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FRACTIONAL INTEGRAL ASSOCIATED WITH SCHRÖDINGER OPERATOR ON VANISHING GENERALIZED MORREY SPACES

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Abstract. Let $L=-\triangle+V$ be a Schrödinger operator, where the non-negative potential V belongs to the reverse Hölder class $RH_{n/2}$, let b belong to a new $BMO_\theta(\rho)$ space, and let $\mathscr{I}_\beta^L$ be the fractional integral operator associated with L. In this paper, we study the boundedness of the operator $\mathscr{I}_\beta^L$ and its commutators $[b,\mathscr{I}_\beta^L]$ with $b\in BMO_\theta(\rho)$ on generalized Morrey spaces associated with Schrödinger operator $M_{p,\varphi}^{\alpha,V}$ and vanishing generalized Morrey spaces associated with Schrödinger operator $VM_{p,\varphi}^{\alpha,V}$. We find the sufficient conditions on the pair (φ_1,φ_2) which ensures the boundedness of the operator $\mathscr{I}_\beta^L$ from $M_{p,\varphi_1}^{\alpha,V}$ to $M_{q,\varphi_2}^{\alpha,V}$ and from $VM_{p,\varphi_1}^{\alpha,V}$ to $VM_{q,\varphi_2}^{\alpha,V}$, $1/p-1/q=\beta/n$. When b belongs to $BMO_\theta(\rho)$ and (φ_1,φ_2) satisfies some conditions, we also show that the commutator operator $[b,\mathscr{I}_\beta^L]$ is bounded from $M_{p,\varphi_1}^{\alpha,V}$ to $M_{q,\varphi_2}^{\alpha,V}$ and from $VM_{p,\varphi_1}^{\alpha,V}$ to $VM_{q,\varphi_2}^{\alpha,V}$, $1/p-1/q=\beta/n$.

1. Introduction and results

Let us consider the Schrödinger operator

$$L=-\Delta+V \ \text{ on } \ \mathbb{R}^n,\ n\geqslant 3,$$

where V is a non-negative, $V\neq 0$, and belongs to the reverse Hölder class RH_q for some $q\geqslant n/2$, i.e., there exists a constant $C>0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}V^q(y)dy\right)^{1/q}\leqslant\frac{C}{|B(x,r)|}\int_{B(x,r)}V(y)dy$$

holds for every $x\in\mathbb{R}^n$ and $0<r<\infty$, where $B(x,r)$ denotes the ball centered at x with radius r. In particular, if V is a nonnegative polynomial, then $V\in RH_\infty$.

Obviously, $RH_{q_2}\subset RH_{q_1}$, if $q_2>q_1$. The most important property of the class RH_q is its self-improvement, that is, if $V\in RH_q$, then $V\in RH_{q+\varepsilon}$ for some $\varepsilon>0$.

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As in [18], for a given potential $V \in RH_q$ with $q \geqslant n/2$, we define the auxiliary function

$$\rho(x) := \frac{1}{m_V(x)} = \sup_{r>0}\left\{ r : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leqslant 1 \right\}.$$

It is well-known that that $0 < \rho(x) < \infty$ for any $x \in \mathbb{R}^n$.

According to [4], the new BMO space $BMO_\theta(\rho)$ with $\theta \geqslant 0$ is defined as a set of all locally integrable functions b such that

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_B|dy \leqslant C\left(1 + \frac{r}{\rho(x)}\right)^{\theta}$$

for all $x \in \mathbb{R}^n$ and $r > 0$, where $b_B = \frac{1}{|B|}\int_B b(y)dy$. A norm for $b \in BMO_\theta(\rho)$, denoted by $[b]_\theta$, is given by the infimum of the constants in the inequalities above. Clearly, $BMO \subset BMO_\theta(\rho)$.

We now present the definition of generalized Morrey spaces (including weak version) related to potential, which introduced by Guliyev in [12].

DEFINITION 1. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$, $1 \leqslant p < \infty$, $\alpha \geqslant 0$, and $V \in RH_q$, $q \geqslant 1$. We denote by $M^{\alpha,V}_{p,\varphi} = M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space associated with Schrödinger operator, the space of all functions $f \in L^p_{loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M^{\alpha,V}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \left(1 + \frac{r}{\rho(x)}\right)^{\alpha} \varphi(x,r)^{-1} r^{-n/p} \|f\|_{L_p(B(x,r))}.$$

Also $WM^{\alpha,V}_{p,\varphi} = WM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space associated with Schrödinger operator,the space of all functions $f \in WL^p_{loc}(\mathbb{R}^n)$ with

$$\|f\|_{WM^{\alpha,V}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \left(1 + \frac{r}{\rho(x)}\right)^{\alpha} \varphi(x,r)^{-1} r^{-n/p} \|f\|_{WL_p(B(x,r))} < \infty.$$

REMARK 1. (i) When $\alpha = 0$, and $\varphi(x,r) = r^{(\lambda-n)/p}$, $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the classical Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ introduced by Morrey in [13];

(ii) When $\varphi(x,r) = r^{(\lambda-n)/p}$, $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the Morrey space associated with Schrödinger operator $L^{\alpha,V}_{p,\lambda}(\mathbb{R}^n)$ studied by Tang and Dong in [21];

(iii) When $\alpha = 0$, $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the generalized Morrey space $M_{p,\varphi}(\mathbb{R}^n)$ introduced by Mizuhara and Nakai in [14, 15].

