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Research Article

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The Arzelà–Ascoli theorem by means of ideal convergence

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Abstract: In this paper, using the concept of ideal convergence, which extends the idea of ordinary convergence and statistical convergence, we are concerned with the I-uniform convergence and the I-pointwise convergence of sequences of functions defined on a set of real numbers D. We present the Arzelà–Ascoli theorem by means of ideal convergence and also the relationship between I-equicontinuity and I-continuity for a family of functions.

Keywords: Pointwise and uniform convergence, ideal convergence, classical Arzelà–Ascoli theorem

MSC 2010: 40A30, 40A35

1 Introduction

Pointwise and uniform convergence of sequences of functions have different applications in functional analysis. One of the most important applications is the Arzelà–Ascoli theorem. There are some generalizations of this theorem in summability theory. Recently, it has been shown that non-matrix summability methods have significant importance for the convergence of sequences of functions (see [1, 2, 7, 14, 15, 19]). Instead of pointwise convergence and uniform convergence, using the concepts of I-pointwise and I-uniform convergence gives certain advantages, since these convergences are more general than the ordinary ones. These types of convergence methods are quite effective, especially when the classical limit does not exist. The concept of ideal convergence is a generalization of statistical convergence and it is based on the notation of the ideal of subsets of $\mathbb{N}$, the set of positive integers. In the present paper, using the concept of ideal convergence, we present an ideal convergence version of the Arzelà–Ascoli theorem by following the proof in [3, 18]. It is also benefical to note that Green and Valentine [12] gave an alternative proof of the Arzelà–Ascoli theorem.

We first recall some notation and basic definitions to be used in the paper.

If K is a set of positive integers, $|K|$ will denote the cardinality of K. The natural density of K is given by

$$\delta(K) = \lim_{n\to\infty} (C_1\chi_K)_n = \lim_{n\to\infty} \frac{1}{n}|\{k \le n : k \in K\}|,$$

if it exists, where C_1 is the Cesàro mean of order one and χ_K is the characteristic function of the set K (see [17])

The number sequence $x = (x_k)$ is statistically convergent to L provided that, for every $\varepsilon > 0$, the set $K_\varepsilon = \{k \in \mathbb{N} : |x_k - L| \ge \varepsilon\}$ has natural density zero (see [4, 8]). In that case we write st-lim $x = L$. This notion has been studied in detail in various directions (see [4, 5, 9–11, 13, 16]).

Recent studies demonstrate that the concept of statistical convergence provides an important contribution to the improvement of classical analysis. Duman and Orhan [7] have extended the concept of ordinary uniform convergence (or pointwise)using μ-statistical uniform (or pointwise)convergence and gave some related results. Also, I-convergence versions of the above notions were examined in [1, 14]. Recall tha

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a family $I \subset P(\mathbb{N})$ of subsets of $\mathbb{N}$ is said to be an ideal in $\mathbb{N}$ if I is closed under subsets and finite unions, i.e., for each $A, B \in I$, we have $A \cup B \in I$ and for each $A \in I$ and each $B \subset A$, we have $B \in I$. The ideal I is said to be a proper ideal in $\mathbb{N}$ if $\mathbb{N} \notin I$. A proper ideal is said to be admissible if $\{n\} \in I$ for each $n \in \mathbb{N}$. One can easily see that an admissible ideal includes all finite subsets of $\mathbb{N}$. Let $I \subset P(\mathbb{N})$ be a proper ideal in $\mathbb{N}$. A sequence $x = (x_k)$ of real numbers is said to be I-convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$

$$K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in I.$$

If $x = (x_k)$ is I-convergent to L, then we write $I\text{-}\lim x = L$ (see [15]).

Example 1.1. (a) Let $I = I_f$ be the class of all finite subsets of $\mathbb{N}$. Then I is an admissible ideal in $\mathbb{N}$, and I-convergence reduces to ordinary convergence.
(b) Let $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$. Then I is an admissible ideal in $\mathbb{N}$, and I-convergence reduces to statistical convergence.

