} u_{\alpha_1\alpha_2}(\alpha_1, \alpha_2, x_3^0)\mathrm{d}\alpha_1\mathrm{d}\alpha_2 + \int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3} u_{\alpha_2\alpha_3}(x_1^0, \alpha_2, \alpha_3)\mathrm{d}\alpha_2\mathrm{d}\alpha_3$$

$$+ \int_{x_1^0}^{x_1}\int_{x_3^0}^{x_3} u_{\alpha_1\alpha_3}(\alpha_1, x_2^0, \alpha_3)\mathrm{d}\alpha_1\mathrm{d}\alpha_3 + f_{0,0,0}u(x_1^0, x_2^0, x_3^0)$$

$$+\int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3} u_{\alpha_1\alpha_2\alpha_3}(\alpha_1, \alpha_2, \alpha_3)\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3\Big]\Big\}\mathrm{d}x$$

$$+ \int_{G_1} f_{1,0,0}(x_1)D_1u(x_1, x_2^0, x_3^0)\mathrm{d}x_1 + \int_{G_2} f_{0,1,0}(x_2)D_2u(x_1^0, x_2, x_3^0)\mathrm{d}x_2$$

$$+ \int_{G_3} f_{0,0,1}(x_3)D_3u(x_1^0, x_2^0, x_3)\mathrm{d}x_3$$

$$+ \int_{G_1}\int_{G_2} f_{1,1,0}(x_1, x_2)D_1D_2u(x_1, x_2, x_3^0)\mathrm{d}x_1\mathrm{d}x_2$$

$$+ \int_{G_2}\int_{G_3} f_{0,1,1}(x_2, x_3)D_2D_3u(x_1^0, x_2, x_3)\mathrm{d}x_2\mathrm{d}x_3$$

$$+ \int_{G_1}\int_{G_3} f_{1,0,1}(x_1, x_3)D_1D_3u(x_1, x_2^0, x_3)\mathrm{d}x_1\mathrm{d}x_3.$$

We denote

$$\omega_{0,0,0}f \equiv \int_G f_{1,1,1}(x)a_{0,0,0}(x)\mathrm{d}x + f_{0,0,0},$$

$$(\omega_{1,0,0}f)(x_1) \equiv \int_{x_1}^{h_1}\int_{x_2^0}^{h_2}\int_{x_3^0}^{h_3} f_{1,1,1}(\alpha_1, x_2, x_3)a_{0,0,0}(\alpha_1, x_2, x_3)\mathrm{d}\alpha_1\mathrm{d}x_2\mathrm{d}x_3$$

$$+ \int_{x_2^0}^{h_2}\int_{x_3^0}^{h_3} f_{1,1,1}(x)a_{1,0,0}(x)\mathrm{d}x_2\mathrm{d}x_3 + f_{1,0,0}(x_1),$$

$$(\omega_{0,1,0}f)(x_2) \equiv \int_{x_1^0}^{h_1}\int_{x_2}^{h_2}\int_{x_3^0}^{h_3} f_{1,1,1}(x_1, \alpha_2, x_3)a_{0,0,0}(x_1, \alpha_2, x_3)\mathrm{d}x_1\mathrm{d}\alpha_2\mathrm{d}x_3$$

$$+ \int_{x_1^0}^{h_1}\int_{x_3^0}^{h_3} f_{1,1,1}(x)a_{0,1,0}(x)\mathrm{d}x_1\mathrm{d}x_3 + f_{0,1,0}(x_2),$$