(iv) The generalized Morrey space associated with Schrödinger operator $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ was introduced by Guliyev in [12].

The classical Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ was introduced by Morrey in [13] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [7, 8, 9, 13]. The generalized Morrey spaces are defined with r^λ replaced by a general non-negative function $\varphi(x,r)$ satisfying some assumptions (see, for example, [10, 14, 15, 19] and etc).

For brevity, in the sequel we use the notations

$$\mathfrak{A}^{\alpha,V}_{p,\varphi}(f;x,r) := \left(1+\frac{r}{\rho(x)}\right)^{\alpha} r^{-n/p}\,\varphi(x,r)^{-1}\|f\|_{L_p(B(x,r))}$$

and

$$\mathfrak{A}^{W,\alpha,V}_{\Phi,\varphi}(f;x,r) := \left(1+\frac{r}{\rho(x)}\right)^{\alpha} r^{-n/p}\,\varphi(x,r)^{-1}\|f\|_{WL_p(B(x,r))}.$$

DEFINITION 2. The vanishing generalized Morrey space associated with Schrödinger operator $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f\in M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}^{\alpha,V}_{p,\varphi}(f;x,r)=0. \tag{1}$$

The vanishing weak generalized Morrey space associated with Schrödinger operator $VWM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is defined as the spaces of functions $f\in WM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\mathfrak{A}^{W,\alpha,V}_{p,\varphi}(f;x,r)=0.$$

The vanishing spaces $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ and $VWM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$\|f\|_{VM^{\alpha,V}_{p,\varphi}} \equiv \|f\|_{M^{\alpha,V}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n,r>0}\mathfrak{A}^{\alpha,V}_{p,\varphi}(f;x,r),$$

$$\|f\|_{VWM^{\alpha,V}_{p,\varphi}} \equiv \|f\|_{WM^{\alpha,V}_{p,\varphi}} = \sup_{x\in\mathbb{R}^n,r>0}\mathfrak{A}^{\alpha,V}_{W,p,\varphi}(f;x,r),$$

respectively.

In the case $\alpha=0$, and $\varphi(x,r)=r^{(\lambda-n)/p}$ $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the vanishing Morrey space $VM_{p,\lambda}$ introduced in [22], where applications to PDE were considered.

We refer to [1, 6, 16, 17] for some properties of vanishing generalized Morrey spaces.

DEFINITION 3. Let $L=-\triangle+V$ with $V\in RH_{n/2}$. The fractional integral associated with L is defined by

$$\mathscr{I}^L_\beta f(x)=L^{-\beta/2}f(x)=\int_0^\infty e^{-tL}(f)(x)\,t^{\beta/2-1}dt$$

for $0<\beta<n$. The commutator of $\mathscr{I}^L_\beta$ is defined by

$$[b,\mathscr{I}^L_\beta]f(x)=b(x)\mathscr{I}^L_\beta f(x)-\mathscr{I}^L_\beta(bf)(x).$$

In this paper, we consider the boundedness of the fractional integral operator $\mathscr{I}_\beta^L$ on the generalized Morrey spaces $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ and the vanishing generalized Morrey spaces $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$. When b belongs to the new BMO space $BMO_\theta(\rho)$, we also show that $[b,\mathscr{I}_\beta^L]$ is bounded from $M_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ to $M_{q,\varphi}^{\alpha,V}(\mathbb{R}^n)$ and from $VM_{p,\varphi}^{\alpha,V}(\mathbb{R}^n)$ to $VM_{q,\varphi}^{\alpha,V}(\mathbb{R}^n)$.

Our main results are as follows.

THEOREM 1. *Let* $V \in RH_{n/2}$, $\alpha \geqslant 0$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ *and* $\varphi_1 \in \Omega_p^{\alpha,V}$, $\varphi_2 \in \Omega_q^{\alpha,V}$ *satisfies the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t} \leqslant c_0 \varphi_2(x,r), \tag{2}$$

where c_0 *does not depend on* x *and* r. *Then the operator* $\mathscr{I}_\beta^L$ *is bounded on* $M_{p,\varphi_1}^{\alpha,V}$ *to* $M_{q,\varphi_2}^{\alpha,V}$ *for* $p > 1$ *and from* $M_{1,\varphi_1}^{\alpha,V}$ *to* $WM_{\frac{n}{n-\beta},\varphi_2}^{\alpha,V}$. *Moreover, for* $p > 1$

$$\|\mathscr{I}_\beta^L f\|_{M_{q,\varphi_2}^{\alpha,V}} \leqslant C \|f\|_{M_{p,\varphi_1}^{\alpha,V}},$$

and for $p = 1$

$$\|\mathscr{I}_\beta^L f\|_{WM_{\frac{n}{n-\beta},\varphi_2}^{\alpha,V}} \leqslant C \|f\|_{M_{1,\varphi_1}^{\alpha,V}},$$

where C *does not depend on* f.

THEOREM 2. *Let* $V \in RH_{n/2}$, $\alpha \geqslant 0$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ *and* $\varphi_1 \in \Omega_p^{\alpha,V}$, $\varphi_2 \in \Omega_q^{\alpha,V}$ *satisfies the condition*

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t} \leqslant c_0 \varphi_2(x,r), \tag{3}$$

where c_0 *does not depend on* x *and* r. *If* $b \in BMO_\theta(\rho)$, *then the operator* $[b,\mathscr{I}_\beta^L]$ *is bounded from* $M_{p,\varphi_1}^{\alpha,V}$ *to* $M_{q,\varphi_2}^{\alpha,V}$ *and*

$$\|[b,\mathscr{I}_\beta^L] f\|_{M_{q,\varphi_2}^{\alpha,V}} \leqslant C [b]_\theta \|f\|_{M_{p,\varphi_1}^{\alpha,V}},$$

where C *does not depend on* f.

