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Commutators of Fractional Maximal Operator on Orlicz Spaces*

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Abstract—In the present paper, we give necessary and sufficient conditions for the boundedness of commutators of fractional maximal operator on Orlicz spaces. The main advance in comparison with the existing results is that we manage to obtain conditions for the boundedness not in integral terms but in less restrictive terms of supremal operators.

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*Dedicated to the 75th birthday
of Professor Nikolai K. Karapetiants (1942-2005).*

1. INTRODUCTION

Let $0 < \alpha < n$. The fractional maximal operator M_α and the Riesz potential operator I_α are defined by

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\alpha/n} \int_{B(x,t)} |f(y)|\,dy, \qquad I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

Hereafter, $B(x,r)$ is the ball of radius r centered at x in $\mathbb{R}^n$ and $|B(x,r)| = v_n r^n$, where v_n is the volume of the unit ball in $\mathbb{R}^n$, is its Lebesgue measure. If $\alpha = 0$, then $M \equiv M_0$ is the well-known Hardy–Littlewood maximal operator. Recall that, for $0 < \alpha < n$, we have

$$M_\alpha f(x) \le v_n^{\alpha/n-1} I_\alpha(|f|)(x).$$

The commutators generated by an appropriate function b and by the operators M_α and I_α are formally defined by

$$[b, M_\alpha]f = M_\alpha(bf) - bM_\alpha(f), \qquad [b, I_\alpha]f = I_\alpha(bf) - bI_\alpha(f),$$

respectively.

Given a measurable function b, the operators $M_{b,\alpha}$ and $|b, I_\alpha|$ are defined by

$$M_{b,\alpha}(f)(x) = \sup_{t>0} |B(x,t)|^{-1+\alpha/n} \int_{B(x,t)} |b(x) - b(y)||f(y)|\,dy,$$

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$$|b, I_\alpha|f(x) = \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|}{|x-y|^{n-\alpha}} f(y)\,dy,$$

respectively. If $\alpha = 0$, then $M_{b,0} \equiv M_b$ is called the *maximal commutator*. Recall that, for $0 < \alpha < n$, we have

$$M_{b,\alpha}(f)(x) \le v_n^{\alpha/n-1} |b, I_\alpha|(|f|)(x), \qquad (1.1)$$
$$|[b, I_\alpha]f(x)| \le |b, I_\alpha|(|f|)(x).$$

For a function b defined on $\mathbb{R}^n$, we set

$$b^-(x) = \begin{cases} 0 & \text{if } b(x) \ge 0, \\ |b(x)| & \text{if } b(x) < 0, \end{cases}$$

and $b^+(x) = |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

The following relations between $[b, M_\alpha]$ and $M_{b,\alpha}$ are valid. Let b be any nonnegative locally integrable function. Then

$$|[b, M_\alpha]f(x)| \le M_{b,\alpha}(f)(x), \qquad x \in \mathbb{R}^n,$$

for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

If b is any locally integrable function on $\mathbb{R}^n$, then

$$|[b, M_\alpha]f(x)| \le M_{b,\alpha}(f)(x) + 2b^-(x) M_\alpha f(x), \qquad x \in \mathbb{R}^n, \qquad (1.2)$$

for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ (see, e.g., [1]).

Suppose that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then f is said to belong to the *class* $\mathrm{BMO}(\mathbb{R}^n)$ if the seminorm given by

$$\|f\|_* = \sup_{x \in \mathbb{R}^n,\ r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}|\,dy$$

is finite.

The following theorem is valid.

Theorem 1.1 ([2], [3]). *Suppose given $0 < \alpha < n$, $1 < p < n/\alpha$, and $1/q = 1/p - \alpha/n$. Then $M_{b,\alpha}$, $[b, I_\alpha]$ and $|b, I_\alpha|$ are bounded operators from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$.*

Remark. The proof of the theorem for $[b, I_\alpha]$ was given in [2] and for $M_{b,\alpha}$ and $|b, I_\alpha|$, in [3]. The boundedness of the operator M_b on the L^p spaces was proved by Garcia-Cuerva, Harboure, Segovia, and Torrea in [4]. In 2000, Bastero, Milman, and Ruiz [5] studied a necessary and sufficient condition for the boundedness of $[b, M]$ on the L^p spaces. In 2009, Zhang and Lu [6] considered the same problem for $[b, M_\alpha]$.

