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Some Characterizations of Lipschitz Spaces via Commutators on Generalized Orlicz–Morrey Spaces

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Abstract. In this paper, we give some new characterizations of the Lipschitz spaces via the boundedness of commutators associated with the fractional maximal operator, Riesz potential and Calderón–Zygmund operator on generalized Orlicz–Morrey spaces.

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1. Introduction

A locally integrable function f is said to be in $BMO(\mathbb{R}^n)$ if the following seminorm is finite:

$$\|f\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| \mathrm{d}y,$$

where

$$f_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) \mathrm{d}y.$$

Here and everywhere in the sequel $B(x,r)$ is the ball in $\mathbb{R}^n$ of radius r centered at x and $|B(x,r)| = v_n r^n$ is its Lebesgue measure, where v_n is the volume of the unit ball in $\mathbb{R}^n$. Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere of $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $\mathrm{d}\sigma = \mathrm{d}\sigma(x')$, $x' = x/|x|$.

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Let $0 < \alpha < n$. The fractional maximal operator M_α and the Riesz potential operator I_α are defined by

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |f(y)|\mathrm{d}y, \;\; I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\mathrm{d}y.$$

If $\alpha = 0$, then $M \equiv M_0$ is the well-known Hardy–Littlewood maximal operator. Recall that, for $0 < \alpha < n$,

$$M_\alpha f(x) \le \upsilon_n^{\frac{\alpha}{n}-1} I_\alpha(|f|)(x).$$

We also deal with Calderón–Zygmund singular integral operator T defined by

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y)f(y)\mathrm{d}y,$$

where $K \in C^\infty(S^{n-1})$ is a Calderón–Zygmund kernel that satisfies

$$K(x) = K(x')/|x|^n \quad \text{for } |x| \neq 0 \tag{1.1}$$

and

$$\int_{S^{n-1}} K(x')\, \mathrm{d}\sigma(x') = 0.$$

A well-known result due to Coifman, Rochberg and Weiss [6] (see also [16]) states that the commutator

$$[b,T]f := bT(f) - T(bf)$$

is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if $b \in BMO(\mathbb{R}^n)$. In 1978, Janson [16] gave some characterizations of the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ (see Definition 4.1) via commutator $[b,T]$ and proved that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)(0 < \beta < 1)$ if and only if $[b,T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1 < p < n/\beta$ and $1/p - 1/q = \beta/n$ (see also Paluszyński [18]).

The commutators generated by a suitable function b and the operators M_α and I_α are formally defined by

$$[b, M_\alpha]f = bM_\alpha(f) - M_\alpha(bf), \;\; [b, I_\alpha]f = bI_\alpha(f) - I_\alpha(bf),$$

respectively.

Given a locally integrable function b, the fractional maximal commutator operator $M_{b,\alpha}$ is defined by

$$M_{b,\alpha}(f)(x) := \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |b(x)-b(y)||f(y)|\mathrm{d}y.$$

If $\alpha = 0$, then $M_{b,0} \equiv M_b$ is the sublinear maximal commutator operator.

For a function b defined on $\mathbb{R}^n$, we denote

$$b^-(x) := \begin{cases} 0\,, & \text{if } b(x) \ge 0 \\ |b(x)|, & \text{if } b(x) < 0 \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

The following relations between $[b, M_\alpha]$ and $M_{b,\alpha}$ are valid.

Let b be any non-negative locally integrable function. Then

$$|[b, M_\alpha]f(x)| \leq M_{b,\alpha}(f)(x), \qquad x \in \mathbb{R}^n,\ 0 \leq \alpha < n \tag{1.2}$$

holds for all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

If b is any locally integrable function on $\mathbb{R}^n$, then

$$|[b, M_\alpha]f(x)| \leq M_{b,\alpha}(f)(x) + 2b^-(x)M_\alpha f(x), \qquad x \in \mathbb{R}^n,\ 0 \leq \alpha < n \tag{1.3}$$

holds for all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ (see, for example, [2,22]).