THEOREM 3. *Let* $V \in RH_{n/2}$, $\alpha \geqslant 0$, $1 \leqslant p < n/\beta$, $1/q = 1/p - \beta/n$ *and* $\varphi_1 \in \Omega^{\alpha,V}_{p,1}$, $\varphi_2 \in \Omega^{\alpha,V}_{q,1}$ *satisfies the conditions*

$$c_\delta := \int_\delta^\infty \sup_{x\in\mathbb{R}^n} \varphi_1(x,t)\frac{dt}{t} < \infty$$

for every $\delta > 0$, *and*

$$\int_r^\infty \varphi_1(x,t)\frac{dt}{t^{1-\beta}} \leqslant C_0\varphi_2(x,r), \tag{4}$$

where C_0 *does not depend on* $x \in \mathbb{R}^n$ *and* $r > 0$. *Then the operator* $\mathscr{I}^L_\beta$ *is bounded from* $VM^{\alpha,V}_{p,\varphi_1}$ *to* $VM^{\alpha,V}_{q,\varphi_2}$ *for* $p > 1$ *and from* $VM^{\alpha,V}_{1,\varphi_1}$ *to* $VWM^{\alpha,V}_{\frac{n}{n-\beta},\varphi_2}$.

THEOREM 4. *Let* $V \in RH_{n/2}$, $b \in BMO_\theta(\rho)$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$, *and* $\varphi_1 \in \Omega^{\alpha,V}_{p,1}$, $\varphi_2 \in \Omega^{\alpha,V}_{q,1}$ *satisfies the conditions*

$$\int_r^\infty \Big(1+\ln\frac{t}{r}\Big)\varphi_1(x,t)\frac{dt}{t^{1-\beta}} \leqslant c_0\varphi_2(x,r), \tag{5}$$

where c_0 *does not depend on* x *and* r,

$$\lim_{r\to 0}\frac{\ln\frac{1}{r}}{\inf\limits_{x\in\mathbb{R}^n}\varphi_2(x,r)} = 0 \tag{6}$$

and

$$c_\delta := \int_\delta^\infty \Big(1+|\ln t|\Big)\sup_{x\in\mathbb{R}^n}\varphi_1(x,t)\frac{dt}{t^{1-\beta}} < \infty \tag{7}$$

for every $\delta > 0$. *Then the operator* $[b,\mathscr{I}^L_\beta]$ *is bounded from* $VM^{\alpha,V}_{p,\varphi_1}$ *to* $VM^{\alpha,V}_{q,\varphi_2}$.

REMARK 2. Note that, Theorems 1 and 2 in the case of $V \equiv 0$ was proved in [11, Corollary 5.5 and 7.5] and in the case of $\varphi(x,r) = r^{(\lambda-n)/p}$ in [21, Theorems 1.3 and 1.4].

REMARK 3. Note that, in [2] the Nikolskii-Morrey type spaces were introduced and the authors studied some embedding theorems. In the next paper, we shall introduce the generalized Nikolskii-Morrey spaces associated with Schrödinger operator and will study some embedding theorems. We will also investigate the boundedness of fractional integral associated with Schrödinger operator on the generalized Nikolskii-Morrey spaces associated with Schrödinger operator.

In this paper, we shall use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \leqslant CB$. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

2. Some preliminaries

We would like to recall the important properties concerning the critical function.

LEMMA 1. [18] *Let $V \in RH_{n/2}$. For the associated function ρ there exist C and $k_0 \geqslant 1$ such that*

$$C^{-1}\rho(x)\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{-k_0} \leqslant \rho(y) \leqslant C\rho(x)\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{\frac{k_0}{1+k_0}} \tag{8}$$

for all $x,y \in \mathbb{R}^n$.

LEMMA 2. [3] *Suppose $x \in B(x_0,r)$. Then for $k \in N$ we have*

$$\frac{1}{\Big(1+\frac{2^k r}{\rho(x)}\Big)^N} \lesssim \frac{1}{\Big(1+\frac{2^k r}{\rho(x_0)}\Big)^{N/(k_0+1)}}.$$

We give some inequalities about the new BMO space $BMO_\theta(\rho)$.

LEMMA 3. [4] *Let $1 \leqslant s < \infty$. If $b \in BMO_\theta(\rho)$, then*

$$\Big(\frac{1}{|B|}\int_B |b(y)-b_B|^s dy\Big)^{1/s} \leqslant [b]_\theta \Big(1+\frac{r}{\rho(x)}\Big)^{\theta'}$$

for all $B = B(x,r)$, with $x \in \mathbb{R}^n$ and $r > 0$, where $\theta' = (k_0+1)\theta$ and k_0 is the constant appearing in (8).

LEMMA 4. [4] *Let $1 \leqslant s < \infty$, $b \in BMO_\theta(\rho)$, and $B = B(x,r)$. Then*

$$\Big(\frac{1}{|2^k B|}\int_{2^k B} |b(y)-b_B|^s dy\Big)^{1/s} \leqslant [b]_\theta k\Big(1+\frac{2^k r}{\rho(x)}\Big)^{\theta'}$$

for all $k \in \mathbb{N}$, with θ' as in Lemma 3.