The main purpose of this paper is to characterize the boundedness of commutators of fractional maximal operator on Orlicz spaces.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of the quantities involved. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$ and say that A and B are equivalent.

2. PRELIMINARIES

The Orlicz spaces were first introduced by Orlicz in [7], [8] as a generalization of the Lebesgue spaces L^p. Since then, these spaces have been an important functional frame in mathematical analysis, especially in real and harmonic analysis. Orlicz spaces are also an appropriate substitute for L^1 when the latter does not work.

First, we recall the definition of Young functions.

Definition 2.1. A function $\Phi\colon [0,\infty) \to [0,\infty]$ is called a *Young function* if Φ is convex left-continuous and

$$\lim_{r\to+0}\Phi(r)=\Phi(0)=0, \qquad \lim_{r\to\infty}\Phi(r)=\infty.$$

Convexity and the condition $\Phi(0)=0$ imply that any Young function is increasing. If there exists an $s\in(0,\infty)$ such that $\Phi(s)=\infty$, then $\Phi(r)=\infty$ for $r\geq s$. The set of Young functions such that

$$0<\Phi(r)<\infty \quad \text{for} \quad 0<r<\infty$$

will be denoted by $\mathcal{Y}$. If $\Phi\in\mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijectively maps $[0,\infty)$ to itself.

For a Young function Φ and $0\leq s\leq\infty$, we set

$$\Phi^{-1}(s)=\inf\{r\geq 0:\Phi(r)>s\}.$$

If $\Phi\in\mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ.

It is well known that

$$r\leq\Phi^{-1}(r)\widetilde{\Phi}^{-1}(r)\leq 2r \qquad \text{for} \quad r\geq 0, \tag{2.1}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r)=\begin{cases}\sup\{rs-\Phi(s):s\in[0,\infty)\}, & r\in[0,\infty),\\ \infty, & r=\infty.\end{cases}$$

A Young function Φ is said to satisfy the Δ_2-*condition* (we write $\Phi\in\Delta_2$ in this case) if

$$\Phi(2r)\leq C\Phi(r), \qquad r>0,$$

for some $C>1$. If $\Phi\in\Delta_2$, then $\Phi\in\mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-*condition* (we write $\Phi\in\nabla_2$ in this case) if

$$\Phi(r)\leq\frac{1}{2C}\Phi(Cr), \qquad r\geq 0,$$

for some $C>1$.

Definition 2.2. For a Young function Φ, the set

$$L^{\Phi}(\mathbb{R}^n)=\left\{f\in L^1_{\mathrm{loc}}(\mathbb{R}^n):\int_{\mathbb{R}^n}\Phi(k|f(x)|)\,dx<\infty \text{ for some } k>0\right\}$$

is called an *Orlicz space*. The space $L^{\Phi}_{\mathrm{loc}}(\mathbb{R}^n)$ is defined as the set of all functions f such that $f\chi_{B}\in L^{\Phi}(\mathbb{R}^n)$ for all balls $B\subset\mathbb{R}^n$.

The space $L^{\Phi}(\mathbb{R}^n)$ is Banach with respect to the norm

$$\|f\|_{L^{\Phi}}=\inf\left\{\lambda>0:\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\lambda}\right)dx\leq 1\right\}.$$

We note that

$$\int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\|f\|_{L^{\Phi}}}\right)dx\leq 1. \tag{2.2}$$

If $\Phi(r)=r^p$, $1\leq p<\infty$, then $L^{\Phi}(\mathbb{R}^n)=L^p(\mathbb{R}^n)$. If $\Phi(r)=0$, $0\leq r\leq 1$, and $\Phi(r)=\infty$, $r>1$, then $L^{\Phi}(\mathbb{R}^n)=L^{\infty}(\mathbb{R}^n)$.

The following analog of Hölder's inequality is well known (see, e.g., [9]).

