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Morrey-type estimates for commutator of fractional integral associated with Schrödinger operators on the Heisenberg group

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Abstract

Let $L = -\Delta_{\mathbb{H}_n} + V$ be a Schrödinger operator on the Heisenberg group $\mathbb{H}_n$, where the nonnegative potential V belongs to the reverse Hölder class RH_{q_1} for some $q_1 \geq Q/2$, and Q is the homogeneous dimension of $\mathbb{H}_n$. Let b belong to a new Campanato space $\Lambda_\nu^\theta(\rho)$, and let $\mathcal{I}_\beta^L$ be the fractional integral operator associated with L. In this paper, we study the boundedness of the commutators $[b,\mathcal{I}_\beta^L]$ with $b \in \Lambda_\nu^\theta(\rho)$ on central generalized Morrey spaces $LM_{p,\varphi}^{\alpha,V}(\mathbb{H}_n)$, generalized Morrey spaces $M_{p,\varphi}^{\alpha,V}(\mathbb{H}_n)$, and vanishing generalized Morrey spaces $VM_{p,\varphi}^{\alpha,V}(\mathbb{H}_n)$ associated with Schrödinger operator, respectively. When b belongs to $\Lambda_\nu^\theta(\rho)$ with $\theta > 0$, $0 < \nu < 1$ and (φ_1, φ_2) satisfies some conditions, we show that the commutator operator $[b,\mathcal{I}_\beta^L]$ is bounded from $LM_{p,\varphi_1}^{\alpha,V}(\mathbb{H}_n)$ to $LM_{q,\varphi_2}^{\alpha,V}(\mathbb{H}_n)$, from $M_{p,\varphi_1}^{\alpha,V}(\mathbb{H}_n)$ to $M_{q,\varphi_2}^{\alpha,V}(\mathbb{H}_n)$, and from $VM_{p,\varphi_1}^{\alpha,V}(\mathbb{H}_n)$ to $VM_{q,\varphi_2}^{\alpha,V}(\mathbb{H}_n)$, $1/p - 1/q = (\beta + \nu)/Q$.

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1 Introduction

Heisenberg groups, in discrete and continuous versions, appear in many parts of mathematics, including Fourier analysis, several complex variables, geometry, and topology. We state some basic results about the Heisenberg group. More detailed information can be found in [5, 12, 13] and the references therein.

Let us consider the Schrödinger operator on Heisenberg group $\mathbb{H}_n$

$$L = -\Delta_{\mathbb{H}_n} + V \quad \text{on } \mathbb{H}_n, n \geq 3,$$

where $V \not\equiv 0$ is nonnegative and belongs to the reverse Hölder class RH_q for some $q \geq Q/2$, that is, there exists a constant $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(g,r)|}\int_{B(g,r)} V^q(h)\,dh\right)^{1/q} \leq \frac{C}{|B(g,r)|}\int_{B(g,r)} V(h)\,dh \tag{1.1}$$

holds for every $g \in \mathbb{H}_n$ and $0 < r < \infty$, where $B(g,r)$ denotes the ball centered at g with radius r.

We also say that a nonnegative function $V \in RH_\infty$ if there exists a constant $C > 0$ such that

$$\sup_{h\in B(g,r)} V(h) \le \frac{C}{|B(g,r)|} \int_{B(g,r)} V(h)\,dh$$

for all $g \in \mathbb{H}_n$ and $0 < r < \infty$.

In particular, if V is a nonnegative polynomial, then $V \in RH_\infty$.

We define the auxiliary function $0 < \rho(g) < \infty$ for a given potential $V \in RH_q$ with $q \ge Q/2$:

$$\rho(g) := \sup_{r>0}\left\{ r : \frac{1}{r^{Q-2}} \int_{B(g,r)} V(h)\,dh \le 1 \right\}$$

for $g \in \mathbb{H}_n$ (for example, see [36]).

Let $\theta > 0$ and $0 < \nu < 1$. In view of [24, 26], the Campanato class associated with the Schrödinger operator $\Lambda_\nu^\theta(\rho)$ consists of locally integrable functions b such that

$$\frac{1}{|B(g,r)|^{1+\nu/Q}} \int_{B(g,r)} \left| b(h) - b_B \right| dh \le C\left(1 + \frac{r}{\rho(g)}\right)^{\theta} \tag{1.2}$$

for all $g \in \mathbb{H}_n$ and $r > 0$, where b_B is the mean integral of b in the ball $B(g,r)$. A seminorm of $b \in \Lambda_\nu^\theta(\mathbb{H}_n, \rho)$, denoted by $[b]_\beta^\theta$, is given as the infimum of the constants in inequality (1.2).

Note that if $\theta = 0$, then $\Lambda_\nu^\theta(\mathbb{H}_n, \rho)$ is the classical Campanato space; if $\nu = 0$, then $\Lambda_\nu^\theta(\mathbb{H}_n, \rho)$ is the space $BMO_\theta(\mathbb{H}_n, \rho)$ introduced in [3]; see also [25].

For brevity, we further use the notations

$$\mathfrak{A}_{p,\varphi}^{\alpha,V}(f;g,r) := \left(1 + \frac{r}{\rho(g)}\right)^{\alpha} r^{-Q/p} \varphi(g,r)^{-1} \|f\|_{L_p(B(g,r))}$$

and

$$\mathfrak{A}_{\Phi,\varphi}^{W,\alpha,V}(f;g,r) := \left(1 + \frac{r}{\rho(g)}\right)^{\alpha} r^{-Q/p} \varphi(g,r)^{-1} \|f\|_{WL_p(B(g,r))}.$$

We give the definition of central (local) and global generalized Morrey spaces (including weak version) associated with the Schrödinger operator; it was introduced by the first author in [18] in the Euclidean setting (see also [1, 3, 39]).

Definition 1.1 Let $\varphi(r)$ be a positive measurable function on $(0,\infty)$, $1 \le p < \infty$, $\alpha \ge 0$, and $V \in RH_q$, $q \ge 1$. We denote by $M_{p,\varphi}^{\alpha,V} = M_{p,\varphi}^{\alpha,V}(\mathbb{H}_n)$ and $LM_{p,\varphi}^{\alpha,V} = LM_{p,\varphi}^{\alpha,V}(\mathbb{H}_n)$ the generalized Morrey space and the central generalized Morrey space associated with the Schrödinger operator, the spaces of all functions $f \in L_{\mathrm{loc}}^p(\mathbb{H}_n)$ with finite quasinorms

$$\|f\|_{M_{p,\varphi}^{\alpha,V}} = \sup_{g\in\mathbb{H}_n, r>0} \mathfrak{A}_{p,\varphi}^{\alpha,V}(f;g,r) \quad \text{and} \quad \|f\|_{LM_{p,\varphi}^{\alpha,V}} = \sup_{r>0} \mathfrak{A}_{p,\varphi}^{\alpha,V}(f;e,r),$$

respectively. Here e is the identity element in $\mathbb{H}_n$.