In the present paper, by using the concepts of I-uniform and I-pointwise convergence, we give some results for the sequences of functions, where I is assumed to be an admissible ideal. The most important theorem related to this topic is the Arzelà–Ascoli theorem. Recall that the classical Arzelà–Ascoli theorem claims that given a pointwise bounded sequence of equicontinuous functions defined on a compact set, we can find its uniformly convergent subsequence. We also establish the relationship between equicontinuity and continuity by means of ideal convergence. It should be mentioned that these results are more powerful than the classical theorems. Let $\mathcal{F} = (f_n)$ be a family of functions defined on $D \subseteq \mathbb{R}$. Thus, one can speak of the equicontinuity of the family on D (see [3, 18]). It is clear that every member of the equicontinuous family is uniformly continuous. The property can be weakened as follows.

Definition 1.2. $\mathcal{F} = (f_n)$ is called I-pointwise bounded on $D \subset \mathbb{R}$ if there exists a set $M \in I$ such that $(f_n)_{n \notin M}$ is pointwise bounded on D.

I-uniform boundedness can be defined in a similar way.

Example 1.3. Let $K = [0, 1]$, $I = I_\delta$, and define $f_n : K \to \mathbb{R}$ as follows:

$$f_n(x) = \begin{cases} n \cos nx, & n = m^2, \\ \sin nx, & n \neq m^2. \end{cases}$$

Since $\delta(\{m^2 : m \in \mathbb{N}\}) = 0$ and $|f_n(x)| \leq 1$, $n \neq m^2$, one can easily see that (f_n) is statistically uniformly bounded, and hence statistically pointwise bounded. But observe that (f_n) is neither pointwise bounded nor uniformly bounded.

Definition 1.4. $\mathcal{F} = (f_n)$ is said to have the I-equicontinuity property on D if there exists a subset $A \in I$ such that the family of functions $\{f_k : k \notin A\} \subset \mathcal{F}$ is equicontinuous on D.

The I-continuity property of the family $\mathcal{F}$ can be given similarly. These properties enhance the applicability of our theorems.

2 Generalization of the Arzelà–Ascoli theorem

In this section we prove the Arzelà–Ascoli theorem by means of ideal convergence.

We first turn to introducing some notations and basic definitions to be used in this section. The next two definitions are slightly more general than the ones given in [7].

Fix an admissible ideal $I \subset P(\mathbb{N})$. Assume that $D \neq \emptyset$ and that the functions $f : D \to \mathbb{R}$, $f_n : D \to \mathbb{R}$, $n \in \mathbb{N}$, are given. The sequence $(f_n)_{n\in\mathbb{N}}$ is said to be I-pointwise convergent to f on D if for all $x \in D$ and all $\varepsilon > 0$ there exists $M_{\varepsilon,x} \in I$ such that for all $n \notin M_{\varepsilon,x}$, we have $|f_n(x) - f(x)| < \varepsilon$, and this convergence is denoted by $f_n \to f(I)$. Next recall the I-uniform convergence of (f_n) to f. It is denoted by $f_n \rightrightarrows f(I)$ and defined as follows:

for all $\varepsilon > 0$, there exists $M_\varepsilon \in I$ such that for all $n \notin M_\varepsilon$ and all $x \in D$, we have $|f_n(x) - f(x)| < \varepsilon$. It is easy to see that I-uniform convergence implies I-pointwise convergence.

In order to give our main result we need the following theorems.

Theorem 2.1. *Let (f_n) be a sequence of functions from a countable set D into $\mathbb{R}$ which is I-pointwise bounded. Then there exists a subsequence (f_{n_k}) which is pointwise convergent on D.*

Proof. Since (f_n) is I-pointwise bounded on D, one can find a subset $M \subseteq \mathbb{N}$ such that $M \in I$ and $\{f_n\}_{n \notin M}$ is pointwise bounded on D. By hypothesis, D is a countable set, so the existence of a pointwise subsequence follows immediately. □

Theorem 2.2. *Let (f_n) be a sequence of functions from $X \subseteq \mathbb{R}$ to $\mathbb{R}$ with the property of I-equicontinuity. If the sequence (f_n) is I-pointwise convergent on a dense subset D of X, then (f_n) is I-pointwise convergent on X, and the limit function f is continuous.*