$$
\begin{aligned}
(\omega_{0,0,1}f)(x_3) \equiv{}& \int_{x_1^0}^{h_1}\int_{x_2^0}^{h_2}\int_{x_3}^{h_3} f_{1,1,1}(x_1,x_2,\alpha_3)a_{0,0,0}(x_1,x_2,\alpha_3)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}\alpha_3\\
&+\int_{x_1^0}^{h_1}\int_{x_2^0}^{h_2} f_{1,1,1}(x)a_{0,0,1}(x)\mathrm{d}x_1\mathrm{d}x_2+f_{0,0,1}(x_3),\\
(\omega_{1,1,0}f)(x_1,x_2) \equiv{}& \int_{x_1}^{h_1}\int_{x_2}^{h_2}\int_{x_3^0}^{h_3} f_{1,1,1}(\alpha_1,\alpha_2,x_3)a_{0,0,0}(\alpha_1,\alpha_2,x_3)\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}x_3\\
&+\int_{x_2}^{h_2}\int_{x_3^0}^{h_3} f_{1,1,1}(x_1,\alpha_2,x_3)a_{1,0,0}(x_1,\alpha_2,x_3)\mathrm{d}\alpha_2\mathrm{d}x_3+f_{1,1,0}(x_1,x_2)\\
&+\int_{x_1}^{h_1}\int_{x_3^0}^{h_3} f_{1,1,1}(\alpha_1,x_2,x_3)a_{0,1,0}(\alpha_1,x_2,x_3)\mathrm{d}\alpha_1\mathrm{d}x_3\\
&+\int_{x_3^0}^{h_3} f_{1,1,1}(x)a_{1,1,0}(x)\mathrm{d}x_3,\\
(\omega_{0,1,1}f)(x_2,x_3) \equiv{}& \int_{x_1^0}^{h_1}\int_{x_2}^{h_2}\int_{x_3}^{h_3} f_{1,1,1}(x_1,\alpha_2,\alpha_3)a_{0,0,0}(x_1,\alpha_2,\alpha_3)\mathrm{d}x_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3\\
&+\int_{x_1^0}^{h_1}\int_{x_3}^{h_3} f_{1,1,1}(x_1,x_2,\alpha_3)a_{0,1,0}(x_1,x_2,\alpha_3)\mathrm{d}x_1\mathrm{d}\alpha_3+f_{0,1,1}(x_2,x_3)\\
&+\int_{x_1^0}^{h_1}\int_{x_2}^{h_2} f_{1,1,1}(x_1,\alpha_2,x_3)a_{0,0,1}(x_1,\alpha_2,x_3)\mathrm{d}x_1\mathrm{d}\alpha_2\\
&+\int_{x_1^0}^{h_1} f_{1,1,1}(x)a_{0,1,1}(x)\mathrm{d}x_1,\\
(\omega_{1,0,1}f)(x_1,x_3) \equiv{}& \int_{x_1}^{h_1}\int_{x_2^0}^{h_2}\int_{x_3}^{h_3} f_{1,1,1}(\alpha_1,x_2,\alpha_3)a_{0,0,0}(\alpha_1,x_2,\alpha_3)\mathrm{d}\alpha_1\mathrm{d}x_2\mathrm{d}\alpha_3\\
&+\int_{x_2^0}^{h_2}\int_{x_3}^{h_3} f_{1,1,1}(x_1,x_2,\alpha_3)a_{1,0,0}(x_1,x_2,\alpha_3)\mathrm{d}x_2\mathrm{d}\alpha_3+f_{1,0,1}(x_1,x_3)\\
&+\int_{x_1}^{h_1}\int_{x_2^0}^{h_2} f_{1,1,1}(\alpha_1,x_2,x_3)a_{0,0,1}(\alpha_1,x_2,x_3)\mathrm{d}\alpha_1\mathrm{d}x_2\\
&+\int_{x_2^0}^{h_2} f_{1,1,1}(x)a_{1,0,1}(x)\mathrm{d}x_2
\end{aligned}
$$