Let K_β be the kernel of $\mathscr{I}_\beta^L$. The following result give the estimate on the kernel $K_\beta(x,y)$.

LEMMA 5. [5] *If $V \in RH_{n/2}$, then for every N, there exists a constant C such that*

$$|K_\beta(x,y)| \leqslant \frac{C}{\Big(1+\frac{|x-y|}{\rho(x)}\Big)^N} \frac{1}{|x-y|^{n-\beta}}. \tag{9}$$

Finally, we recall a relationship between essential supremum and essential infimum.

LEMMA 6. [23] *Let f be a real-valued nonnegative function and measurable on E. Then*

$$\Big(\operatorname*{ess\,inf}_{x\in E} f(x)\Big)^{-1} = \operatorname*{ess\,sup}_{x\in E} \frac{1}{f(x)}.$$

LEMMA 7. [3] *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n\times(0,\infty)$, $1\leqslant p<\infty$, $\alpha\geqslant 0$, and $V\in RH_q$, $q\geqslant 1$.*

(i) If $\sup_{t<r<\infty}\Big(1+\frac{r}{\rho(x)}\Big)^{\alpha}\frac{r^{-\frac{n}{p}}}{\varphi(x,r)}=\infty$ *for some $t>0$ and for all $x\in\mathbb{R}^n$, then* $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)=\Theta$.

(ii) If $\sup_{0<r<\tau}\Big(1+\frac{r}{\rho(x)}\Big)^{\alpha}\varphi(x,r)^{-1}=\infty$ *for some $\tau>0$ and for all $x\in\mathbb{R}^n$, then* $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)=\Theta$.

REMARK 4. We denote by $\Omega^{\alpha,V}_p$ the sets of all positive measurable functions φ on $\mathbb{R}^n\times(0,\infty)$ such that for all $t>0$,

$$\sup_{x\in\mathbb{R}^n}\Big\|\Big(1+\frac{r}{\rho(x)}\Big)^{\alpha}\frac{r^{-\frac{n}{p}}}{\varphi(x,r)}\Big\|_{L_\infty(t,\infty)}<\infty,\quad\text{and}\quad \sup_{x\in\mathbb{R}^n}\Big\|\Big(1+\frac{r}{\rho(x)}\Big)^{\alpha}\varphi(x,r)^{-1}\Big\|_{L_\infty(0,t)}<\infty,$$

respectively. In what follows, keeping in mind Lemma 7, we always assume that $\varphi\in\Omega^{\alpha,V}_p$.

REMARK 5. We denote by $\Omega^{\alpha,V}_{p,1}$ the sets of all positive measurable functions φ on $\mathbb{R}^n\times(0,\infty)$ such that

$$\inf_{x\in\mathbb{R}^n}\inf_{r>\delta}\Big(1+\frac{r}{\rho(x)}\Big)^{-\alpha}\varphi(x,r)>0,\ \text{for some }\delta>0, \tag{10}$$

and

$$\lim_{r\to 0}\Big(1+\frac{r}{\rho(x)}\Big)^{\alpha}\frac{r^{n/p}}{\varphi(x,r)}=0.$$

For the non-triviality of the space $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ we always assume that $\varphi\in\Omega^{\alpha,V}_{p,1}$.

3. Proof of Theorem 1

We first prove the following conclusions

THEOREM 5. *Let $V\in RH_{n/2}$. If $1<p<n/\beta$, $1/q=1/p-\beta/n$ then the inequality*

$$\|\mathscr{I}^L_\beta(f)\|_{L_q(B(x_0,r))}\lesssim r^{\frac{n}{q}}\int_{2r}^{\infty}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}$$

holds for any $f \in L^p_{loc}(\mathbb{R}^n)$. *Moreover, for* $p=1$ *the inequality*

$$\|\mathscr{I}^L_\beta(f)\|_{WL_{\frac{n}{n-\beta}}(B(x_0,r))} \lesssim r^{n-\beta} \int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t}$$

holds for any $f \in L^1_{loc}(\mathbb{R}^n)$.

Proof. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ and $\lambda B = B(x_0,\lambda r)$ for any $\lambda > 0$. We write f as $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{B(x_0,2r)}(y)$, and $\chi_{B(x_0,2r)}$ denotes the characteristic function of $B(x_0,2r)$. Then

$$\|\mathscr{I}^L_\beta(f)\|_{L_q(B(x_0,r))} \leqslant \|\mathscr{I}^L_\beta(f_1)\|_{L_q(B(x_0,r))} + \|\mathscr{I}^L_\beta(f_2)\|_{L_q(B(x_0,r))}.$$

Since $f_1 \in L_p(\mathbb{R}^n)$ and from the boundedness of $\mathscr{I}^L_\beta$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ (see [20]) it follows that

$$\begin{aligned} \|\mathscr{I}^L_\beta(f_1)\|_{L_q(B(x_0,r))} &\lesssim \|f\|_{L_p(B(x_0,2r))} \\ &\lesssim r^{\frac{n}{q}} \|f\|_{L_p(B(x_0,2r))} \int_{2r}^{\infty} \frac{dt}{t^{\frac{n}{q}+1}} \\ &\lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}. \end{aligned} \tag{11}$$