Also, by $WM^{\alpha,V}_{p,\varphi} = WM^{\alpha,V}_{p,\varphi}(\mathbb{H}_n)$ and $LWM^{\alpha,V}_{p,\varphi} = LWM^{\alpha,V}_{p,\varphi}(\mathbb{H}_n)$ we denote the weak generalized Morrey space and central weak generalized Morrey space associated with the Schrödinger operator, the spaces of all functions $f \in WL^p_{\mathrm{loc}}(\mathbb{H}_n)$ with

$$\|f\|_{WM^{\alpha,V}_{p,\varphi}} = \sup_{g\in\mathbb{H}_n,r>0} \mathfrak{A}^{W,\alpha,V}_{\Phi,\varphi}(f;g,r) < \infty \quad \text{and}$$

$$\|f\|_{LWM^{\alpha,V}_{p,\varphi}} = \sup_{r>0} \mathfrak{A}^{W,\alpha,V}_{\Phi,\varphi}(f;e,r) < \infty,$$

respectively.

Remark 1.1

(i) When $\alpha = 0$ and $\varphi(r) = r^{(\lambda-Q)/p}$, $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the classical Morrey space $M_{p,\lambda}(\mathbb{R}^n)$ introduced by Morrey [28], and $LM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the central Morrey space $LM_{p,\lambda}(\mathbb{R}^n)$ studied by Alvarez et al. [2] in the Euclidean setting.

(ii) When $\varphi(r) = r^{(\lambda-Q)/p}$, $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the Morrey space associated with Schrödinger operator $M^{\alpha,V}_{p,\lambda}(\mathbb{R}^n)$ studied by Tang and Dong in [39] on the Euclidean setting.

(iii) When $\alpha = 0$, $M^{\alpha,V}_{p,\varphi}(\mathbb{H}_n)$ is the generalized Morrey space $M_{p,\varphi}(\mathbb{H}_n)$ studied by Guliyev et al. [20], and $LM^{\alpha,V}_{p,\varphi}(\mathbb{H}_n)$ is the central generalized Morrey space $LM_{p,\varphi}(\mathbb{H}_n)$ studied by first author in [14]; see also [10, 15, 17, 19, 21, 23, 34, 35].

(iv) $M^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ and $LM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ are the generalized Morrey space and the central generalized Morrey space associated with the Schrödinger operator, respectively, studied by first author in [18] in the Euclidean setting; see also [1].

Definition 1.2 The vanishing generalized Morrey space $VM^{\alpha,V}_{p,\varphi}(\mathbb{H}_n)$ associated with the Schrödinger operator is defined as the space of functions $f \in M^{\alpha,V}_{p,\varphi}(\mathbb{H}_n)$ such that

$$\lim_{r\to 0} \sup_{g\in\mathbb{H}_n} \mathfrak{A}^{\alpha,V}_{p,\varphi}(f;g,r) = 0. \tag{1.3}$$

The vanishing weak generalized Morrey space $VWM^{\alpha,V}_{p,\varphi}(\mathbb{H}_n)$ associated with the Schrödinger operator is defined as the space of functions $f \in WM^{\alpha,V}_{p,\varphi}(\mathbb{H}_n)$ such that

$$\lim_{r\to 0} \sup_{g\in\mathbb{H}_n} \mathfrak{A}^{W,\alpha,V}_{p,\varphi}(f;g,r) = 0.$$

The classical Morrey spaces $M_{p,\lambda}(\mathbb{R}^n)$ were introduced by Morrey in [28] to study the local behavior of solutions to second-order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [7, 9, 11, 28, 31]. The generalized Morrey spaces are defined with r^λ replaced by a general nonnegative function $\varphi(r)$ satisfying some assumptions (see, for example, [16, 20, 27, 29, 37], etc.).

In the case $\alpha = 0$, $\varphi(x,r) = r^{(\lambda-n)/p}$ $VM^{\alpha,V}_{p,\varphi}(\mathbb{R}^n)$ is the vanishing Morrey space $VM_{p,\lambda}$ introduced in [40], where applications to PDE were considered.

We refer to [1, 8, 22, 32, 33] for some properties of vanishing generalized Morrey spaces.

Definition 1.3 Let $L = -\Delta_{\mathbb{H}_n} + V$ with $V \in RH_{Q/2}$. The fractional integral associated with L is defined by

$$\mathcal{I}^L_\beta f(g) = L^{-\beta/2} f(g) = \int_0^\infty e^{-tL} f(g) t^{\beta/2-1}\, dt$$

for $0 < \beta < Q$. The commutator of $\mathcal{I}_\beta^L$ is defined by

$$[b,\mathcal{I}_\beta^L]f(g) = b(g)\mathcal{I}_\beta^L f(g) - \mathcal{I}_\beta^L(bf)(g).$$

Note that, if $L = -\Delta_{\mathbb{H}_n}$ is the sub-Laplacian on $\mathbb{H}_n$, then $\mathcal{I}_\beta^L$ and $[b,\mathcal{I}_\beta^L]$ are the Riesz potential I_β and the commutator of the Riesz potential $[b,I_\beta]$, respectively, that is,

$$I_\beta f(g) = \int_{\mathbb{H}_n} \frac{f(h)}{|h^{-1}g|^{Q-\beta}}\,dh, \qquad [b,I_\beta]f(g) = \int_{\mathbb{H}_n} \frac{b(g)-b(h)}{|h^{-1}g|^{Q-\beta}} f(h)\,dh.$$

When $b \in BMO$, Chanillo proved in [6] that $[b,I_\beta]$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $1/q = 1/p - \beta/n$, $1 < p < n/\beta$. When b belongs to the Campanato space Λ_ν, $0 < \nu < 1$, Paluszynski [30] showed that $[b,I_\beta]$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $1/q = 1/p - (\beta + \nu)/n$, $1 < p < n/(\beta+\nu)$. When $b \in BMO_\theta(\rho)$, Bui [4] obtained the boundedness of $[b,\mathcal{I}_\beta^L]$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $1/q = 1/p - \beta/n$, $1 < p < n/\beta$.

Inspired by the results mentioned, we are interested in the boundedness of $[b,\mathcal{I}_\beta^L]$ on the generalized Morrey spaces $M_{p,\varphi}^{\alpha,V}(\mathbb{H}_n)$ and the vanishing generalized Morrey spaces $VM_{p,\varphi}^{\alpha,V}(\mathbb{H}_n)$ when b belongs to the new Campanato class $\Lambda_\nu^\theta(\rho)$.