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set, and let f and g be measurable functions on Ω. For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid:*

$$\int_{\Omega} |f(x)g(x)|\,dx \le 2\|f\|_{L^{\Phi}(\Omega)}\|g\|_{L^{\widetilde{\Phi}}(\Omega)}.$$

Elementary calculations yield the following property.

Lemma 2.4. *Let Φ be a Young function, and let B be a set in $\mathbb{R}^n$ with finite Lebesgue measure. Then*

$$\|\chi_B\|_{L^{\Phi}} = \|\chi_B\|_{WL^{\Phi}} = \frac{1}{\Phi^{-1}(|B|^{-1})}.$$

Using Theorem 2.3, Lemma 2.4, and (2.1) we obtain the following estimate.

Lemma 2.5. *For a Young function Φ and $B = B(x,r)$, the following inequality is valid:*

$$\int_B |f(y)|\,dy \le 2|B|\Phi^{-1}(|B|^{-1})\|f\|_{L^{\Phi}(B)}. \tag{2.3}$$

3. COMMUTATORS OF FRACTIONAL MAXIMAL FUNCTION IN ORLICZ SPACES

In this section, we find necessary and sufficient conditions for the boundedness of the commutators of fractional maximal operators on Orlicz spaces.

To prove our theorems, we need the following lemmas and theorem.

Lemma 3.1 ([10]). *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that*

$$|b_{B(x,r)} - b_{B(x,t)}| \le C\|b\|_* \ln\frac{t}{r} \qquad \text{for} \quad 0 < 2r < t \tag{3.1}$$

and C is independent of b, x, r, and t.

Lemma 3.2 ([11]). *Let $f \in \mathrm{BMO}(\mathbb{R}^n)$, and let Φ be a Young function such that $\Phi \in \Delta_2$. Then*

$$\|f\|_* \approx \sup_{x\in\mathbb{R}^n,\, r>0} \Phi^{-1}(|B(x,r)|^{-1})\|f(\,\cdot\,) - f_{B(x,r)}\|_{L^{\Phi}(B(x,r))}. \tag{3.2}$$

Theorem 3.3 ([12], [13]). *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$, and let Φ be a Young function. Then the condition $\Phi \in \nabla_2$ is necessary and sufficient for the boundedness of the operator M_b on $L^{\Phi}(\mathbb{R}^n)$, i.e., for the inequality*

$$\|M_b f\|_{L^{\Phi}} \le C_0\|b\|_*\|f\|_{L^{\Phi}} \tag{3.3}$$

to hold with a constant C_0 independent of f.

Remark 3.4. The sufficiency part of Theorem 3.3 was proved in [12], and the necessity part was proved in [13].

The following lemma is an analog of Hedberg's trick for $[b, I_\alpha]$.

Lemma 3.5 ([13]). *If $0 < \alpha < n$ and $f, b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then*

$$\int_{B(x,r)} \frac{|f(y)|}{|x-y|^{n-\alpha}}|b(x) - b(y)|\,dy \lesssim r^{\alpha} M_b f(x)$$

for all $x \in \mathbb{R}^n$ and $r > 0$.

The following theorem completely characterizes the boundedness of M_α on Orlicz spaces.

Theorem 3.6 ([14]). *Let α, $0<\alpha<n$, be given. Let Φ and Ψ be Young functions such that $\Phi \in \mathcal{Y} \cap \nabla_2$. Then the condition*

$$r^{-\alpha/n}\Phi^{-1}(r) \le C\Psi^{-1}(r) \tag{3.4}$$

for all $r>0$, where $C>0$ does not depend on r, is necessary and sufficient for the boundedness of M_α as an operator from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

To prove our main theorems, we need the following estimate.