To estimate $\|\mathscr{I}^L_\beta(f_2)\|_{L_p(B(x_0,r))}$, obverse that $x \in B$, $y \in (2B)^c$ implies $|x-y| \approx |x_0 - y|$. Then by (9) we have

$$\begin{aligned} \sup_{x\in B} |\mathscr{I}^L_\beta(f_2)(x)| &\leqslant \sup_{x\in B} \int_{(2B)^c} |K_\beta(x,y) f(y)| dy \\ &\lesssim \int_{(2B)^c} \frac{|f(y)|}{|x_0-y|^{n-\beta}} dy \\ &\lesssim \sum_{k=1}^{\infty} (2^{k+1}r)^{-n+\beta} \int_{2^{k+1}B} |f(y)| dy. \end{aligned}$$

By Hölder's inequality we get

$$\begin{aligned} \sup_{x\in B} |\mathscr{I}^L_\beta(f_2)(x)| &\lesssim \sum_{k=1}^{\infty} \|f\|_{L_p(2^{k+1}B)} (2^{k+1}r)^{-1-\frac{n}{p}+\beta} \int_{2^k r}^{2^{k+1}r} dt \\ &\lesssim \sum_{k=1}^{\infty} \int_{2^k r}^{2^{k+1}r} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \\ &\lesssim \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}. \end{aligned} \tag{12}$$

Then

$$\|\mathscr{I}_\beta^L(f_2)\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \tag{13}$$

holds for $1 \leqslant p < n/\beta$. Therefore, by (11) and (13) we get

$$\|\mathscr{I}_\beta^L(f)\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \tag{14}$$

holds for $1 \leqslant p < n/\beta$.

When $p=1$, by the boundedness of $\mathscr{I}_\beta^L$ from $L_1(\mathbb{R}^n)$ to $WL_{\frac{n}{n-\beta}}(\mathbb{R}^n)$, we get

$$\|\mathscr{I}_\beta^L(f_1)\|_{WL_{\frac{n}{n-\beta}}(B(x_0,r))} \lesssim \|f\|_{L_1(B(x_0,2r))} \lesssim r^{n-\beta} \int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t}.$$

By (13) we have

$$\|\mathscr{I}_\beta^L(f_2)\|_{WL_{\frac{n}{n-\beta}}(B(x_0,r))} \leqslant \|\mathscr{I}_\beta^L(f_2)\|_{L_{\frac{n}{n-\beta}}(B(x_0,2r))} \lesssim r^{n-\beta} \int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t}.$$

Then

$$\|\mathscr{I}_\beta^L(f)\|_{WL_{\frac{n}{n-\beta}}(B(x_0,r))} \lesssim r^{n-\beta} \int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t}. \qquad \square$$

Proof of Theorem 1. From Lemma 6, we have

$$\frac{1}{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}} = \operatorname*{ess\,sup}_{t<s<\infty} \frac{1}{\varphi_1(x,s)s^{\frac{n}{p}}}.$$

Note the fact that $\|f\|_{L_p(B(x_0,t))}$ is a nondecresing function of t, and $f \in M_{p,\varphi_1}^{\alpha,V}$, then

$$\begin{aligned}
\frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha} \|f\|_{L_p(B(x_0,t))}}{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}}} &\lesssim \operatorname*{ess\,sup}_{t<s<\infty} \frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha} \|f\|_{L_p(B(x_0,t))}}{\varphi_1(x_0,s)s^{\frac{n}{p}}} \\
&\lesssim \sup_{0<s<\infty} \frac{\left(1+\frac{s}{\rho(x_0)}\right)^{\alpha} \|f\|_{L_p(B(x_0,s))}}{\varphi_1(x_0,s)s^{\frac{n}{p}}} \\
&\lesssim \|f\|_{M_{p,\varphi_1}^{\alpha,V}}.
\end{aligned}$$

Since $\alpha \geqslant 0$, and (φ_1, φ_2) satisfies the condition (2), then

$$\int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} = \int_{2r}^{\infty} \frac{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha} \|f\|_{L_p(B(x_0,t))}}{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}}} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}}}{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha} t^{\frac{n}{q}}} \frac{dt}{t}$$

$$\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}} \int_{2r}^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s) s^{\frac{n}{p}}}{\left(1+\frac{t}{\rho(x_0)}\right)^{\alpha} t^{\frac{n}{q}}} \frac{dt}{t}$$

$$\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}} \left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \int_{r}^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s) s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t}$$

$$\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}} \left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \varphi_2(x_0,r). \tag{15}$$

Then by Theorem 5 we get

$$\|\mathscr{I}^L_\beta(f)\|_{M^{\alpha,V}_{q,\varphi_2}} \lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} r^{-n/q} \|\mathscr{I}^L_\beta(f)\|_{L_p(B(x_0,r))}$$

$$\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}$$

$$\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}}.$$

Let $q = \frac{n}{n-\beta}$, similar to the estimates of (15) we have

$$\int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t} \lesssim \|f\|_{M^{\alpha,V}_{1,\varphi_1}} \left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \varphi_2(x_0,r).$$

Thus by Theorem 5 we get

$$\|\mathscr{I}^L_\beta(f)\|_{WM^{\alpha,V}_{\frac{n}{n-\beta},\varphi_2}} \lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} r^{\beta-n} \|\mathscr{I}^L_\beta(f)\|_{WL_{\frac{n}{n-\beta}}(B(x_0,r))}$$

$$\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} \int_{2r}^{\infty} \frac{\|f\|_{L_1(B(x_0,t))}}{t^{n-\beta}} \frac{dt}{t}$$

$$\lesssim \|f\|_{M^{\alpha,V}_{1,\varphi_1}}. \quad \square$$

4. Proof of Theorem 2

As the proof of Theorem 1, it suffices to prove the following result.