The classical result by Hardy–Littlewood–Sobolev states that the operator I_α is of weak type $(p, np/(n - \alpha p))$ if $1 \leq p < n/\alpha$ and of strong type $(p, np/(n - \alpha p))$ if $1 < p < n/\alpha$. Also the operator M_α is of weak type $(p, np/(n - \alpha p))$ if $1 \leq p \leq n/\alpha$ and of strong type $(p, np/(n - \alpha p))$ if $1 < p \leq n/\alpha$.

Around the 1970s, the Hardy–Littlewood–Sobolev theorem is extended from Lebesgue spaces to Morrey spaces by Spanne [19] and Adams [1], respectively. Although Adams type theorems provide a stronger estimate, theorems of Spanne type with a weaker estimate have a wider range of applicability. For more details, we refer to survey paper [17].

Commutators of classical operators of harmonic analysis play an important role in various topics of analysis and PDE, see for instance [4–7,16].

A natural step in the theory of functions spaces was to study generalized Orlicz–Morrey spaces $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$, where the "Morrey-type measuring" of regularity of functions is realized with respect to the Orlicz norm over balls instead of the Lebesgue one.

Adams and Spanne type results for fractional maximal operator M_α and Riesz potential I_α in generalized Orlicz–Morrey spaces were obtained in [10,11,13,14]. In this paper, as an application of these results, we give some new characterizations of the Lipschitz spaces via the boundedness of commutators associated with the operators M_α, I_α and T on generalized Orlicz–Morrey spaces.

The structure of the remaining part of the present paper is as follows. Section 2 provides the definitions and some preliminaries on Young functions, Orlicz spaces and generalized Orlicz–Morrey spaces. In Sect. 3, we present Adams and Spanne type results for the operators M_α and I_α in generalized Orlicz–Morrey spaces which were obtained in [10,11,13,14]. As an application of these results, we consider the boundedness of $[b, I_\alpha]$, $M_{b,\alpha}$, $[b, M_\alpha]$ and $[b, T]$ on generalized Orlicz–Morrey spaces when b belongs to the Lipschitz space, by which some new characterizations of the Lipschitz spaces are given in Sect. 4.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Preliminaries

We recall the definition of Young functions.

Definition 2.1. A function $\Phi\colon [0,\infty) \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim_{r\to+0}\Phi(r) = \Phi(0) = 0$ and $\lim_{r\to\infty}\Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0,\infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \geq s$. The set of Young functions such that

$$0 < \Phi(r) < \infty \qquad \text{for} \qquad 0 < r < \infty$$

will be denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to itself.

For a Young function Φ and $0 \leq s \leq \infty$, let

$$\Phi^{-1}(s) = \inf\{r \geq 0 : \Phi(r) > s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. We note that

$$\Phi(\Phi^{-1}(r)) \leq r \leq \Phi^{-1}(\Phi(r)) \quad \text{for } 0 \leq r < \infty.$$

It is well known that

$$r \leq \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \leq 2r \qquad \text{for } r \geq 0, \tag{2.1}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\}, & r \in [0,\infty) \\ \infty \qquad\qquad\qquad\qquad\qquad\quad , & r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \leq k\Phi(r) \text{ for } r > 0$$

for some $k > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2k}\Phi(kr), \qquad r \geq 0$$

for some $k > 1$.

Definition 2.2 (*Orlicz space*). For a Young function Φ, the set

$$L^{\Phi}(\mathbb{R}^n) = \left\{ f \in L^1_{\rm loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)\mathrm{d}x < \infty \text{ for some } k > 0 \right\}$$

is called Orlicz space. The space $L^{\Phi}_{\rm loc}(\mathbb{R}^n)$ is defined as the set of all functions f such that $f\chi_B \in L^{\Phi}(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$.

$L^{\Phi}(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}} = \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)\mathrm{d}x \leq 1 \right\}.$$

If $\Phi(r) = r^p$, $1 \leq p < \infty$, then $L^{\Phi}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\Phi(r) = 0$, $(0 \leq r \leq 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L^{\Phi}(\mathbb{R}^n) = L^{\infty}(\mathbb{R}^n)$.