In this paper, we consider the boundedness of the commutator of $\mathcal{I}_\beta^L$ on the central generalized Morrey spaces $LM_{p,\varphi}^{\alpha,V}(\mathbb{H}_n)$, the generalized Morrey spaces $M_{p,\varphi}^{\alpha,V}(\mathbb{H}_n)$, and the vanishing generalized Morrey spaces $VM_{p,\varphi}^{\alpha,V}(\mathbb{H}_n)$. When b belongs to the new Campanato space $\Lambda_\nu^\theta(\rho)$, $0 < \nu < 1$, we show that $[b,\mathcal{I}_\beta^L]$ are bounded from $LM_{p,\varphi_1}^{\alpha,V}(\mathbb{H}_n)$ to $LM_{q,\varphi_2}^{\alpha,V}(\mathbb{H}_n)$, from $M_{p,\varphi}^{\alpha,V}(\mathbb{H}_n)$ to $M_{q,\varphi}^{\alpha,V}(\mathbb{H}_n)$, and from $VM_{p,\varphi}^{\alpha,V}(\mathbb{H}_n)$ to $VM_{q,\varphi}^{\alpha,V}(\mathbb{H}_n)$ with $1/q = 1/p - (\beta + \nu)/Q$, $1 < p < Q/(\beta+\nu)$.

Our main results are as follows.

Theorem 1.1 *Let $x_0 \in \mathbb{H}_n$, $b \in \Lambda_\nu^\theta(\rho)$, $V \in RH_{q_1}$, $q_1 > Q/2$, $0 < \nu < 1$, $\alpha \ge 0$, $1 \le p < Q/(\beta + \nu)$, $1/q = 1/p - (\beta+\nu)/Q$, and let $\varphi_1, \varphi_2 \in \Omega_{p,loc}^{\alpha,V}$ satisfy the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(g_0,s)s^{\frac{Q}{p}}}{t^{\frac{Q}{q}}}\frac{dt}{t} \le c_0\varphi_2(g_0,r), \tag{1.4}$$

where c_0 does not depend on g_0 and r. Then the operator $[b,\mathcal{I}_\beta^L]$ is bounded from $M_{p,\varphi_1}^{\alpha,V}(\mathbb{H}_n)$ to $M_{q,\varphi_2}^{\alpha,V}(\mathbb{H}_n)$ for $p > 1$ and from $M_{1,\varphi_1}^{\alpha,V}(\mathbb{H}_n)$ to $WM_{\frac{Q}{Q-\beta-\nu},\varphi_2}^{\alpha,V}(\mathbb{H}_n)$. Moreover, for $p > 1$,

$$\|[b,\mathcal{I}_\beta^L]f\|_{M_{q,\varphi_2}^{\alpha,V}} \le C[b]_\nu^\theta \|f\|_{M_{p,\varphi_1}^{\alpha,V}},$$

and for $p = 1$,

$$\|[b,\mathcal{I}_\beta^L]f\|_{WM_{\frac{Q}{Q-\beta-\nu},\varphi_2}^{\alpha,V}} \le C[b]_\nu^\theta \|f\|_{M_{1,\varphi_1}^{\alpha,V}},$$

where C does not depend on f.

Corollary 1.1 *Let $b \in \Lambda_\nu^\theta(\rho)$, $V \in RH_{q_1}$, $q_1 > Q/2$, $0 < \nu < 1$, $\alpha \geq 0$, $1 \leq p < Q/(\beta + \nu)$, $1/q = 1/p - (\beta + \nu)/Q$, and let $\varphi_1 \in \Omega_p^{\alpha,V}$, $\varphi_2 \in \Omega_q^{\alpha,V}$ satisfy the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(g,s)s^{\frac{Q}{p}}}{t^{\frac{Q}{q}}} \frac{dt}{t} \leq c_0\varphi_2(g,r), \tag{1.5}$$

where c_0 does not depend on x and r. Then the operator $[b,\mathcal{I}_\beta^L]$ is bounded from $M_{p,\varphi_1}^{\alpha,V}(\mathbb{H}_n)$ to $M_{q,\varphi_2}^{\alpha,V}(\mathbb{H}_n)$ for $p > 1$ and from $M_{1,\varphi_1}^{\alpha,V}(\mathbb{H}_n)$ to $WM_{\frac{Q}{Q-\beta-\nu},\varphi_2}^{\alpha,V}(\mathbb{H}_n)$. Moreover, for $p > 1$,

$$\big\|[b,\mathcal{I}_\beta^L]f\big\|_{M_{q,\varphi_2}^{\alpha,V}} \leq C[b]_\theta\|f\|_{M_{p,\varphi_1}^{\alpha,V}},$$

and for $p = 1$,

$$\big\|[b,\mathcal{I}_\beta^L]f\big\|_{WM_{\frac{Q}{Q-\beta-\nu},\varphi_2}^{\alpha,V}} \leq C\|f\|_{M_{1,\varphi_1}^{\alpha,V}},$$

where C does not depend on f.

Theorem 1.2 *Let $b \in \Lambda_\nu^\theta(\rho)$, $V \in RH_{q_1}$, $q_1 > Q/2$, $0 < \nu < 1$, $\alpha \geq 0$, $b \in \Lambda_\nu^\theta(\rho)$, $1 < p < Q/(\beta + \nu)$, $1/q = 1/p - (\beta + \nu)/Q$, and let $\varphi_1 \in \Omega_{p,1}^{\alpha,V}$, $\varphi_2 \in \Omega_{q,1}^{\alpha,V}$ satisfy the conditions*

$$c_\delta := \int_\delta^\infty \sup_{g\in\mathbb{H}_n} \varphi_1(g,t)\frac{dt}{t} < \infty$$

for every $\delta > 0$ and

$$\int_r^\infty \varphi_1(g,t)\frac{dt}{t^{1-\beta-\nu}} \leq C_0\varphi_2(g,r), \tag{1.6}$$

where C_0 does not depend on $g \in \mathbb{H}_n$ and $r > 0$. Then the operator $[b,\mathcal{I}_\beta^L]$ is bounded from $VM_{p,\varphi_1}^{\alpha,V}(\mathbb{H}_n)$ to $VM_{q,\varphi_2}^{\alpha,V}(\mathbb{H}_n)$ for $p > 1$ and from $VM_{1,\varphi_1}^{\alpha,V}(\mathbb{H}_n)$ to $VWM_{\frac{Q}{Q-\beta-\nu},\varphi_2}^{\alpha,V}(\mathbb{H}_n)$.