and

$$
\begin{aligned}
(\omega_{1,1,1}f)(x) \equiv{}& \int_{x_1}^{h_1}\int_{x_2}^{h_2}\int_{x_3}^{h_3} f_{1,1,1}(\alpha_1,\alpha_2,\alpha_3)a_{0,0,0}(\alpha_1,\alpha_2,\alpha_3)\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3\\
&+f_{1,1,1}(x)+\int_{x_2}^{h_2}\int_{x_3}^{h_3} f_{1,1,1}(x_1,\alpha_2,\alpha_3)a_{1,0,0}(x_1,\alpha_2,\alpha_3)\mathrm{d}\alpha_2\mathrm{d}\alpha_3\\
&+\int_{x_1}^{h_1}\int_{x_3}^{h_3} f_{1,1,1}(\alpha_1,x_2,\alpha_3)a_{0,1,0}(\alpha_1,x_2,\alpha_3)\mathrm{d}\alpha_1\mathrm{d}\alpha_3\\
&+\int_{x_1}^{h_1}\int_{x_2}^{h_2} f_{1,1,1}(\alpha_1,\alpha_2,x_3)a_{0,0,1}(\alpha_1,\alpha_2,x_3)\mathrm{d}\alpha_1\mathrm{d}\alpha_2
\end{aligned}
$$

$$+\int_{x_3}^{h_3} f_{1,1,1}(x_1, x_2, \alpha_3) a_{1,1,0}(x_1, x_2, \alpha_3) \mathrm{d}\alpha_3$$
$$+\int_{x_1}^{h_1} f_{1,1,1}(\alpha_1, x_2, x_3) a_{0,1,1}(\alpha_1, x_2, x_3) \mathrm{d}\alpha_1$$
$$+\int_{x_2}^{h_2} f_{1,1,1}(x_1, \alpha_2, x_3) a_{1,0,1}(x_1, \alpha_2, x_3) \mathrm{d}\alpha_2.$$

Finally, we have

$$\begin{aligned} f(Vu) &= u(x_1^0, x_2^0, x_3^0)\omega_{0,0,0} f + \int_{G_1} (\omega_{1,0,0} f)(x_1) D_1 u(x_1, x_2^0, x_3^0) \mathrm{d}G_1 \\ &+ \int_{G_2} (\omega_{0,1,0} f)(x_2) D_2 u(x_1^0, x_2, x_3^0) \mathrm{d}x_2 \\ &+ \int_{G_3} (\omega_{0,0,1} f)(x_3) D_3 u(x_1^0, x_2^0, x_3) \mathrm{d}x_3 \\ &+ \int_{G_1}\int_{G_2} (\omega_{1,1,0} f)(x_1, x_2) D_1 D_2 u(x_1, x_2, x_3^0) \mathrm{d}x_1 \mathrm{d}x_2 \\ &+ \int_{G_2}\int_{G_3} (\omega_{0,1,1} f)(x_2, x_3) D_2 D_3 u(x_1^0, x_2, x_3) \mathrm{d}x_2 \mathrm{d}x_3 \\ &+ \int_{G_1}\int_{G_3} (\omega_{1,0,1} f)(x_1, x_3) D_1 D_3 u(x_1, x_2^0, x_3) \mathrm{d}x_1 \mathrm{d}x_3 \\ &+ \int_{G} (\omega_{1,1,1} f)(x) D_1 D_2 D_3 u(x) \mathrm{d}x \equiv (V^* f)(u). \end{aligned} \tag{9}$$

Thus, $(V^* f)(u)$ is a finite sum of the Hardy-type operators. It is well known that, if variable exponent $p(x)$ satisfies Dini-Lipschitz condition, then Hardy-type operators are bounded on variable Lebesgue spaces $L_{q(x)}(G)$ (see [16,17]).

Thus, we proved the following lemma.

Lemma 4.1 *Let* $p \in B(G) \bigcap \mathcal{P}(G)$ *and* $1 < \underline{p} \le \overline{p} < \infty$. *Then, the operator*

$$V : SW_{p(x)}^{(1,1,1)}(G) \mapsto E_{p(x)}(G)$$

has an adjoint operator $V^\star$, *which acts boundedly in the spaces* $E_{q(x)}(G)$.

Also, we need the following lemma.