Lemma 2.1 [8]. *For a Young function Φ and a ball B in $\mathbb{R}^n$, the following inequality is valid*

$$\|f\|_{L^1(B)} \le 2|B|\Phi^{-1}\left(|B|^{-1}\right)\|f\|_{L^\Phi(B)},$$

where $\|f\|_{L^\Phi(B)} = \|f\chi_B\|_{L^\Phi}$.

We find it convenient to define generalized Orlicz–Morrey space as follows.

Definition 2.3. Let $\varphi(r)$ be a positive measurable function on $(0,\infty)$ and Φ any Young function. We denote by $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ the generalized Orlicz–Morrey space, the space of all functions $f \in L^\Phi_{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{\mathcal{M}^{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(r)^{-1}\Phi^{-1}(r^{-n})\|f\|_{L^\Phi(B(x,r))} < \infty.$$

A function $\varphi : (0,\infty) \to (0,\infty)$ is said to be almost increasing (resp. almost decreasing) if there exists a constant $C > 0$ such that

$$\varphi(r) \le C\varphi(s) \qquad (\text{resp. } \varphi(r) \ge C\varphi(s)) \quad \text{for } r \le s.$$

For a Young function Φ, we denote by $\mathcal{G}_\Phi$ the set of all almost decreasing functions $\varphi : (0,\infty) \to (0,\infty)$ such that $t \in (0,\infty) \mapsto \frac{1}{\Phi^{-1}(t^{-n})}\varphi(t)$ is almost increasing.

Lemma 2.2 [10]. *Let B be a ball in $\mathbb{R}^n$. If $\varphi \in \mathcal{G}_\Phi$, then there exists $C > 0$ such that*

$$\frac{1}{\varphi(r_B)} \le \|\chi_B\|_{\mathcal{M}^{\Phi,\varphi}} \le \frac{C}{\varphi(r_B)},$$

where r_B denotes the radius of the ball.

3. Auxiliary Results

In the next sections where we prove our main estimates, we use the following results.

Theorem 3.1 [10] (Adams type result).

Let $0 < \alpha < n$, $\Phi \in \mathcal{Y}$, $\gamma \in (0,1)$ and $\eta(t) \equiv \varphi(t)^\gamma$ and $\Psi(t) \equiv \Phi(t^{1/\gamma})$.

1. *If $\Phi \in \nabla_2$ and $\varphi(t)$ satisfies*

$$\sup_{r<t<\infty} \Phi^{-1}\big(t^{-n}\big) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi(s)}{\Phi^{-1}\big(s^{-n}\big)} \le C\,\varphi(r), \tag{3.1}$$

then the condition

$$t^\alpha\varphi(t) + \int_t^\infty r^\alpha\,\varphi(r)\frac{\mathrm{d}r}{r} \le C\varphi(t)^\gamma$$

for all $t > 0$, where $C > 0$ does not depend on t, is sufficient for the boundedness of I_α from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

2. *If $\varphi \in \mathcal{G}_\Phi$, then the condition*

$$t^\alpha\varphi(t) \le C\varphi(t)^\gamma \tag{3.2}$$

for all $t > 0$, where $C > 0$ does not depend on t, is necessary for the boundedness of I_α from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

3. *Let $\Phi \in \nabla_2$. If $\varphi \in \mathcal{G}_\Phi$ satisfies the regularity condition*

$$\int_t^\infty r^\alpha \varphi(r) \frac{\mathrm{d}r}{r} \le C t^\alpha \varphi(t)$$

for all $t > 0$, where $C > 0$ does not depend on t, then the condition (3.2) is necessary and sufficient for the boundedness of I_α from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

Theorem 3.2 [10,13] (Spanne type result).

Let $\Phi, \Psi \in \mathcal{Y}$ and $0 < \alpha < n$.