Remark 1.2 Note that Theorems 1.1 and 1.2 and Corollary 1.1 were proved in [19, Theorems 1.1, 1.2; Corollary 1.1] in the Euclidean setting.

In this paper, we use the symbol $A \lesssim B$ to indicate that there exists a universal positive constant C, independent of all important parameters, such that $A \leq CB$.

2 Some preliminaries

Let $\mathbb{H}_n$ be a Heisenberg group of dimension $2n + 1$, that is, a nilpotent Lie group with underlying manifold $\mathbb{R}^{2n} \times \mathbb{R}$. The group structure is given by

$$(x,t)(y,s) = \left(x + y, t + s + 2\sum_{j=1}^n (x_{n+j}y_j - x_jy_{n+j})\right).$$

The Lie algebra of left-invariant vector fields on $\mathbb{H}_n$ is spanned by

$$X_{2n+1} = \frac{\partial}{\partial t}, \qquad X_j = \frac{\partial}{\partial x_j} + 2x_{n+j}\frac{\partial}{\partial t}, \qquad X_{n+j} = \frac{\partial}{\partial x_{n+j}} - 2x_j\frac{\partial}{\partial t}, \quad j = 1,\dots,n.$$

The nontrivial commutation relations are given by $[X_j, X_{n+j}] = -4X_{2n+1}, j = 1, \dots, n$. The sub-Laplacian $\triangle_{\mathbb{H}_n}$ is defined by $\Delta_{\mathbb{H}_n} = \sum_{j=1}^{2n} X_j^2$. The Haar measure on $\mathbb{H}_n$ is simply the Lebesgue measure on $\mathbb{R}^{2n} \times \mathbb{R}$. The measure of any measurable set $E \subset \mathbb{H}_n$ is denoted by $|E|$. The homogeneous norm on $\mathbb{H}_n$ is defined by

$$|g| = \left(|x|^4 + |t|^2\right)^{\frac{1}{4}}, \quad g = (x, t) \in \mathbb{H}_n,$$

which leads to the left-invariant distance $d(g, h) = |g^{-1}h|$ on $\mathbb{H}_n$. The dilations on $\mathbb{H}_n$ have the form $\delta_r(x, t) = (rx, r^2 t)$, $r > 0$. The Haar measure on this group coincides with the Lebesgue measure $dx = dx_1 \dots dx_{2n}\, dt$. The identity element in $\mathbb{H}_n$ is $e = 0 \in \mathbb{R}^{2n+1}$, whereas the element g^{-1} inverse to $g = (x, t)$ is $(-x, -t)$.

The ball of radius r and centered at g is $B(g, r) = \{h \in \mathbb{H}_n : |g^{-1}h| < r\}$. Note that $|B(g, r)| = r^Q |B(0, 1)|$, where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}_n$. If $B = B(g, r)$, then λB denotes $B(g, \lambda r)$ for $\lambda > 0$. Clearly, we have $|\lambda B| = \lambda^Q |B|$.

For background on the analysis on the Heisenberg groups, we refer the reader to [13, 38].

We would like to recall the important properties concerning the critical function.

Lemma 2.1 ([24]) *Let $V \in RH_{Q/2}$. For the associated function ρ, there exist C and $k_0 \geq 1$ such that*

$$C^{-1} \rho(g) \left(1 + \frac{|h^{-1}g|}{\rho(g)}\right)^{-k_0} \leq \rho(h) \leq C \rho(g) \left(1 + \frac{|h^{-1}g|}{\rho(g)}\right)^{\frac{k_0}{1+k_0}} \tag{2.1}$$

for all $g, h \in \mathbb{H}_n$.

Lemma 2.2 ([1]) *Suppose $g \in B(g_0, r)$. Then, for $k \in \mathbb{N}$, we have*

$$\frac{1}{(1 + \frac{2^k r}{\rho(g)})^N} \lesssim \frac{1}{(1 + \frac{2^k r}{\rho(g_0)})^{N/(k_0+1)}}.$$

The BMO space $BMO_\theta(\mathbb{H}_n, \rho)$ associated with the Schrödinger operator with $\theta \geq 0$ is defined as the set of all locally integrable functions b such that

$$\frac{1}{|B(g, r)|} \int_{B(g,r)} \left|b(h) - b_B\right| dh \leq C \left(1 + \frac{r}{\rho(g)}\right)^\theta$$

for all $g \in \mathbb{H}_n$ and $r > 0$, where $b_B = \frac{1}{|B|} \int_B b(h)\, dh$ (see [3]). The norm for $b \in BMO_\theta(\mathbb{H}_n, \rho)$, denoted by $[b]_\theta$, is given by the infimum of the constants in the inequality above. Clearly, $BMO(\mathbb{H}_n) \subset BMO_\theta(\mathbb{H}_n, \rho)$.

Let $\theta > 0$ and $0 < \nu < 1$. The seminorm on Campanato class $\Lambda_\nu^\theta(\rho)$ is

$$[b]_\nu^\theta := \sup_{g \in \mathbb{H}_n, r > 0} \frac{\frac{1}{|B(g,r)|^{1+\nu/Q}} \int_{B(g,r)} |b(h) - b_B|\, dh}{(1 + \frac{r}{\rho(g)})^\theta} < \infty.$$

The Lipschitz space associated with the Schrödinger operator (see [26]) consists of the functions f satisfying

$$\|f\|_{\mathrm{Lip}_\nu^\theta(\rho)} := \sup_{g\in\mathbb{H}_n, r>0} \frac{|f(g)-f(h)|}{|h^{-1}g|^\nu(1+\frac{|h^{-1}g|}{\rho(g)}+\frac{|h^{-1}g|}{\rho(h)})^\theta} < \infty.$$

It is easy to see that this space is exactly the Lipschitz space when $\theta = 0$.

Note that if $\theta = 0$ in (1.2), then $\Lambda_\nu^\theta(\rho)$ is the classical Campanato space; if $\nu = 0$, then $\Lambda_\nu^\theta(\rho)$ is the space $BMO_\theta(\rho)$; and if $\theta = 0$ and $\nu = 0$, then it is the John–Nirenberg space BMO.

The following embedding between $\mathrm{Lip}_\nu^\theta(\rho)$ and $\Lambda_\nu^\theta(\rho)$ was proved in [26, Theorem 5].

Lemma 2.3 ([26]) *Let $\theta > 0$ and $0 < \nu < 1$. Then we have the following embedding*:

$$\Lambda_\nu^\theta(\rho) \subseteq \mathrm{Lip}_\nu^\theta(\rho) \subseteq \Lambda_\nu^{(k_0+1)\theta}(\rho),$$

where k_0 is the constant appearing in Lemma 2.1.