Lemma 3.7. *If $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $B_0 := B(x_0, r_0)$, then*

$$r_0^\alpha |b(x) - b_{B_0}| \le C M_{b,\alpha}\chi_{B_0}(x) \qquad \textit{for every} \quad x \in B_0.$$

Proof. It is well known that

$$\mathrm{M}_{b,\alpha} f(x) \le 2^{n-\alpha} M_{b,\alpha} f(x), \tag{3.5}$$

where

$$\mathrm{M}_{b,\alpha}(f)(x) = \sup_{B \ni x} |B|^{-1+\alpha/n} \int_B |b(x)-b(y)||f(y)|\,dy.$$

Now let $x \in B_0$. Using (3.5), we obtain

$$\begin{aligned} M_{b,\alpha}\chi_{B_0}(x) &\ge C\mathrm{M}_{b,\alpha} f(x) = C\sup_{B\ni x}|B|^{-1+\alpha/n}\int_B |b(x)-b(y)|\chi_{B_0}\,dy \\ &= C\sup_{B\ni x}|B|^{-1+\alpha/n}\int_{B\cap B_0}|b(x)-b(y)|\,dy \\ &\ge C|B_0|^{-1+\alpha/n}\int_{B_0\cap B_0}|b(x)-b(y)|\,dy \\ &\ge \left| C|B_0|^{-1+\alpha/n}\int_{B_0}(b(x)-b(y))\,dy\right| = Cr_0^\alpha|b(x)-b_{B_0}|. \quad \square \end{aligned}$$

The following theorem gives necessary and sufficient conditions for the boundedness of $M_{b,\alpha}$ as an operator from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Theorem 3.8. *Let α, $0<\alpha<n$, and $b \in \mathrm{BMO}(\mathbb{R}^n)$ be given. Let Φ and Ψ be Young functions such that $\Phi \in \mathcal{Y}$.*

(1) *If $\Phi \in \nabla_2$ and $\Psi \in \Delta_2$, then the condition*

$$r^\alpha \Phi^{-1}(r^{-n}) + \sup_{r<t<\infty}\left(1+\ln\frac{t}{r}\right)\Phi^{-1}(t^{-n})t^\alpha \le C\Psi^{-1}(r^{-n}) \tag{3.6}$$

for all $r>0$, where $C>0$ does not depend on r, is sufficient for the boundedness of $M_{b,\alpha}$ as an operator from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

(2) *If $\Psi \in \Delta_2$, then condition* (3.4) *is necessary for the boundedness of $M_{b,\alpha}$ as an operator from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

(3) *If $\Phi \in \nabla_2$, $\Psi \in \Delta_2$, and the condition*

$$\sup_{r<t<\infty}\left(1+\ln\frac{t}{r}\right)\Phi^{-1}(t^{-n})t^\alpha \le Cr^\alpha\Phi^{-1}(r^{-n}) \tag{3.7}$$

holds for all $r>0$, where $C>0$ does not depend on r, then condition (3.4) *is necessary and sufficient for the boundedness of $M_{b,\alpha}$ as an operator from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

Proof. (1) Given any $x_0 \in \mathbb{R}^n$, consider the ball $B = B(x_0, r)$ of radius r centered at x_0 and the function $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement(2B)}$.

Let x be any point in B. If $B(x,t) \cap \{\complement(2B)\} \neq \varnothing$, then $t > r$. Indeed, if $y \in B(x,t) \cap \{\complement(2B)\}$, then

$$t > |x-y| \geq |x_0 - y| - |x_0 - x| > 2r - r = r.$$

On the other hand, $B(x,t) \cap \{\complement(2B)\} \subset B(x_0, 2t)$. Indeed, if $y \in B(x,t) \cap \{\complement(2B)\}$, then we have

$$|x_0 - y| \leq |x-y| + |x_0 - x| < t + r < 2t.$$

Hence

$$\begin{aligned} M_{b,\alpha}(f_2)(x) &= \sup_{t>0} \frac{1}{|B(x,t)|^{1-\alpha/n}} \int_{B(x,t)\cap\complement(2B)} |b(y) - b(x)||f(y)|\,dy \\ &\leq 2^{n-\alpha} \sup_{t>r} \frac{1}{|B(x_0,2t)|^{1-\alpha/n}} \int_{B(x_0,2t)} |b(y) - b(x)||f(y)|\,dy \\ &= 2^{n-\alpha} \sup_{t>2r} \frac{1}{|B(x_0,t)|^{1-\alpha/n}} \int_{B(x_0,t)} |b(y) - b(x)||f(y)|\,dy. \end{aligned}$$