Lemma 4.2 *Let* $f \in E_{q(x)}(G)$. *Then, the increment of the functional* (3) *has the integral form*

$$\Delta F(\nu) = -\int_G f_{1,1,1}(x) \Delta\varphi(x)\, \mathrm{d}x.$$

Proof Now in (9) instead of $u(x)$, we substitute the solution of the problem (5)–(6), i.e., replace a function $u(x)$ by $\Delta u(x)$. Then, the equality

$$f(V\Delta u) = \int_G f_{1,1,1}(x) \Delta\varphi(x) \mathrm{d}x$$

$$= \int_G \left(\omega_{1,1,1} f\right)(x) D_1 D_2 D_3 \Delta u(x) dx \equiv \left(V^\star f\right)(\Delta u)$$

holds for all $f \in E_{q(x)}(G)$. In other words,

$$-\int_G f_{1,1,1}(x) \Delta\varphi(x) dx + \int_G \left(\omega_{1,1,1} f\right)(x) D_1 D_2 D_3 \Delta u(x) \mathrm{d}x = 0. \tag{10}$$

Therefore, the function $\Delta u(x)$ as an element of $SW^{(1,1,1)}_{p(x)}(G)$ satisfies condition (6). Using the integral representation (7), we have

$$\begin{aligned}
&\alpha_k^{(1,0,0)} \Delta u\big(x_1^{(k)}, h_2, h_3\big) + \alpha_k^{(0,1,0)} \Delta u\big(h_1, x_2^{(k)}, h_3\big) + \alpha_k^{(0,0,1)} \Delta u\big(h_1, h_2, x_3^{(k)}\big) \\
&\quad + \alpha_k^{(1,1,0)} \Delta u\big(x_1^{(k)}, x_2^{(k)}, h_3\big) + \alpha_k^{(0,1,1)} \Delta u\big(h_1, x_2^{(k)}, x_3^{(k)}\big) \\
&\quad + \alpha_k^{(1,0,1)} \Delta u\big(x_1^{(k)}, h_2, x_3^{(k)}\big) + \quad \alpha_k^{(1,1,1)} \Delta u\big(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\big) \\
&= \int_G B_k(x) D_1 D_2 D_3 \Delta u(x) \mathrm{d}x,
\end{aligned}$$

where

$$\begin{aligned}
B_k(x) = {}& \alpha_k^{(1,0,0)} \theta\big(x_1^{(k)} - x_1\big) + \alpha_k^{(1,0,1)} \theta\big(x_1^{(k)} - x_1\big) \theta\big(x_3^{(k)} - x_3\big) \\
&+ \alpha_k^{(0,1,0)} \theta\big(x_2^{(k)} - x_2\big) + \alpha_k^{(1,1,0)} \theta\big(x_1^{(k)} - x_1\big) \theta\big(x_2^{(k)} \\
&+ \alpha_k^{(0,0,1)} \theta\big(x_3^{(k)} - x_3\big) - x_2\big) + \alpha_k^{(0,1,1)} \theta\big(x_2^{(k)} - x_2\big) \theta\big(x_3^{(k)} - x_3\big) \\
&+ \alpha_k^{(1,1,1)} \theta\big(x_1^{(k)} - x_1\big) \theta\big(x_2^{(k)} - x_2\big) \theta\big(x_3^{(k)} - x_3\big)
\end{aligned}$$

and $\theta(t) := \left\{\begin{matrix} 1, & t > 0 \\ 0, & t \le 0 \end{matrix}\right\}$ is the Heaviside function. Therefore, the increment (4) of the functional (3) can be represented as

$$\Delta F(v) = \int_G \sum_{k=1}^{N} B_k(x) D_1 D_2 D_3 \Delta u(x) \mathrm{d}x,$$

or

$$\Delta F(\nu) = \int_G B(x) D_1 D_2 D_3 \Delta u(x) \mathrm{d}x, \tag{11}$$

and

$$B(x) = \sum_{k=1}^{N} B_k(x).$$

By (10), the increment (11) can be represented in the form

$$\begin{aligned} \Delta F(\nu) = &\int_G \left[B(x) + \left(\omega_{1,1,1} f\right)(x)\right] D_1 D_2 D_3 \Delta u(x) \mathrm{d}x \\ &- \int_G f_{1,1,1}(x) \Delta\varphi(x)\, \mathrm{d}x. \end{aligned} \tag{12}$$