Proof. By the definition of I-equicontinuity, there exists a set $M_1 \subseteq \mathbb{N}$ such that $M_1 \in I$ and $\{f_n : n \notin M_1\}$ is equicontinuous on X. Then, using the definition of equicontinuity, for every $\varepsilon > 0$, one can find $\tau(\varepsilon) > 0$ such that $|x - y| < \tau$ for every $x, y \in X$ implies $|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$ for every $n \notin M_1$. Now for given $x \in X$ and $\varepsilon > 0$, one can also find an open set A containing x such that $|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$ for all y in A and $n \notin M_1$. Since D is dense in X, there must be a point $y \in D \cap A$. By hypothesis, $(f_n(y))$ is I-convergent, and hence it must be an I-Cauchy sequence. Thus, there exists a subset $M_2 \subseteq \mathbb{N}$ such that $M_2 \in I$, and for every $n, m \notin M_2$, we have $|f_n(y) - f_m(y)| < \frac{\varepsilon}{3}$ (see [6]). Let $M = M_1 \cup M_2$. For every $n, m \notin M$, we get

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(y)| + |f_n(y) - f_m(y)| + |f_m(y) - f_m(x)| < \varepsilon.$$

So $(f_n(x))$ is an I-convergent sequence. Since $x \in X$ is arbitrary, (f_n) is I-pointwise convergent on X. Let f be a function to which (f_n) is I-pointwise convergent. Hence, for every $y \in X$ such that $|x - y| < \tau$ and for every $n \notin M$, we have

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \varepsilon.$$

Hence, f is continuous at x. □

Theorem 2.3. *Let K be a compact set and (f_n) a sequence of functions with the property of I-equicontinuity. If (f_n) is I-pointwise convergent on K to a function f, then (f_n) is I-uniformly convergent to f.*

Proof. By the I-equicontinuity property, there exists a subset $M_1 \subseteq \mathbb{N}$, with $M_1 \in I$, and for given $\varepsilon > 0$, each x in K is contained in an open set A_x such that $|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$ for all y in A_x and $n \notin M_1$.

By the compactness of K, there exists a finite collection of these sets $\{A_{x_1}, A_{x_2}, \ldots, A_{x_k}\}$ which covers K. Since (f_n) is I-pointwise convergent, we have $|f_n(x_i) - f(x_i)| < \frac{\varepsilon}{3}$ for all $n \notin M_{x_i}$ such that $M_{x_i} \in I$ for each x_i corresponding to this finite collection. Then, for any $y \in K$, there exists $i \leq k$ such that $y \in A_{x_i}$. Let $M = M_1 \cup M_{x_1} \cup M_{x_2} \cup \cdots \cup M_{x_k}$. Hence, for every $n \notin M$,

$$|f_n(y) - f(y)| \leq |f_n(y) - f_n(x_i)| + |f_n(x_i) - f(x_i)| + |f(x_i) - f(y)| < \varepsilon.$$

Thus, (f_n) is I-uniformly convergent to f on K. □

As a consequence of the above theorems, we formulate the following analog of the Arzelà–Ascoli theorem by means of ideal convergence.

Theorem 2.4. *Let K be a compact set and (f_n) a sequence of functions from K to $\mathbb{R}$ which is I-pointwise bounded with the property of I-equicontinuity. Then (f_n) has a subsequence (f_{n_k}) which is I-uniformly convergent to a continuous function f.*

Proof. By the compactness of K, there exists a subset D of K which is dense and countable. We define $D = \{x_1, x_2, \ldots, x_k, \ldots\}$. By Theorem 2.1, (f_n) has a subsequence which is pointwise convergent. $f_{n_k} \to f$ implies that $f_{n_k} \to f(I)$ on D. Since D is dense in K, by Theorem 2.2, $f_{n_k} \to f(I)$ on K and f is continuous. Then $f_{n_k} \rightrightarrows f(I)$ on K, by Theorem 2.3. This completes the proof. □

Also, from [1], we know that every statistically uniformly convergent sequence (f_n) has a subsequence $(f_{n'_k})$ that converges uniformly. So we can immediately give the following result.

Corollary 2.5. *Under the hypothesis of Theorem 2.4 with $I = I_\delta$, (f_n) has a subsequence $(f_{n'_k})$ that is I_δ-uniformly convergent on K.*

Now we present examples of a sequence of functions satisfying Theorem 2.4 but not satisfying the classical Arzelà–Ascoli theorem.