Therefore, for all $x \in B$, we have

$$\begin{aligned} M_{b,\alpha}(f_2)(x) &\lesssim \sup_{t>2r} t^{\alpha-n} \int_{B(x_0,t)} |b(y) - b(x)||f(y)|\,dy \\ &\lesssim \sup_{t>2r} t^{\alpha-n} \int_{B(x_0,t)} |b(y) - b_{B(x_0,t)}||f(y)|\,dy \\ &\qquad + \sup_{t>2r} t^{\alpha-n} \int_{B(x_0,t)} |b_{B(x_0,t)} - b_B||f(y)|\,dy \\ &\qquad + \sup_{t>2r} t^{\alpha-n} \int_{B(x_0,t)} |b_B - b(x)||f(y)|\,dy \\ &= J_1 + J_2 + J_3. \end{aligned}$$

Applying Hölder's inequality and using (2.1), (3.1), (3.2), and (2.3), we obtain

$$\begin{aligned} J_1 + J_2 &\lesssim \sup_{t>2r} t^{\alpha-n} \int_{B(x_0,t)} |b(y) - b_{B(x_0,t)}||f(y)|\,dy \\ &\qquad + \sup_{t>2r} t^{\alpha-n} |b_{B(x_0,r)} - b_{B(x_0,t)}| \int_{B(x_0,t)} |f(y)|\,dy \\ &\lesssim \sup_{t>2r} t^{\alpha-n} \|b(\cdot) - b_{B(x_0,t)}\|_{L^{\widetilde{\Phi}}(B(x_0,t))} \|f\|_{L^\Phi(B(x_0,t))} \\ &\qquad + \sup_{t>2r} t^{\alpha-n} |b_{B(x_0,r)} - b_{B(x_0,t)}| t^n \Phi^{-1}(|B(x_0,t)|^{-1}) \|f\|_{L^\Phi(B(x_0,t))} \\ &\lesssim \|b\|_* \sup_{t>2r} \Phi^{-1}(|B(x_0,t)|^{-1}) t^\alpha \left(1 + \ln\frac{t}{r}\right) \|f\|_{L^\Phi(B(x_0,t))} \\ &\lesssim \|b\|_* \|f\|_{L^\Phi} \sup_{t>2r} \left(1 + \ln\frac{t}{r}\right) t^\alpha \Phi^{-1}(t^{-n}). \end{aligned}$$

From geometric considerations, $2B \subset B(x,3r)$ for all $x \in B$. Using Lemma 3.5, we obtain

$$\begin{aligned} J_0(x) := M_{b,\alpha}(f_1)(x) \lesssim |b, I_\alpha|(|f_1|)(x) &= \int_{2B} \frac{|b(y) - b(x)|}{|x-y|^{n-\alpha}} |f(y)|\,dy \\ &\lesssim \int_{B(x,3r)} \frac{|b(y) - b(x)|}{|x-y|^{n-\alpha}} |f(y)|\,dy \lesssim r^\alpha M_b f(x). \end{aligned}$$

Consequently, for all $x \in B$, we have

$$J_0(x) + J_1 + J_2 \lesssim \|b\|_* r^\alpha M_b f(x) + \|b\|_* \|f\|_{L^\Phi} \sup_{t>2r} \Big(1 + \ln \frac{t}{r}\Big) t^\alpha \Phi^{-1}(t^{-n}).$$

Thus, (3.6) implies

$$J_0(x) + J_1 + J_2 \lesssim \|b\|_* \bigg(M_b f(x) \frac{\Psi^{-1}(r^{-n})}{\Phi^{-1}(r^{-n})} + \Psi^{-1}(r^{-n}) \|f\|_{L^\Phi} \bigg).$$

Choose $r > 0$ so that $\Phi^{-1}(r^{-n}) = M_b f(x)/(C_0 \|b\|_* \|f\|_{L^\Phi})$. Then

$$\frac{\Psi^{-1}(r^{-n})}{\Phi^{-1}(r^{-n})} = \frac{(\Psi^{-1} \circ \Phi)(M_b f(x)/(C_0 \|b\|_* \|f\|_{L^\Phi}))}{M_b f(x)/(C_0 \|b\|_* \|f\|_{L^\Phi})}.$$