THEOREM 6. *Let $V \in RH_{n/2}$, $b \in BMO_\theta(\rho)$. If $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ then the inequality*

$$\|[b,\mathscr{I}^L_\beta(f)]\|_{L_q(B(x_0,r))} \lesssim [b]_\theta r^{\frac{n}{q}} \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t} \tag{16}$$

holds for any $f \in L^p_{loc}(\mathbb{R}^n)$.

Proof. We write f as $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{B(x_0,2r)}(y)$. Then

$$\|[b,\mathscr{I}_\beta^L](f)\|_{L_q(B(x_0,r))} \leqslant \|[b,\mathscr{I}_\beta^L](f_1)\|_{L_q(B(x_0,r))} + \|[b,\mathscr{I}_\beta^L](f_2)\|_{L_q(B(x_0,r))}.$$

By the boundedness of $[b,\mathscr{I}_\beta^L]$ on $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ (see [21]) and (11) we get

$$\begin{aligned}\|[b,\mathscr{I}_\beta^L](f_1)\|_{L_q(B(x_0,r))} &\lesssim [b]_\theta \|f\|_{L_p(B(x_0,2r))}\\ &\lesssim [b]_\theta r^{\frac{n}{q}} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}\\ &\lesssim [b]_\theta r^{\frac{n}{q}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}. \end{aligned} \tag{17}$$

We now turn to deal with the term $\|[b,\mathscr{I}_\beta^L](f_2)\|_{L_q(B(x_0,r))}$. For any given $x \in B(x_0,r)$ we have

$$|[b,\mathscr{I}_\beta^L]f_2(x)| \leqslant |b(x)-b_{2B}|\,|\mathscr{I}_\beta^L(f_2)(x)| + |\mathscr{I}_\beta^L((b-b_{2B})f_2)(x)|.$$

Then by (12), Lemma 3, and taking $N \geqslant (k_0+1)\theta$ we get

$$\begin{aligned}\|(b(x)-b_{2B})\mathscr{I}_\beta^L(f_2)\|_{L_q(B(x_0,r))} &\lesssim [b]_\theta r^{\frac{n}{q}} \Big(1+\frac{2r}{\rho(x_0)}\Big)^{\theta-N/(k_0+1)} \int_{2r}^{\infty} \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}\\ &\lesssim [b]_\theta r^{\frac{n}{q}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}. \end{aligned} \tag{18}$$

Finally, let us estimate $\|\mathscr{I}_\beta^L((b-b_{2B})f_2)\|_{L_q(B(x_0,r))}$. By (9), Lemma 2 and (12) we have

$$\begin{aligned}\sup_{x\in B}|\mathscr{I}_\beta^L((b-b_{2B})f_2)(x)| &\lesssim \sup_{x\in B}\int_{(2B)^c} \frac{1}{\Big(1+\frac{|x-y|}{\rho(x)}\Big)^N}\frac{|b(y)-b_{2B}||f(y)|}{|x_0-y|^{n-\beta}}dy\\ &\lesssim \sup_{x\in B}\sum_{k=1}^{\infty}\frac{1}{(2^k r)^{n-\beta}\Big(1+\frac{2^k r}{\rho(x)}\Big)^N}\int_{2^{k+1}B}|b(y)-b_{2B}||f(y)|dy\\ &\lesssim \sum_{k=1}^{\infty}\frac{1}{(2^k r)^{n-\beta}\Big(1+\frac{2^k r}{\rho(x_0)}\Big)^{N/(k_0+1)}}\int_{2^{k+1}B}|b(y)-b_{2B}||f(y)|dy. \end{aligned}$$

Note that

$$\begin{aligned}\int_{2^{k+1}B}|b(y)-b_{2B}||f(y)|dy &\lesssim \Big(\int_{2^{k+1}B}|b(y)-b_{2B}|^{p'}\Big)^{1/p'}\|f\|_{L_p(B(x_0,2^{k+1}r))}\\ &\lesssim [b]_\theta k\Big(1+\frac{2^k r}{\rho(x_0)}\Big)^{\theta'}(2^k r)^{\frac{n}{p'}}\|f\|_{L_p(B(x_0,2^{k+1}r))}.\end{aligned}$$

Then

$$\begin{aligned}\sup_{x\in B}|\mathscr{I}^L_\beta((b-b_{2B})f_2)(x)| &\lesssim [b]_\theta\sum_{k=1}^\infty \frac{k(2^k r)^{-\frac{n}{p}+\beta}}{\Big(1+\frac{2^k r}{\rho(x_0)}\Big)^{N/(k_0+1)-\theta'}}\|f\|_{L_p(B(x_0,2^{k+1}r))}\\ &\lesssim [b]_\theta\sum_{k=1}^\infty k(2^k r)^{-\frac{n}{q}}\|f\|_{L_p(B(x_0,2^{k+1}r))}\\ &\lesssim [b]_\theta\sum_{k=1}^\infty k\int_{2^k r}^{2^{k+1}r}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}.\end{aligned}$$