1. *Let $\Phi \in \nabla_2$. If the functions (Φ, Ψ) satisfy the condition*

$$r^\alpha \Phi^{-1}\big(r^{-n}\big) + \int_r^\infty \Phi^{-1}\big(t^{-n}\big) t^\alpha \frac{\mathrm{d}t}{t} \le C \Psi^{-1}\big(r^{-n}\big), \tag{3.3}$$

then the condition

$$\int_t^\infty \operatorname*{ess\,inf}_{r<s<\infty} \frac{\varphi_1(s)}{\Phi^{-1}\big(s^{-n}\big)} \Psi^{-1}\big(r^{-n}\big) \frac{\mathrm{d}r}{r} \le C\, \varphi_2(t) \tag{3.4}$$

for all $t > 0$, where $C > 0$ does not depend on t, is sufficient for the boundedness of I_α from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

2. *If the function $\varphi_1 \in \mathcal{G}_\Phi$, then the condition*

$$t^\alpha \varphi_1(t) \le C \varphi_2(t) \tag{3.5}$$

for all $t > 0$, where $C > 0$ does not depend on t, is necessary for the boundedness of I_α from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

3. *Let $\Phi \in \nabla_2$. Let also the functions (Φ, Ψ) satisfy the condition (3.3). If $\varphi_1 \in \mathcal{G}_\Phi$ satisfies the regularity type condition*

$$\int_t^\infty \frac{\Psi^{-1}\big(r^{-n}\big)}{\Phi^{-1}\big(r^{-n}\big)} \varphi_1(r) \frac{\mathrm{d}r}{r} \le C t^\alpha \varphi_1(t)$$

for all $t > 0$, where $C > 0$ does not depend on t, then the condition (3.5) is necessary and sufficient for the boundedness of I_α from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

Theorem 3.3 [11] (Adams type result).

Let $0 < \alpha < n$, $\Phi \in \mathcal{Y}$, $\gamma \in (0,1)$, $\eta(t) \equiv \varphi(t)^\gamma$ and $\Psi(t) \equiv \Phi(t^{1/\gamma})$.

1. *If $\Phi \in \nabla_2$ and $\varphi(t)$ satisfies (3.1), then the condition*

$$t^\alpha \varphi(t) + \sup_{t<r<\infty} r^\alpha \varphi(r) \le C \varphi(t)^\gamma$$

for all $t > 0$, where $C > 0$ does not depend on t, is sufficient for the boundedness of M_α from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

2. *If $\varphi \in \mathcal{G}_\Phi$, then the condition (3.2) is necessary for the boundedness of M_α from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.*

3. *Let $\Phi \in \nabla_2$ and $\varphi \in \mathcal{G}_\Phi$. Then, the condition (3.2) is necessary and sufficient for the boundedness of M_α from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.*

Theorem 3.4 [14] (Spanne type result).
Let $\Phi, \Psi \in \mathcal{Y}$ *and* $0 \le \alpha < n$.

1. *Let* $\Phi \in \nabla_2$. *If the functions* (Φ, Ψ) *satisfy the condition*

$$r^{\alpha}\Phi^{-1}\big(r^{-n}\big) \le C\Psi^{-1}\big(r^{-n}\big), \tag{3.6}$$

then the condition

$$\sup_{t<r<\infty} \Psi^{-1}\big(r^{-n}\big) \operatorname*{ess\,inf}_{r<s<\infty} \frac{\varphi_1(s)}{\Phi^{-1}\big(s^{-n}\big)} \le C\,\varphi_2(t) \tag{3.7}$$

for all $t > 0$, *where* $C > 0$ *does not depend on* t, *is sufficient for the boundedness of* M_α *from* $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

2. *If the function* $\varphi_1 \in \mathcal{G}_\Phi$, *then the condition* (3.5) *is necessary for the boundedness of* M_α *from* $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

3. *Let* $\Phi \in \nabla_2$. *Let also the functions* (Φ, Ψ) *satisfy the condition* (3.6). *If* $\varphi_1 \in \mathcal{G}_\Phi$ *satisfies the condition*

$$\sup_{t<r<\infty} \frac{\Psi^{-1}\big(r^{-n}\big)}{\Phi^{-1}\big(r^{-n}\big)}\,\varphi_1(r) \le Ct^{\alpha}\varphi_1(t)$$

for all $t > 0$, *where* $C > 0$ *does not depend on* t, *then the condition* (3.5) *is necessary and sufficient for the boundedness of* M_α *from* $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

4. Main Results

In this section, as an application of theorems of the previous section we consider the boundedness of $[b, I_\alpha]$, $M_{b,\alpha}$, $[b, M_\alpha]$ and $[b, T]$ on generalized Orlicz–Morrey spaces when b belongs to the Lipschitz space, by which some new characterizations of the Lipschitz spaces are given. Such a characterization was given in [23] for the boundedness of M_b on Lebesgue and Morrey spaces.