Therefore, we obtain

$$J_0(x) + J_1 + J_2 \le C_1 \|b\|_* \|f\|_{L^\Phi} (\Psi^{-1} \circ \Phi) \bigg(\frac{M_b f(x)}{C_0 \|b\|_* \|f\|_{L^\Phi}} \bigg).$$

Let C_0 be as in (3.3). Theorem 3.3 and (2.2) give

$$\int_B \Psi\bigg(\frac{J_0(x) + J_1 + J_2}{C_1 \|b\|_* \|f\|_{L^\Phi}} \bigg)\, dx \le \int_B \Phi\bigg(\frac{M_b f(x)}{C_0 \|b\|_* \|f\|_{L^\Phi}} \bigg)\, dx \le \int_{\mathbb{R}^n} \Phi\bigg(\frac{M_b f(x)}{\|M_b f\|_{L^\Phi}} \bigg)\, dx \le 1,$$

i.e.,

$$\|J_0(\,\cdot\,) + J_1 + J_2\|_{L^\Psi(B)} \lesssim \|b\|_* \|f\|_{L^\Phi}. \tag{3.8}$$

Using relations (3.2) and (2.3), and condition (3.6), we also obtain

$$\begin{aligned}
\|J_3\|_{L^\Psi(B)} &= \bigg\| \sup_{t>2r} \frac{1}{|B(x_0,t)|^{1-\alpha/n}} \int_{B(x_0,t)} |b(\,\cdot\,) - b_B| |f(y)|\, dy \bigg\|_{L^\Psi(B)} \\
&\approx \|b(\,\cdot\,) - b_B\|_{L^\Psi(B)} \sup_{t>2r} t^{\alpha-n} \int_{B(x_0,t)} |f(y)|\, dy \\
&\lesssim \|b\|_* \frac{1}{\Psi^{-1}(|B|^{-1})} \sup_{t>2r} \Phi^{-1}(|B(x_0,t)|^{-1}) t^\alpha \|f\|_{L^\Phi(B(x_0,t))} \\
&\lesssim \|b\|_* \frac{1}{\Psi^{-1}(|B|^{-1})} \|f\|_{L^\Phi} \sup_{t>2r} t^\alpha \Phi^{-1}(|B(x_0,t)|^{-1}) \\
&\lesssim \|b\|_* \|f\|_{L^\Phi}.
\end{aligned}$$

Consequently, we have

$$\|J_3\|_{L^\Psi(B)} \lesssim \|b\|_* \|f\|_{L^\Phi}. \tag{3.9}$$

Combining (3.8) and (3.9), we obtain

$$\|M_{b,\alpha} f\|_{L^\Psi(B)} \lesssim \|b\|_* \|f\|_{L^\Phi}. \tag{3.10}$$

Taking the supremum over B in (3.10), we see that

$$\|M_{b,\alpha} f\|_{L^\Psi} \lesssim \|b\|_* \|f\|_{L^\Phi},$$

since the constants in (3.10) do not depend on x_0 and r.

(2) Let us prove the second part. Consider $B_0 = B(x_0, r_0)$ and take $x \in B_0$. By Lemma 3.7, we have

$$r_0^\alpha |b(x) - b_{B_0}| \le C M_{b,\alpha} \chi_{B_0}(x).$$

Therefore, by Lemmas 3.2 and 2.4,

$$r_0^\alpha \lesssim \frac{\|M_{b,\alpha} \chi_{B_0}\|_{L^\Psi(B_0)}}{\|b(\,\cdot\,) - b_{B_0}\|_{L^\Psi(B_0)}} \lesssim \Psi^{-1}(|B_0|^{-1}) \|M_{b,\alpha} \chi_{B_0}\|_{L^\Psi(B_0)}$$

$$\lesssim \Psi^{-1}(|B_0|^{-1})\|M_{b,\alpha}\chi_{B_0}\|_{L^\Psi} \lesssim \Psi^{-1}(|B_0|^{-1})\|\chi_{B_0}\|_{L^\Phi} \lesssim \frac{\Psi^{-1}(r_0^{-n})}{\Phi^{-1}(r_0^{-n})}.$$

Since this is true for every $r_0 > 0$, we are done.