Since $2^k r\leqslant t\leqslant 2^{k+1}r$, then $k\approx \ln\frac{t}{r}$. Thus

$$\begin{aligned}\sup_{x\in B}|\mathscr{I}^L_\beta((b-b_{2B})f_2)(x)| &\lesssim [b]_\theta\sum_{k=1}^\infty k\int_{2^k r}^{2^{k+1}r}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}\\ &\lesssim [b]_\theta\sum_{k=1}^\infty \int_{2^k r}^{2^{k+1}r}\ln\frac{t}{r}\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}\\ &\lesssim [b]_\theta\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}.\end{aligned}$$

Then

$$\|\mathscr{I}^L_\beta((b-b_{2B})f_2)\|_{L_q(B(x_0,r))}\lesssim [b]_\theta r^{\frac{n}{q}}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}. \tag{19}$$

Combining (17), (18) and (19), the proof of Theorem 6 is completed. □

Proof of Theorem 2. Since $f\in M^{\alpha,V}_{p,\varphi_1}$ and (φ_1,φ_2) satisfies the condition (3), by (15) we have

$$\begin{aligned}&\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}\\ &=\int_{2r}^{\infty}\frac{\Big(1+\frac{t}{\rho(x_0)}\Big)^{\alpha}\|f\|_{L_p(B(x_0,t))}}{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{\frac{n}{p}}}\Big(1+\ln\frac{t}{r}\Big)\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{\frac{n}{p}}}{\Big(1+\frac{t}{\rho(x_0)}\Big)^{\alpha}t^{\frac{n}{q}}}\frac{dt}{t}\\ &\lesssim\|f\|_{M^{\alpha,V}_{p,\varphi_1}}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x_0,s)s^{\frac{n}{p}}}{\Big(1+\frac{t}{\rho(x_0)}\Big)^{\alpha}t^{\frac{n}{q}}}\frac{dt}{t}\end{aligned}$$

$$\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}} \left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \int_r^\infty \left(1+\ln\frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s) s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t}$$
$$\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}} \left(1+\frac{r}{\rho(x_0)}\right)^{-\alpha} \varphi_2(x_0,r). \tag{20}$$

Then from Theorem 6 and by (20) we get

$$\|[b,\mathscr{I}^L_\beta](f)\|_{M^{\alpha,V}_{q,\varphi_2}}$$
$$\lesssim \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} r^{-n/q} \|[b,\mathscr{I}^L_\beta](f)\|_{L_q(B(x_0,r))}$$
$$\lesssim [b]_\theta \sup_{x_0\in\mathbb{R}^n, r>0} \left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} \varphi_2(x_0,r)^{-1} \int_{2r}^\infty \left(1+\ln\frac{t}{r}\right) \frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}} \frac{dt}{t}$$
$$\lesssim [b]_\theta \|f\|_{M^{\alpha,V}_{p,\varphi_1}}. \quad \square$$

5. Proof of Theorem 3

The statement is derived from the estimate (14). The estimation of the norm of the operator, that is, the boundedness in the non-vanishing space, immediately follows from by Theorem 1. So we only have to prove that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{p,\varphi_1}(f;x,r) = 0 \;\Rightarrow\; \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{q,\varphi_2}(\mathscr{I}^L_\beta(f);x,r) = 0 \tag{21}$$

and

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{\alpha,V}_{1,\varphi_1}(f;x,r) = 0 \;\Rightarrow\; \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{A}^{W,\alpha,V}_{n/(n-\beta),\varphi_2}(\mathscr{I}^L_\beta(f);x,r) = 0. \tag{22}$$

To show that $\sup_{x\in\mathbb{R}^n} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi_2(x,r)^{-1} r^{-n/p} \|\mathscr{I}^L_\beta(f)\|_{L_q(B(x,r))} < \varepsilon$ for small r, we split the right-hand side of (14):

$$\left(1+\frac{r}{\rho(x)}\right)^{\alpha} \varphi_2(x,r)^{-1} r^{-n/p} \|\mathscr{I}^L_\beta(f)\|_{L_q(B(x,r))} \leqslant C[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)], \tag{23}$$

where $\delta_0>0$ (we may take $\delta_0>1$), and

$$I_{\delta_0}(x,r) := \frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x,r)} \int_r^{\delta_0} t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x,t))} dt$$

and

$$J_{\delta_0}(x,r) := \frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x,r)} \int_{\delta_0}^\infty t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x,t))} dt$$

and it is supposed that $r < \delta_0$. We use the fact that $f \in VM^{\alpha,V}_{p,\varphi_1}(\mathbb{R}^n)$ and choose any fixed $\delta_0 > 0$ such that

$$\sup_{x\in\mathbb{R}^n}\left(1+\frac{t}{\rho(x)}\right)^{\alpha}\varphi_1(x,t)^{-1}t^{-n/p}\|f\|_{L_p(B(x,t))} < \frac{\varepsilon}{2CC_0},$$

where C and C_0 are constants from (4) and (23). This allows to estimate the first term uniformly in $r \in (0,\delta_0)$:

$$\sup_{x\in\mathbb{R}^n} CI_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \quad 0 < r < \delta_0.$$

The estimation of the second term now my be made already by the choice of r sufficiently small. Indeed, thanks to the condition (10) we have

$$J_{\delta_0}(x,r) \leqslant c_{\sigma_0}\,\frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_1(x,r)}\,\|f\|_{VM^{\alpha,V}_{p,\varphi_1}},$$

where c_{σ_0} is the constant from (1). Then, by (10) it suffices to choose r small enough such that

$$\sup_{x\in\mathbb{R}^n}\frac{\left(1+\frac{r}{\rho(x)}\right)^{\alpha}}{\varphi_2(x,r)} \leqslant \frac{\varepsilon}{2c_{\sigma_0}\|f\|_{VM^{\alpha,V}_{p,\varphi_1}}},$$

which completes the proof of (21).