Example 2.6. Let $K = [0, 1]$, $I = I_\delta$, and define $f_n : K \to \mathbb{R}$ as follows:

$$f_n(x) = \begin{cases} \dfrac{x^2}{x^2 + (1 - nx)^2}, & n = m^2, \\ x^2 + \dfrac{x}{n}, & n \neq m^2. \end{cases}$$

Since $|f_n(x)| \leq 2$, for all $x \in K$ and all $n \in \mathbb{N}$, (f_n) is uniformly bounded, and hence pointwise bounded. But note that (f_n) is not equicontinuous. So we can not use the classical Arzelà–Ascoli theorem but Theorem 2.4 can be applied to get an I_δ-uniformly convergent subsequence, since (f_n) has the I_δ-equicontinuity property.

Example 2.7. Let $K = [0, 1]$, $I = I_\delta$, and define $f_n : K \to \mathbb{R}$ as follows:

$$f_n(x) = \begin{cases} \dfrac{1}{n} + x, & n = m^2, \\ x + n, & n \neq m^2. \end{cases}$$

Note that (f_n) is I_δ-pointwise bounded and has the I_δ-equicontinuity property. So we cannot use the classical Arzelà–Ascoli theorem but Theorem 2.4 can be applied easily.

Let us give the next theorem without proof. Note that the proof can be obtained similarly to the proof of [7, Theorem 2.1].

Theorem 2.8. *Let (f_n) be a sequence of functions from A into $\mathbb{R}$ with the property of I-continuity. If the sequence (f_n) is I-uniformly convergent to a function f, then f is continuous.*

Corollary 2.9. *If $\mathcal{F} = (f_n)$ is a sequence of functions with the property of I-continuity on the compact set K into $\mathbb{R}$ which is I-uniformly convergent to f on K, then $\mathcal{F}$ has the property of I-uniform boundedness.*

Proof. Since f is the I-uniform limit of the sequence $\mathcal{F}$, for every $\varepsilon > 0$, there exists a set $M \subset \mathbb{N}$ such that $M \in I$, and for every $n \notin M$ and every $x \in K$, we have $|f_n(x) - f(x)| < \varepsilon$. Moreover, it follows from Theorem 2.8 that f is continuous on K. Since K is compact, f is bounded. Also, since, for all $x \in K$,

$$||f_n(x)| - |f(x)|| \leq |f_n(x) - f(x)| < 1,$$

we have $|f_n(x)| \leq |f(x)| + 1 \leq L$. Thus, $\mathcal{F}$ has the property of I-uniform boundedness. □

Theorem 2.10. *If $\mathcal{F} = (f_n)$ is a sequence of functions on a compact set K into $\mathbb{R}$ which is I-uniformly convergent and has the property of I-continuity, then $\mathcal{F}$ has the I-equicontinuity property.*

Proof. Let f be the I-uniform limit of the sequence $\mathcal{F}$. For every $\varepsilon > 0$, there exists a subset $M \subseteq \mathbb{N}$ such that $M \in I$. Then, for every $n \notin M$ and for every $x \in K$, we have $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$. By Theorem 2.8, f is continuous on K, and since K is compact, f is uniformly continuous. This means that for every $\varepsilon > 0$, there exists $\tau(\varepsilon) > 0$ such that for every $x, y \in K$, with $|x - y| < \tau$, we have

$$|f(x) - f(y)| < \frac{\varepsilon}{3}.$$

For $n \notin M$, it follows that

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f(x)| + |f(x) - f(y)| + |f(y) - f_n(y)| < \varepsilon.$$

This completes the proof. □

The following corollary can be regarded as the converse of Theorem 2.4.

Corollary 2.11. *Let K be a compact set and $\mathcal{F} = (f_n)$ be a sequence of functions with the property of I-continuity. If (f_n) has a subsequence (f_{n_k}) which is I-uniformly convergent to a function on K, then $\mathcal{F}$ has the property of I-uniform boundedness and I-equicontinuity.*

Proof. The proof can be easily obtained by using Theorem 2.10 and Corollary 2.9, respectively. □

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