Since $\omega_{1,1,1}$ depends only on f, equality (12) holds for all $f_{1,1,1} \in L_{q(x)}(G)$. For the integro-differential expression (12), we consider the equation

$$\left(\omega_{1,1,1} f\right)(x) + B(x) = 0, \quad x \in G \tag{13}$$

that is said to be an adjoint equation for the optimal control problem (1)–(3). As the function $f_{1,1,1}(x)$ we take the solution of equation (13) in $L_{q(x)}(G)$. Then, equality (12) has the integral form

$$\Delta F(\nu) = -\int_G f_{1,1,1}(x) \Delta\varphi(x)\, \mathrm{d}x.$$

This completes the proof. □

5 Main Result

Now, for a fixed $(\tau_1, \tau_2, \tau_3) \in G$, we consider the following needle variation of the admissible control $\nu(x)$:

$$\Delta\nu_\varepsilon(x) = \begin{cases} \widehat{\nu} - \nu(x), & x \in G_\varepsilon \\ 0, & x \in G \setminus G_\varepsilon, \end{cases}$$

where $\widehat{\nu} \in \Omega_\partial$, $\varepsilon > 0$ is a sufficiently small parameter, and $G_\varepsilon = \left(\tau_1 - \frac{\varepsilon}{2}, \tau_1 + \frac{\varepsilon}{2}\right) \times \left(\tau_2 - \frac{\varepsilon}{2}, \tau_2 + \frac{\varepsilon}{2}\right) \times \left(\tau_3 - \frac{\varepsilon}{2}, \tau_3 + \frac{\varepsilon}{2}\right) \subset G$. A control $\nu_\varepsilon(x)$ defined by the equality $\nu_\varepsilon(x) = \nu(x) + \Delta\nu_\varepsilon(x)$ is an admissible control for all sufficiently small $\varepsilon > 0$ and all $\widehat{\nu} \in \Omega_\partial$ called a needle perturbation given by control $\nu(x)$, where $(\tau_1, \tau_2, \tau_3) \in G$ is some fixed point. Obviously,

$$\begin{aligned} F(\nu_\varepsilon) - F(\nu) &= -\int_{G_\varepsilon} f_{1,1,1}(x) \left[\varphi\left(x, \nu(x) + \Delta\nu_\varepsilon(x)\right) - \varphi\left(x, \nu(x)\right)\right] dx \\ &= -\int_{G_\varepsilon} f_{1,1,1}(x) \left[\varphi\left(x, \widehat{\nu}(x)\right) - \varphi\left(x, \nu(x)\right)\right] \mathrm{d}x. \end{aligned} \tag{14}$$

Since the optimal control problem is linear, the following theorem follows from (14).

Theorem 5.1 *Let $f_{1,1,1} \in L_{q(x)}(G)$ be a solution of the adjoint equation (13). Then, for the optimality of the admissible control $\nu(x)$, it is necessary and sufficient that for almost all $x \in G$ the Pontryagin's maximum condition*

$$\max_{\widehat{\nu} \in \Omega_\partial} H\left(x, f_{1,1,1}(x), \widehat{\nu}\right) = H\left(x, f_{1,1,1}(x), \nu\right)$$

be satisfied, where $H\left(x, f_{1,1,1}(x), \nu\right) = f_{1,1,1}(x) \cdot \varphi(x, \nu)$ is the Hamilton-Pontryagin function.