(3) The third assertion of the theorem follows from the first and second ones. □

Setting $\Phi(t) = t^p$ and $\Psi(t) = t^q$ in Theorem 3.8, we obtain the following corollary.

Corollary 3.9. *Suppose given $1 < p < \infty$, $0 < \alpha < n/p$, and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then $M_{b,\alpha}$ is bounded as an operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $1/q = 1/p - \alpha/n$.*

Inequality (1.2) and Theorems 3.8 and 3.6 imply the following corollary.

Corollary 3.10. *Suppose given $0 < \alpha < n$, $b \in \mathrm{BMO}(\mathbb{R}^n)$, and $b^- \in L^\infty(\mathbb{R}^n)$. Let Φ and Ψ be Young functions such that $\Phi \in \nabla_2 \cap \mathcal{Y}$ and $\Psi \in \Delta_2$. Suppose also that condition* (3.6) *is satisfied. Then $[b, M_\alpha]$ is bounded as an operator from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

The following theorem is valid.

Theorem 3.11. *Suppose given $0 < \alpha < n$ and $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Let Φ and Ψ be Young functions such that $\Phi \in \mathcal{Y}$.*

(1) *If $\Phi \in \nabla_2$, $\Psi \in \Delta_2$, and condition* (3.6) *holds, then the condition $b \in \mathrm{BMO}(\mathbb{R}^n)$ is sufficient for the boundedness of $M_{b,\alpha}$ as an operator from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

(2) *If $\Psi^{-1}(t) \lesssim \Phi^{-1}(t)t^{-\alpha/n}$, then the condition $b \in \mathrm{BMO}(\mathbb{R}^n)$ is necessary for the boundedness of $M_{b,\alpha}$ as an operator from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

(3) *If $\Phi \in \nabla_2$, $\Psi \in \Delta_2$, $\Psi^{-1}(t) \approx \Phi^{-1}(t)t^{-\alpha/n}$, and condition* (3.7) *holds, then the condition $b \in \mathrm{BMO}(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of $M_{b,\alpha}$ as an operator from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

Proof. (1) The first assertion of the theorem follows from the first assertion of Theorem 3.8.

(2) Let us prove the second assertion. Suppose that $M_{b,\alpha}$ is bounded as an operator from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$. Consider any ball $B = B(x,r)$ in $\mathbb{R}^n$; by (2.1), we have

$$\begin{aligned}
\frac{1}{|B|}\int_B |b(y) - b_B|\,dy &= \frac{1}{|B|}\int_B \left|\frac{1}{|B|}\int_B (b(y) - b(z))\,dz\right| dy \\
&\le \frac{1}{|B|^2}\int_B \int_B |b(y) - b(z)|\,dz\,dy \\
&= \frac{1}{|B|^{1+\alpha/n}}\int_B \frac{1}{|B|^{1-\alpha/n}}\int_B |b(y) - b(z)|\chi_B(z)\,dz\,dy \\
&\le \frac{1}{|B|^{1+\alpha/n}}\int_B M_{b,\alpha}(\chi_B)(y)\,dy \le \frac{2}{|B|^{1+\alpha/n}}\|M_{b,\alpha}(\chi_B)\|_{L^\Psi(B)}\|1\|_{L^{\tilde{\Psi}}(B)} \\
&\le \frac{C}{|B|^{\alpha/n}}\frac{\Psi^{-1}(|B|^{-1})}{\Phi^{-1}(|B|^{-1})} \le C.
\end{aligned}$$

Thus, $b \in \mathrm{BMO}(\mathbb{R}^n)$.

(3) The third assertion of the theorem follows from the first and second ones. □

Remark 3.12. Note that, in the case where $\Phi(t) = t^p$ and $\Psi(t) = t^q$, Theorem 3.11 implies Theorem 1.1 for the operator $M_{b,\alpha}$.

For comparison, we formulate the following theorem, which was proved in [13], and make a remark.