Definition 4.1. Let $0 < \beta < 1$, we say a function b belongs to the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ if there exists a constant C such that for all $x, y \in \mathbb{R}^n$,

$$|b(x) - b(y)| \le C|x - y|^{\beta}.$$

The smallest such constant C is called the $\dot{\Lambda}_\beta(\mathbb{R}^n)$ norm of b and is denoted by $\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}$.

To prove the theorems, we need auxiliary results. The first one is the following characterizations of Lipschitz space, which is due to DeVore and Sharply [9].

Lemma 4.1. *Let* $0 < \beta < 1$, *we have*

$$\|f\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \approx \sup_{B} \frac{1}{|B|^{1+\beta/n}} \int_B |f(x) - f_B| \mathrm{d}x.$$

Lemma 4.2 [10]. *Let* $0 < \beta < 1$, $0 < \alpha$, $\alpha + \beta < n$ *and* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, *then the following pointwise estimate holds:*

$$|[b, I_\alpha](f)(x)| \lesssim \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} I_{\alpha+\beta}(|f|)(x).$$

The following theorem is valid.

Theorem 4.1 (Adams type result). *Let* $0 < \beta, \gamma < 1$, $0 < \alpha$, $\alpha + \beta < n$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $\Phi \in \mathcal{Y}$, $\eta(t) \equiv \varphi(t)^\gamma$ *and* $\Psi(t) \equiv \Phi(t^{1/\gamma})$.

1. *If* $\Phi \in \nabla_2$ *and* $\varphi(t)$ *satisfies* (3.1) *and*

$$\int_t^\infty r^{\alpha+\beta}\,\varphi(r)\frac{\mathrm{d}r}{r} \le Ct^{\alpha+\beta}\varphi(t), \tag{4.1}$$

$$t^{\alpha+\beta}\varphi(t) \le C\varphi(t)^\gamma \tag{4.2}$$

hold for all $t > 0$, *where* $C > 0$ *does not depend on* t, *then the condition* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *is sufficient for the boundedness of* $[b, I_\alpha]$ *from* $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

2. *If* $\varphi \in \mathcal{G}_\Phi$ *and the condition*

$$\varphi(t)^\gamma \le Ct^{\alpha+\beta}\varphi(t) \tag{4.3}$$

holds for all $t > 0$, *where* $C > 0$ *does not depend on* t, *then the condition* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *is necessary for the boundedness of* $[b, I_\alpha]$ *from* $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

3. *If* $\Phi \in \nabla_2$, $\varphi \in \mathcal{G}_\Phi$, *condition* (4.1) *holds and* $\varphi(t)^\gamma \approx t^{\alpha+\beta}\varphi(t)$, *then the condition* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *is necessary and sufficient for the boundedness of* $[b, I_\alpha]$ *from* $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

Proof. (1) The first statement of the theorem follows from Theorem 3.1 and Lemma 4.2.

(2) We shall now prove the second part. We use the idea given in [16] (see also [12,18]). Choose $z_0 \in \mathbb{R}^n$ and $\delta > 0$ such that in the neighborhood $\{z : |z - z_0| < \sqrt{n}\delta\}$, function $|z|^{n-\alpha}$ can be represented as a Fourier series which absolutely converges. That is

$$|z|^{n-\alpha} = \sum_{n=0}^{\infty} a_n e^{iv_n \cdot z}.$$

Let $z_1 = \frac{z_0}{\delta}$. For any ball $B = B(x_0, r)$, let $y_0 = x_0 - 2rz_1$ and $B' = B(y_0, r)$. Then for $x \in B$ and $y \in B'$, we have that

$$\left|\frac{x-y}{2r} - z_1\right| \le \left|\frac{x-x_0}{2r}\right| + \left|\frac{y-y_0}{2r}\right| \le 1.$$