Proof Suppose that a control $\nu\left(x_1, x_2, x_3\right) \in \Omega_\partial$ gives the minimum value of the functional (3). Then by (14), we have

$$-\int_{G_\varepsilon} \left[H\left(x_1, x_2, x_3, f_{1,1,1}\left(x_1, x_2, x_3\right), \widehat{\nu}\right)\right.$$
$$\left. - H\left(x_1, x_2, x_3, f_{1,1,1}\left(x_1, x_2, x_3\right), \nu\left(x_1, x_2, x_3\right)\right)\right] \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \geq 0. \tag{15}$$

Dividing the both sides of (15) by ε^3 and passing to the limit as $\varepsilon \to +0$, for almost all $(\tau_1, \tau_2, \tau_3) \in G$ and using analog of the Lebesgue differentiation theorem in $L_{p(x)}$ (see [17]) for all $\nu \in \Omega_\partial$, we get

$$H\left(\tau_1, \tau_2, \tau_3, f_{1,1,1}\left(\tau_1, \tau_2, \tau_3\right), \nu\left(\tau_1, \tau_2, \tau_3\right)\right)$$
$$- H\left(\tau_1, \tau_2, \tau_3, f_{1,1,1}\left(\tau_1, \tau_2, \tau_3\right), \widehat{\nu}\right) \geq 0. \tag{16}$$

Thus, for optimal control of $\nu\left(x_1, x_2, x_3\right) \in \Omega_\partial$, it is necessary to satisfy condition (16). Besides, the equality

$$\Delta F(\nu) = -\int_G \Delta H\left(x_1, x_2, x_3, f_{1,1,1}\left(x_1, x_2, x_3\right), \nu\left(x_1, x_2, x_3\right)\right) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3$$

shows that this condition is also sufficient for optimal control $\nu\left(x_1, x_2, x_3\right)$, where $\Delta H\left(x_1, x_2, x_3, f_{1,1,1}, \nu\right) = H\left(x_1, x_2, x_3, f_{1,1,1}, \nu + \Delta\nu\right) - H\left(x_1, x_2, x_3, f_{1,1,1}, \nu\right)$. This completes the proof. □

Remark 5.1 Theorem 5.1 shows that the solution to the optimal control problem (1)–(3), it is sufficient to find a solution $f_{1,1,1}(x) \in L_{q(x)}(G)$ of the integral equation (13). Then, the optimal control $\nu(x)$ can be found as an element of Ω_∂, which gives the maximum value to the functional $H\left(x, f_{1,1,1}(x), \nu(x)\right)$ in Ω_∂ with respect to the function ν.

Remark 5.2 In [13, p. 198] a theorem in the form of the Pontryagin's maximum principle is proved for control systems with a nonsmooth right-hand side. In this paper, an admissible control is taken as a nonsmooth function that belongs to the class L_∞. In addition, in the present article, the control function enters the right-hand side of the equation in a nonlinear form. More exactly, to obtain necessary and sufficient optimality conditions in the form of the Pontryagin's maximum principle in the 3D nonsmooth optimal control problem, needle variation is used.

Example 5.1 Obviously, Equation (1) generalizes the three-dimensional analog of vibrating string equation and the three-dimensional telegraph equation. Indeed, if we take $a_{0,0,0}(x) = -k$, $k = const \geq 0$ and $a_{i_1,i_2,i_3}(x) \equiv 0$, $0 < i_1 + i_2 + i_3 \leq 2$ in the right-hand side of the Bianchi equation (1), we get

$$D_1 D_2 D_3 u(x) - k\, u(x) = \varphi(x, \nu(x)). \tag{17}$$

It is well known that (17) is a controlled process described by the three-dimensional telegraph equation. The 3D telegraph equation arises in 3D mathematical modeling of filtering and telecommunication. Let $k = 0$. Then, the adjoint equation (13) for 3D optimal control problem (1)–(3) takes a simpler form

$$f_{1,1,1}(x) + B(x) = 0, \quad x \in G.$$

6 Conclusions

In this paper, a new approach to the Pontryagin's maximum principle for an optimal control problem with distributed parameters, is given by the third-order Bianchi equation with coefficients from variable exponent Lebesgue spaces. The statement of an optimal control problem is studied by using a new version of the increment method, that essentially uses the concept of the adjoint equation of the integral form.

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