Theorem 3.13. *Suppose given $0<\alpha<n$ and $b\in \mathrm{BMO}(\mathbb{R}^n)$. Let Φ and Ψ be Young functions such that $\Phi\in\mathcal{Y}$.*

(1) *If $\Phi\in\nabla_2$ and $\Psi\in\Delta_2$, then the condition*

$$r^{\alpha}\Phi^{-1}(r^{-n})+\int_r^{\infty}\Big(1+\ln\frac{t}{r}\Big)\Phi^{-1}(t^{-n})t^{\alpha}\,\frac{dt}{t}\le C\Psi^{-1}(r^{-n}) \tag{3.11}$$

for all $r>0$, where $C>0$ does not depend on r, is sufficient for the boundedness of $[b,I_\alpha]$ as an operator from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Psi}(\mathbb{R}^n)$.

(2) *If $\Psi\in\Delta_2$, then condition* (3.4) *is necessary for the boundedness of $|b,I_\alpha|$ as an operator from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Psi}(\mathbb{R}^n)$.*

(3) *If $\Phi\in\nabla_2$, $\Psi\in\Delta_2$, and the condition*

$$\int_r^{\infty}\Big(1+\ln\frac{t}{r}\Big)\Phi^{-1}(t^{-n})t^{\alpha}\,\frac{dt}{t}\le Cr^{\alpha}\Phi^{-1}(r^{-n}) \tag{3.12}$$

holds for all $r>0$, where $C>0$ does not depend on r, then condition (3.4) *is necessary and sufficient for the boundedness of $|b,I_\alpha|$ as an operator from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Psi}(\mathbb{R}^n)$.*

Remark 3.14. Although the operator $M_{b,\alpha}$ is pointwise dominated by $|b,I_\alpha|$ (see formula (1.1)) and, consequently, the results for the former could be derived from those for the latter, we consider these operators separately, because we can study the boundedness of $M_{b,\alpha}$ under weaker assumptions than those required for the operator $|b,I_\alpha|$. To be more precise, the integral condition (3.12) implies the supremum condition (3.7). Indeed, (2.1) and the monotonicity of $\widetilde{\Phi}^{-1}(s)$ imply

$$\Phi^{-1}(s^{-n})\approx\Phi^{-1}(s^{-n})s^n\int_s^{\infty}\frac{dt}{t^{n+1}}\approx\frac{1}{\widetilde{\Phi}^{-1}(s^{-n})}\int_s^{\infty}\frac{dt}{t^{n+1}}$$
$$\lesssim\int_s^{\infty}\frac{1}{\widetilde{\Phi}^{-1}(t^{-n})t^n}\,\frac{dt}{t}\approx\int_s^{\infty}\Phi^{-1}(t^{-n})\,\frac{dt}{t}\,.$$

This follows from the inequality

$$r^{\alpha}\Phi^{-1}(r^{-n})\gtrsim\int_r^{\infty}\Big(1+\ln\frac{t}{r}\Big)t^{\alpha}\Phi^{-1}(t^{-n})\,\frac{dt}{t}\gtrsim\int_s^{\infty}\Big(1+\ln\frac{t}{r}\Big)t^{\alpha}\Phi^{-1}(t^{-n})\,\frac{dt}{t}$$
$$\gtrsim\Big(1+\ln\frac{s}{r}\Big)s^{\alpha}\int_s^{\infty}\Phi^{-1}(t^{-n})\,\frac{dt}{t}\gtrsim\Big(1+\ln\frac{s}{r}\Big)s^{\alpha}\Phi^{-1}(s^{-n}),$$

where $s\in(r,\infty)$ is arbitrary, so that

$$\sup_{s>r}\Big(1+\ln\frac{s}{r}\Big)\Phi^{-1}(s^{-n})s^{\alpha}\lesssim r^{\alpha}\Phi^{-1}(r^{-n}).$$

On the other hand, the Young function Φ whose inverse is given by

$$\Phi^{-1}(t)=\frac{t^{\alpha/n}}{1-(\ln t)/n-\ln r}\qquad\text{for}\quad 0<\alpha<n,\quad t\ge r,$$

satisfies condition (3.7), but does not satisfy condition (3.12).

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