The proof of (22) is similar to the proof of (21).

6. Proof of Theorem 4

The norm inequality having already been provided by Theorem 2, we only have to prove the implication

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi_1(x,r)^{-1}r^{-n/p}\|f\|_{L_p(B(x,r))} = 0$$
$$\Longrightarrow \lim_{r\to 0}\sup_{x\in\mathbb{R}^n}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi_2(x,r)^{-1}r^{-n/p}\|[b,\mathscr{I}^L_\beta(f)]\|_{L_q(B(x,r))} = 0.$$

To check that

$$\sup_{x\in\mathbb{R}^n}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\varphi_2(x,r)^{-1}r^{-n/p}\|[b,\mathscr{I}^L_\beta(f)]\|_{L_q(B(x,r))} < \varepsilon \quad \text{for small } r,$$

we use the estimate (16):

$$\varphi_2(x,r)^{-1}r^{-n/p}\|[b,\mathscr{I}_\beta^L(f)]\|_{L_q(B(x,r))} \lesssim \frac{[b]_\theta}{\varphi_2(x,r)}\int_r^\infty \left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}.$$

We take $r<\delta_0$, where δ_0 will be chosen small enough and split the integration:

$$\left(1+\frac{r}{\rho(x)}\right)^\alpha \varphi_2(x,r)^{-1}r^{-n/p}\|[b,\mathscr{I}_\beta^L(f)]\|_{L_q(B(x,r))} \leqslant C[I_{\delta_0}(x,r)+J_{\delta_0}(x,r)], \tag{24}$$

where

$$I_{\delta_0}(x,r) := \frac{\left(1+\frac{r}{\rho(x)}\right)^\alpha}{\varphi_2(x,r)}\int_r^{\delta_0}\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}$$

and

$$J_{\delta_0}(x,r) := \frac{\left(1+\frac{r}{\rho(x)}\right)^\alpha}{\varphi_2(x,r)}\int_{\delta_0}^\infty\left(1+\ln\frac{t}{r}\right)\frac{\|f\|_{L_p(B(x_0,t))}}{t^{\frac{n}{q}}}\frac{dt}{t}.$$

We choose a fixed $\delta_0>0$ such that

$$\sup_{x\in\mathbb{R}^n}\left(1+\frac{r}{\rho(x)}\right)^\alpha \varphi_1(x,r)^{-1}r^{-n/p}\|f\|_{L_p(B(x,r))} < \frac{\varepsilon}{2CC_0}, \quad r\leqslant\delta_0,$$

where C and C_0 are constants from (24) and (5), which yields the estimate of the first term uniform in $r\in(0,\delta_0)$: $\sup\limits_{x\in\mathbb{R}^n} CI_{\delta_0}(x,r)<\frac{\varepsilon}{2}$, $0<r<\delta_0$.

For the second term, writing $1+\ln\frac{t}{r}\leqslant 1+|\ln t|+\ln\frac{1}{r}$, we obtain

$$J_{\delta_0}(x,r)\leqslant \frac{c_{\delta_0}+\widetilde{c_{\delta_0}}\ln\frac{1}{r}}{\varphi_2(x,r)}\|f\|_{M^{\alpha,V}_{p,\varphi_1}},$$

where c_{δ_0} is the constant from (7) with $\delta=\delta_0$ and $\widetilde{c_{\delta_0}}$ is a similar constant with omitted logarithmic factor in the integrand. Then, by (6) we can choose small r such that $\sup_{x\in\mathbb{R}^n} J_{\delta_0}(x,r)<\frac{\varepsilon}{2}$, which completes the proof.

7. Conclusions

In this paper, we study the boundedness of the of the fractional integral operator $\mathscr{I}_\beta^L$ associated with Schrödinger operator and its commutators $[b,\mathscr{I}_\beta^L]$ with $b\in BMO_\theta(\rho)$ on generalized Morrey spaces $M^{\alpha,V}_{p,\varphi}$ associated with Schrödinger operator and vanishing generalized Morrey spaces $VM^{\alpha,V}_{p,\varphi}$ associated with Schrödinger operator. We find the sufficient conditions on the pair (φ_1,φ_2) which ensures the boundedness of the operator $\mathscr{I}_\beta^L$ from $M^{\alpha,V}_{p,\varphi_1}$ to $M^{\alpha,V}_{q,\varphi_2}$ and from $VM^{\alpha,V}_{p,\varphi_1}$ to $VM^{\alpha,V}_{q,\varphi_2}$, $1/p-1/q=\beta/n$. When b belongs to $BMO_\theta(\rho)$ and (φ_1,φ_2) satisfies some conditions, we also show that the commutator operator $[b,\mathscr{I}_\beta^L]$ is bounded from $M_{p,\varphi_1}^{\alpha,V}$ to $M_{q,\varphi_2}^{\alpha,V}$ and from $VM_{p,\varphi_1}^{\alpha,V}$ to $VM_{q,\varphi_2}^{\alpha,V}$, $1/p-1/q=\beta/n$.

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