We give some inequalities about the Campanato space associated with Schrödinger operator $\Lambda_\nu^\theta(\rho)$.

Lemma 2.4 ([26]) *Let $\theta > 0$ and $1 \le s < \infty$. If $b \in \Lambda_\nu^\theta(\rho)$, then there exists a constant $C > 0$ such that*

$$\left(\frac{1}{|B|}\int_B \left|b(h)-b_B\right|^s dh\right)^{1/s} \le C[b]_\nu^\theta r^\nu \left(1+\frac{r}{\rho(g)}\right)^{\theta'}$$

for all $B = B(g,r)$ with $g \in \mathbb{H}_n$ and $r > 0$, where $\theta' = (k_0+1)\theta$, and k_0 is the constant appearing in (2.1).

Let K_β be the kernel of $\mathcal{I}_\beta^L$. The following result gives an estimate of the kernel $K_\beta(g,y)$.

Lemma 2.5 ([4]) *If $V \in RH_{Q/2}$, then, for every N, there exists a constant C such that*

$$\left|K_\beta(g,y)\right| \le \frac{C}{(1+\frac{|h^{-1}g|}{\rho(g)})^N}\frac{1}{|h^{-1}g|^{Q-\beta}}. \tag{2.2}$$

Finally, we recall a relationship between essential supremum and essential infimum.

Lemma 2.6 ([41]) *Let f be a real-valued nonnegative measurable function on E. Then*

$$\left(\operatorname*{ess\,inf}_{g\in E} f(g)\right)^{-1} = \operatorname*{ess\,sup}_{g\in E}\frac{1}{f(g)}.$$

Lemma 2.7 *Let φ be a positive measurable function on $(0,\infty)$, $1 \le p < \infty$, $\alpha \ge 0$, and $V \in RH_q$, $q \ge 1$. If*

$$\sup_{t<r<\infty}\left(1+\frac{r}{\rho(e)}\right)^\alpha \frac{r^{-\frac{n}{p}}}{\varphi(r)} = \infty \quad \textit{for some } t > 0, \tag{2.3}$$

then $LM_{p,\varphi}^{\alpha,V}(\mathbb{H}_n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{H}_n$.

Lemma 2.8 ([1]) *Let φ be a positive measurable function on $(0,\infty)$, $1 \le p < \infty$, $\alpha \ge 0$, and $V \in RH_q$, $q \ge 1$.*

(i) *If*

$$\sup_{t<r<\infty}\left(1+\frac{r}{\rho(g)}\right)^{\alpha}\frac{r^{-\frac{Q}{p}}}{\varphi(r)} = \infty \quad \text{for some } t>0 \text{ and for all } g \in \mathbb{H}_n, \tag{2.4}$$

then $M^{\alpha,V}_{p,\varphi}(\mathbb{H}_n) = \Theta$.

(ii) *If*

$$\sup_{0<r<\tau}\left(1+\frac{r}{\rho(g)}\right)^{\alpha}\varphi(r)^{-1} = \infty \quad \text{for some } \tau>0 \text{ and for all } g \in \mathbb{H}_n, \tag{2.5}$$

then $M^{\alpha,V}_{p,\varphi}(\mathbb{H}_n) = \Theta$.

Remark 2.1 We denote by $\Omega^{\alpha,V}_{p,loc}$ the sets of all positive measurable functions φ on $(0,\infty)$ such that, for all $t>0$,

$$\left\|\left(1+\frac{r}{\rho(e)}\right)^{\alpha}\frac{r^{-\frac{n}{p}}}{\varphi(r)}\right\|_{L_\infty(t,\infty)} < \infty.$$

Moreover, we denote by $\Omega^{\alpha,V}_{p}$ (see [1]) the sets of all positive measurable functions φ on $(0,\infty)$ such that, for all $t>0$,

$$\sup_{g\in\mathbb{H}_n}\left\|\left(1+\frac{r}{\rho(g)}\right)^{\alpha}\frac{r^{-\frac{Q}{p}}}{\varphi(r)}\right\|_{L_\infty(t,\infty)} < \infty \quad \text{and} \quad \sup_{g\in\mathbb{H}_n}\left\|\left(1+\frac{r}{\rho(g)}\right)^{\alpha}\varphi(r)^{-1}\right\|_{L_\infty(0,t)} < \infty.$$

For the nontriviality of the spaces $LM^{\alpha,V}_{p,\varphi}(\mathbb{H}_n)$ and $M^{\alpha,V}_{p,\varphi}(\mathbb{H}_n)$, we always assume that $\varphi \in \Omega^{\alpha,V}_{p,\mathrm{loc}}$, $\varphi \in \Omega^{\alpha,V}_{p}$, respectively.

Remark 2.2 We denote by $\Omega^{\alpha,V}_{p,1}$ the set of all positive measurable functions φ on $\mathbb{H}_n \times (0,\infty)$ such that

$$\inf_{g\in\mathbb{H}_n}\inf_{r>\delta}\left(1+\frac{r}{\rho(g)}\right)^{-\alpha}\varphi(g,r) > 0 \quad \text{for some } \delta>0 \tag{2.6}$$

and

$$\lim_{r\to 0}\left(1+\frac{r}{\rho(g)}\right)^{\alpha}\frac{r^{Q/p}}{\varphi(g,r)} = 0.$$

For the nontriviality of the space $VM^{\alpha,V}_{p,\varphi}(\mathbb{H}_n)$, we always assume that $\varphi \in \Omega^{\alpha,V}_{p,1}$.

3 Proof of Theorem 1.1

We first prove the following conclusions.

Lemma 3.1 *Let $0<\nu<1$, $0<\beta+\nu<Q$, and $b \in \Lambda^{\theta}_{\nu}(\rho)$, then the following pointwise estimate holds*:

$$\left|\left[b,\mathcal{I}^{L}_{\beta}\right]f(g)\right| \lesssim [b]^{\theta}_{\nu} I_{\beta+\nu}\left(|f|\right)(g).$$

Proof Note that

$$[b,\mathcal{I}_\beta^L]f(g)=b(g)\mathcal{I}_\beta^L f(g)-\mathcal{I}_\beta^L(bf)(g)=\int_{\mathbb{H}_n}[b(g)-b(h)]K_\beta(g,h)f(h)\,dy.$$

If $b\in\Lambda_\nu^\theta(\rho)$, then from Lemma 2.5 we have

$$\begin{aligned}\left|[b,\mathcal{I}_\beta^L]f(g)\right| &\le \int_{\mathbb{H}_n}|b(g)-b(h)||K_\beta(g,h)||f(h)|\,dy\\ &\lesssim [b]_\nu^\theta\int_{\mathbb{H}_n}|h^{-1}g|^\nu|K_\beta(g,h)||f(h)|\,dy=[b]_\nu^\theta I_{\beta+\nu}(|f|)(g).\end{aligned}$$

□

From Lemma 3.1 we get the following:

Corollary 3.1 *Suppose $V\in RH_{q_1}$ with $q_1>Q/2$ and $b\in\Lambda_\nu^\theta(\rho)$ with $0<\nu<1$. Let $0<\beta+\nu<Q$, and let $1\le p<q<\infty$ satisfy $1/q=1/p-(\beta+\nu)/Q$. Then, for all f in $L_p(\mathbb{H}_n)$, we have*

$$\|[b,\mathcal{I}_\beta^L]f\|_{L_q}\lesssim\|f\|_{L_p}$$

when $p>1$ and

$$\|[b,\mathcal{I}_\beta^L]f\|_{WL_q}\lesssim\|f\|_{L_1}$$

when $p=1$.