Now set $s(x) = sgn(b(x) - b_{B'})$, then

$$\begin{aligned}
&\int_B |b(x) - b_{B'}|\mathrm{d}x = \int_B (b(x) - b_{B'})s(x)dx \\
&= \frac{1}{|B'|}\int_B\int_{B'}(b(x) - b(y))s(x)\mathrm{d}y\mathrm{d}x \\
&\approx 2^{n-\alpha}\delta^{\alpha-n}r^{-\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{b(x)-b(y)}{|x-y|^{n-\alpha}}\left|\frac{\delta(x-y)}{2r}\right|^{n-\alpha} s(x)\chi_B(x)\chi_{B'}(y)\mathrm{d}y\mathrm{d}x \\
&\approx r^{-\alpha}\sum_{n=0}^{\infty} a_n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{b(x)-b(y)}{|x-y|^{n-\alpha}} e^{iv_n\cdot\frac{\delta}{2r}(x-y)}s(x)\chi_B(x)\chi_{B'}(y)\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

Taking

$$g_n(y) = e^{-i(\delta/2r)v_n\cdot y}\chi_{B'}(y) \text{ and } h_n(x) = e^{i(\delta/2r)v_n\cdot x}s(x)\chi_B(x),$$

we obtain

$$\begin{aligned}\int_B |b(x)-b_{B'}|\mathrm{d}x &\approx r^{-\alpha}\sum_{n=0}^{\infty} a_n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{b(x)-b(y)}{|x-y|^{n-\alpha}} g_n(y)h_n(x)\mathrm{d}y\mathrm{d}x\\ &\le Cr^{-\alpha}\sum_{n=0}^{\infty}|a_n|\int_{\mathbb{R}^n}|[b,I_\alpha]g_n(x)||h_n(x)|\mathrm{d}x\\ &= Cr^{-\alpha}\sum_{n=0}^{\infty}|a_n|\int_B |[b,I_\alpha]g_n(x)|dx.\end{aligned}$$

Applying Lemma 2.1, we have

$$\begin{aligned}&\int_B |[b,I_\alpha]g_n(x)|dx \le 2|B|\Psi^{-1}\left(|B|^{-1}\right)\|[b,I_\alpha]g_n\|_{L^\Psi(B)}\\ &\quad\lesssim |B|\eta(r)\|[b,I_\alpha]g_n\|_{\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)} \lesssim r^n\eta(r)\|g_n\|_{\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)} \lesssim r^n\varphi(r)^{\gamma-1}\\ &\quad\lesssim r^{n+\alpha+\beta},\end{aligned}$$

since $\|g_n\|_{\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)} \lesssim \varphi(r)^{-1}$ by Lemma 2.2.

Thus, we have obtained

$$\frac{1}{|B|^{1+\frac{\beta}{n}}}\int_B |b(x)-b_B|\mathrm{d}x \le \frac{2}{|B|^{1+\frac{\beta}{n}}}\int_B |b(x)-b_{B'}|\mathrm{d}x \lesssim 1$$

which completes the proof of Theorem by Lemma 4.1.

(3) The third statement of the theorem follows from the first and second parts of the theorem. □

Remark 4.1. The condition $\varphi \in \mathcal{G}_\Phi$ implies the condition (3.1). Therefore, we do not need the hypothesis that φ satisfies (3.1) in the third part of the Theorem 4.1.

The following theorem is valid.