To prove Theorem 1.1, we need the following new result.

Theorem 3.1 *Suppose $V\in RH_{q_1}$ with $q_1>Q/2$, $b\in\Lambda_\nu^\theta(\rho)$, $\theta>0$, $0<\nu<1$. Let $0<\beta+\nu<Q$, and let $1\le p<q<\infty$ satisfy $1/q=1/p-(\beta+\nu)/Q$. Then*

$$\|[b,\mathcal{I}_\beta^L]f\|_{L_q(B(g_0,r))}\lesssim\|I_{\beta+\nu}(|f|)\|_{L_q(B(g_0,r))}\lesssim r^{\frac{Q}{q}}\int_{2r}^\infty\frac{\|f\|_{L_p(B(g_0,t))}}{t^{\frac{Q}{q}}}\frac{dt}{t}$$

for all $f\in L_{\mathrm{loc}}^p(\mathbb{H}_n)$. Moreover, for $p=1$,

$$\|[b,\mathcal{I}_\beta^L f]\|_{WL_{\frac{Q}{Q-\beta-\nu}}(B(g_0,r))}\lesssim\|I_{\beta+\nu}(|f|)\|_{WL_{\frac{Q}{Q-\beta-\nu}}(B(g_0,r))}\lesssim r^{n-\beta}\int_{2r}^\infty\frac{\|f\|_{L_1(B(g_0,t))}}{t^{Q-\beta-\nu}}\frac{dt}{t}$$

for all $f\in L_{\mathrm{loc}}^1(\mathbb{H}_n)$.

Proof For arbitrary $g_0\in\mathbb{H}_n$, set $B=B(g_0,r)$ and $\lambda B=B(g_0,\lambda r)$ for any $\lambda>0$. We write f as $f=f_1+f_2$, where $f_1(h)=f(h)\chi_{B(g_0,2r)}(h)$, and $\chi_{B(g_0,2r)}$ denotes the characteristic function of $B(g_0,2r)$. Then

$$\begin{aligned}\|[b,\mathcal{I}_\beta^L]f\|_{L_q(B(g_0,r))} &\lesssim\|I_{\beta+\nu}(|f|)\|_{L_q(B(g_0,r))}\\ &\le\|I_{\beta+\nu}f_1\|_{L_q(B(g_0,r))}+\|I_{\beta+\nu}f_2\|_{L_q(B(g_0,r))}.\end{aligned}$$

Since $f_1 \in L_p(\mathbb{H}_n)$, from the boundedness of $I_{\beta+\nu}$ from $L_p(\mathbb{H}_n)$ to $L_q(\mathbb{H}_n)$ (see [38]) it follows that

$$\begin{aligned}\|I_{\beta+\nu}f_1\|_{L_q(B(g_0,r))} &\lesssim \|f\|_{L_p(B(g_0,2r))} \\ &\lesssim r^{\frac{Q}{q}}\|f\|_{L_p(B(g_0,2r))}\int_{2r}^{\infty}\frac{dt}{t^{\frac{Q}{q}+1}} \lesssim r^{\frac{Q}{q}}\int_{2r}^{\infty}\frac{\|f\|_{L_p(B(g_0,t))}}{t^{\frac{Q}{q}}}\frac{dt}{t}.\end{aligned} \tag{3.1}$$

To estimate $\|I_{\beta+\nu}f_2\|_{L_p(B(g_0,r))}$, obverse that $g \in B$ and $h \in (2B)^c$ imply $|h^{-1}g| \approx |h^{-1}g_0|$. Then by (2.2) we have

$$\sup_{g\in B}\bigl|I_{\beta+\nu}f_2(g)\bigr| \lesssim \int_{(2B)^c}\frac{|f(h)|}{|h^{-1}g_0|^{Q-\beta-\nu}}\,dh \lesssim \sum_{k=1}^{\infty}\bigl(2^{k+1}r\bigr)^{-n+\beta}\int_{2^{k+1}B}\bigl|f(h)\bigr|\,dh.$$

By Hölder's inequality we get

$$\begin{aligned}\sup_{g\in B}\bigl|I_{\beta+\nu}f_2(g)\bigr| &\lesssim \sum_{k=1}^{\infty}\|f\|_{L_p(2^{k+1}B)}\bigl(2^{k+1}r\bigr)^{-1-\frac{Q}{p}+\beta}\int_{2^k r}^{2^{k+1}r}dt \\ &\lesssim \sum_{k=1}^{\infty}\int_{2^k r}^{2^{k+1}r}\frac{\|f\|_{L_p(B(g_0,t))}}{t^{\frac{Q}{q}}}\frac{dt}{t} \lesssim \int_{2r}^{\infty}\frac{\|f\|_{L_p(B(g_0,t))}}{t^{\frac{Q}{q}}}\frac{dt}{t}.\end{aligned} \tag{3.2}$$

Then

$$\|I_{\beta+\nu}f_2\|_{L_q(B(g_0,r))} \lesssim r^{\frac{Q}{q}}\int_{2r}^{\infty}\frac{\|f\|_{L_p(B(g_0,t))}}{t^{\frac{Q}{q}}}\frac{dt}{t} \tag{3.3}$$

for $1 \le p < Q/\beta$. Therefore by (3.1) and (3.3) we get

$$\bigl\|I_{\beta+\nu}\bigl(|f|\bigr)\bigr\|_{L_q(B(g_0,r))} \lesssim r^{\frac{Q}{q}}\int_{2r}^{\infty}\frac{\|f\|_{L_p(B(g_0,t))}}{t^{\frac{Q}{q}}}\frac{dt}{t} \tag{3.4}$$

for $1 < p < Q/\beta$.

When $p = 1$, by the boundedness of $I_{\beta+\nu}$ from $L_1(\mathbb{H}_n)$ to $WL_{\frac{Q}{Q-\beta-\nu}}(\mathbb{H}_n)$ we get

$$\|I_{\beta+\nu}f_1\|_{WL_{\frac{Q}{Q-\beta-\nu}}(B(g_0,r))} \lesssim \|f\|_{L_1(B(g_0,2r))} \lesssim r^{Q-\beta-\nu}\int_{2r}^{\infty}\frac{\|f\|_{L_1(B(g_0,t))}}{t^{Q-\beta-\nu}}\frac{dt}{t}.$$

By (3.3) we have

$$\|I_{\beta+\nu}f_2\|_{WL_{\frac{Q}{Q-\beta-\nu}}(B(g_0,r))} \le \|I_{\beta+\nu}f_2\|_{L_{\frac{Q}{Q-\beta-\nu}}(B(g_0,2r))} \lesssim r^{Q-\beta-\nu}\int_{2r}^{\infty}\frac{\|f\|_{L_1(B(g_0,t))}}{t^{Q-\beta-\nu}}\frac{dt}{t}.$$

Then

$$\bigl\|I_{\beta+\nu}\bigl(|f|\bigr)\bigr\|_{WL_{\frac{Q}{Q-\beta-\nu}}(B(g_0,r))} \lesssim r^{Q-\beta-\nu}\int_{2r}^{\infty}\frac{\|f\|_{L_1(B(g_0,t))}}{t^{Q-\beta-\nu}}\frac{dt}{t}. \qquad \square$$

Proof of Theorem 1.1 From Lemma 2.6 we have

$$\frac{1}{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(g,s)s^{\frac{Q}{p}}} = \operatorname*{ess\,sup}_{t<s<\infty}\frac{1}{\varphi_1(g,s)s^{\frac{Q}{p}}}.$$

Since $\|f\|_{L_p(B(g_0,t))}$ is a nondecreasing function of t and $f \in M^{\alpha,V}_{p,\varphi_1}(\mathbb{H}_n)$, we have

$$\frac{(1+\frac{t}{\rho(g_0)})^\alpha \|f\|_{L_p(B(g_0,t))}}{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(g_0,s)s^{\frac{Q}{p}}} \lesssim \operatorname*{ess\,sup}_{t<s<\infty} \frac{(1+\frac{t}{\rho(g_0)})^\alpha \|f\|_{L_p(B(g_0,t))}}{\varphi_1(g_0,s)s^{\frac{Q}{p}}}$$
$$\lesssim \sup_{0<s<\infty} \frac{(1+\frac{s}{\rho(g_0)})^\alpha \|f\|_{L_p(B(g_0,s))}}{\varphi_1(g_0,s)s^{\frac{Q}{p}}} \lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}}.$$

Since $\alpha \geq 0$ and (φ_1,φ_2) satisfies condition (1.5), we have

$$\begin{aligned}
&\int_{2r}^\infty \frac{\|f\|_{L_p(B(g_0,t))}}{t^{\frac{Q}{q}}}\frac{dt}{t} \\
&= \int_{2r}^\infty \frac{(1+\frac{t}{\rho(g_0)})^\alpha \|f\|_{L_p(B(g_0,t))}}{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(g_0,s)s^{\frac{Q}{p}}} \frac{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(g_0,s)s^{\frac{Q}{p}}}{(1+\frac{t}{\rho(g_0)})^\alpha t^{\frac{Q}{q}}}\frac{dt}{t} \\
&\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}} \int_{2r}^\infty \frac{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(g_0,s)s^{\frac{Q}{p}}}{(1+\frac{t}{\rho(g_0)})^\alpha t^{\frac{Q}{q}}}\frac{dt}{t} \\
&\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}} \left(1+\frac{r}{\rho(g_0)}\right)^{-\alpha} \int_r^\infty \frac{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1(g_0,s)s^{\frac{Q}{p}}}{t^{\frac{Q}{q}}}\frac{dt}{t} \\
&\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}} \left(1+\frac{r}{\rho(g_0)}\right)^{-\alpha} \varphi_2(g_0,r).
\end{aligned} \tag{3.5}$$

Then by Theorem 3.1 we get

$$\begin{aligned}
\big\|[b,\mathcal{I}^L_\beta]f\big\|_{M^{\alpha,V}_{q,\varphi_2}} &\lesssim \big\|I_{\beta+\nu}(|f|)\big\|_{M^{\alpha,V}_{q,\varphi_2}} \\
&\lesssim \sup_{g_0\in\mathbb{H}_n,r>0}\left(1+\frac{r}{\rho(g_0)}\right)^\alpha \varphi_2(g_0,r)^{-1}r^{-Q/q}\big\|I_{\beta+\nu}(|f|)\big\|_{L_p(B(g_0,r))} \\
&\lesssim \sup_{g_0\in\mathbb{H}_n,r>0}\left(1+\frac{r}{\rho(g_0)}\right)^\alpha \varphi_2(g_0,r)^{-1}\int_{2r}^\infty \frac{\|f\|_{L_p(B(g_0,t))}}{t^{\frac{Q}{q}}}\frac{dt}{t} \\
&\lesssim \|f\|_{M^{\alpha,V}_{p,\varphi_1}}.
\end{aligned}$$

Let $q = \frac{Q}{Q-\beta-\nu}$. Similarly to estimates (3.5), we have

$$\int_{2r}^\infty \frac{\|f\|_{L_1(B(g_0,t))}}{t^{Q-\beta-\nu}}\frac{dt}{t} \lesssim \|f\|_{M^{\alpha,V}_{1,\varphi_1}}\left(1+\frac{r}{\rho(g_0)}\right)^{-\alpha}\varphi_2(g_0,r).$$

Thus by Theorem 3.1 we get

$$\begin{aligned}
&\big\|[b,\mathcal{I}^L_\beta]f\big\|_{WM^{\alpha,V}_{\frac{Q}{Q-\beta-\nu},\varphi_2}} \\
&\lesssim \big\|I_{\beta+\nu}(|f|)\big\|_{WM^{\alpha,V}_{\frac{Q}{Q-\beta-\nu},\varphi_2}} \\
&\lesssim \sup_{g_0\in\mathbb{H}_n,r>0}\left(1+\frac{r}{\rho(g_0)}\right)^\alpha \varphi_2(g_0,r)^{-1}r^{\beta-n}\big\|I_{\beta+\nu}(|f|)\big\|_{WL_{\frac{Q}{Q-\beta-\nu}}(B(g_0,r))}
\end{aligned}$$

$$\lesssim \sup_{g_0\in\mathbb{H}_n,r>0}\left(1+\frac{r}{\rho(g_0)}\right)^{\alpha}\varphi_2(g_0,r)^{-1}\int_{2r}^{\infty}\frac{\|f\|_{L_1(B(g_0,t))}}{t^{Q-\beta-\nu}}\frac{dt}{t}$$

$$\lesssim \|f\|_{M^{\alpha,V}_{1,\varphi_1}}. \qquad \square$$

4 Proof of Theorem 1.2

We derive the statement from estimate (3.4). The estimation of the norm of the operator, that is, the boundedness in the nonvanishing space, immediately follows by Theorem 1.1. So we only have to prove that

$$\lim_{r\to 0}\sup_{g\in\mathbb{H}_n}\mathfrak{A}^{\alpha,V}_{p,\varphi_1}(f;g,r)=0 \quad\Rightarrow\quad \lim_{r\to 0}\sup_{g\in\mathbb{H}_n}\mathfrak{A}^{\alpha,V}_{q,\varphi_2}\big([b,\mathcal{I}^L_\beta]f;g,r\big)=0 \tag{4.1}$$

and

$$\lim_{r\to 0}\sup_{g\in\mathbb{H}_n}\mathfrak{A}^{\alpha,V}_{1,\varphi_1}(f;g,r)=0 \quad\Rightarrow\quad \lim_{r\to 0}\sup_{g\in\mathbb{H}_n}\mathfrak{A}^{W,\alpha,V}_{Q/(Q-\beta),\varphi_2}\big([b,\mathcal{I}^L_\beta]f;g,r\big)=0. \tag{4.2}$$

To show that $\sup_{g\in\mathbb{H}_n}(1+\frac{r}{\rho(g)})^{\alpha}\varphi_2(g,r)^{-1}r^{-Q/p}\|[b,\mathcal{I}^L_\beta]f\|_{L_q(B(g,r))}<\varepsilon$ for small r, we split the right-hand side of (3.4):

$$\left(1+\frac{r}{\rho(g)}\right)^{\alpha}\varphi_2(g,r)^{-1}r^{-Q/p}\big\|[b,\mathcal{I}^L_\beta]f\big\|_{L_q(B(g,r))}\le C\big[I_{\delta_0}(g,r)+J_{\delta_0}(g,r)\big], \tag{4.3}$$

where $\delta_0>0$ (we may take $\delta_0>1$), and

$$I_{\delta_0}(g,r):=\frac{(1+\frac{r}{\rho(g)})^{\alpha}}{\varphi_2(g,r)}\int_r^{\delta_0}t^{-\frac{Q}{q}-1}\|f\|_{L_p(B(g,t))}\,dt$$

and

$$J_{\delta_0}(g,r):=\frac{(1+\frac{r}{\rho(g)})^{\alpha}}{\varphi_2(g,r)}\int_{\delta_0}^{\infty}t^{-\frac{Q}{q}-1}\|f\|_{L_p(B(g,t))}\,dt,$$

and we suppose that $r<\delta_0$. We use the fact that $f\in VM^{\alpha,V}_{p,\varphi_1}(\mathbb{H}_n)$ and choose any fixed $\delta_0>0$ such that

$$\sup_{g\in\mathbb{H}_n}\left(1+\frac{t}{\rho(g)}\right)^{\alpha}\varphi_1(g,t)^{-1}t^{-n/p}\|f\|_{L_p(B(g,t))}<\frac{\varepsilon}{2CC_0},$$

where C and C_0 are constants from (1.6) and (4.3). This allows us to estimate the first term uniformly in $r\in(0,\delta_0)$:

$$\sup_{g\in\mathbb{H}_n}CI_{\delta_0}(g,r)<\frac{\varepsilon}{2},\quad 0<r<\delta_0.$$

The estimation of the second term now can be made by the choice of r sufficiently small. Indeed, thanks to condition (2.6), we have

$$J_{\delta_0}(g,r)\le c_{\sigma_0}\frac{(1+\frac{r}{\rho(g)})^{\alpha}}{\varphi_1(g,r)}\|f\|_{VM^{\alpha,V}_{p,\varphi_1}},$$

where c_{σ_0} is the constant from (1.3). Then, by (2.6) it suffices to choose r small enough such that

$$\sup_{x\in\mathbb{R}^n} \frac{(1+\frac{r}{\rho(g)})^{\alpha}}{\varphi_2(g,r)} \leq \frac{\varepsilon}{2c_{\sigma_0}\|f\|_{VM^{\alpha,V}_{p,\varphi_1}}},$$

which completes the proof of (4.1).

The proof of (4.2) is similar to that of (4.1).

5 Conclusions

In this paper, we study the boundedness of the commutators $[b,\mathcal{I}^L_\beta]$ with $b \in \Lambda^\theta_\nu(\rho)$ on the central generalized Morrey spaces $LM^{\alpha,V}_{p,\varphi}(\mathbb{H}_n)$, generalized Morrey spaces $M^{\alpha,V}_{p,\varphi}(\mathbb{H}_n)$, and vanishing generalized Morrey spaces $VM^{\alpha,V}_{p,\varphi}(\mathbb{H}_n)$ associated with the Schrödinger operator. When b belongs to $\Lambda^\theta_\nu(\rho)$ with $\theta > 0$, $0 < \nu < 1$ and (φ_1,φ_2) satisfies some conditions, we show that the commutator operator $[b,\mathcal{I}^L_\beta]$ is bounded from $LM^{\alpha,V}_{p,\varphi_1}(\mathbb{H}_n)$ to $LM^{\alpha,V}_{q,\varphi_2}(\mathbb{H}_n)$, from $M^{\alpha,V}_{p,\varphi_1}(\mathbb{H}_n)$ to $M^{\alpha,V}_{q,\varphi_2}(\mathbb{H}_n)$, and from $VM^{\alpha,V}_{p,\varphi_1}(\mathbb{H}_n)$ to $VM^{\alpha,V}_{q,\varphi_2}(\mathbb{H}_n)$, $1/p - 1/q = (\beta+\nu)/Q$.

Our result about the boundedness of $[b,\mathcal{I}^L_\beta]$ with $b \in \Lambda^\theta_\nu(\rho)$ from $LM^{\alpha,V}_{p,\varphi_1}(\mathbb{H}_n)$ to $LM^{\alpha,V}_{q,\varphi_2}(\mathbb{H}_n)$ (Theorem 1.1) is based on the local estimate for the commutators $[b,\mathcal{I}^L_\beta]$ (Theorem 3.1).

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
This work was carried out in collaboration between all authors. VSG raised these interesting problems in the research. VSG, AA, and FMN proved the theorems, interpreted the results, and wrote the paper. All authors defined the research theme and read and approved the manuscript.

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