Theorem 4.2 (Spanne type result). *Let $0<\beta<1$, $0<\alpha$, $\alpha+\beta<n$, $b\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\Phi,\Psi\in\mathcal{Y}$.*

1. *Let $\Phi\in\nabla_2$, (3.4) holds and (Φ,Ψ) satisfy the condition*

$$r^{\alpha+\beta}\Phi^{-1}\big(r^{-n}\big) + \int_r^\infty \Phi^{-1}\big(t^{-n}\big)t^{\alpha+\beta}\frac{\mathrm{d}t}{t} \le C\Psi^{-1}\big(r^{-n}\big) \tag{4.4}$$

holds for all $r>0$, where $C>0$ does not depend on r, then the condition $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$ is sufficient for the boundedness of $[b,I_\alpha]$ from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

2. *If $\varphi_1\in\mathcal{G}_\Phi$ and the condition*

$$\varphi_2(t) \le Ct^{\alpha+\beta}\varphi_1(t) \tag{4.5}$$

holds for all $t>0$, where $C>0$ does not depend on t, then the condition $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary for the boundedness of $[b,I_\alpha]$ from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

3. *Let* $\Phi \in \nabla_2$*, condition* (4.4) *holds and* $\varphi_2(t) \approx t^{\alpha+\beta}\varphi_1(t)$*. If* $\varphi_1 \in \mathcal{G}_\Phi$ *satisfies the regularity type condition*

$$\int_t^\infty \frac{\Psi^{-1}\big(r^{-n}\big)}{\Phi^{-1}\big(r^{-n}\big)}\, \varphi_1(r)\frac{\mathrm{d}r}{r} \le C t^{\alpha+\beta}\varphi_1(t) \tag{4.6}$$

for all $t > 0$*, where* $C > 0$ *does not depend on* t*, then the condition* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *is necessary and sufficient for the boundedness of* $[b, I_\alpha]$ *from* $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

Proof. Similar to the proof of Theorem 4.1, we can obtain Theorem 4.2. □

Lemma 4.3 [15]. *Let* $0 < \beta < 1$, $0 \le \alpha < n$, $0 < \alpha + \beta < n$ *and* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$*, then the following pointwise estimate holds:*

$$M_{b,\alpha} f(x) \lesssim \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} M_{\alpha+\beta} f(x).$$

The following theorem is valid.

Theorem 4.3 (Adams type result). *Let* $0 < \beta, \gamma < 1$, $0 < \alpha$, $\alpha + \beta < n$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $\Phi \in \mathcal{Y}$, $\eta(t) \equiv \varphi(t)^\gamma$ *and* $\Psi(t) \equiv \Phi(t^{1/\gamma})$.

1. *If* $\Phi \in \nabla_2$ *and* $\varphi(t)$ *satisfies* (3.1), (4.2) *and*

$$\sup_{t<r<\infty} r^{\alpha+\beta}\, \varphi(r) \le C\varphi(t)^\gamma \tag{4.7}$$

holds for all $t > 0$*, where* $C > 0$ *does not depend on* t*, then the condition* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *is sufficient for the boundedness of* $M_{b,\alpha}$ *from* $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

2. *If* $\varphi \in \mathcal{G}_\Phi$ *and the condition* (4.3) *holds, then the condition* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *is necessary for the boundedness of* $M_{b,\alpha}$ *from* $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\eta}$ $(\mathbb{R}^n)$.

3. *If* $\Phi \in \nabla_2$, $\varphi \in \mathcal{G}_\Phi$ *holds and* $\varphi(t)^\gamma \approx t^{\alpha+\beta}\varphi(t)$*, then the condition* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *is necessary and sufficient for the boundedness of* $M_{b,\alpha}$ *from* $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

Proof. (1) The first statement of the theorem follows from Theorem 3.3 and Lemma 4.3.

(2) We shall now prove the second part. Suppose that (4.3) holds and $M_{b,\alpha}$ is bounded from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$. Choose any ball B in $\mathbb{R}^n$, by Lemmas 2.1 and 2.2

$$\begin{aligned}
&\frac{1}{|B|^{1+\frac{\beta}{n}}}\int_B |b(y) - b_B|\mathrm{d}y = \frac{1}{|B|^{1+\frac{\alpha+\beta}{n}}}\int_B \left|\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B (b(y) - b(z))\mathrm{d}z\right| dy \\
&\le \frac{1}{|B|^{1+\frac{\alpha+\beta}{n}}}\int_B M_{b,a}(\chi_B)(y)\mathrm{d}y \le \frac{2\Psi^{-1}(|B|^{-1})}{|B|^{\frac{\alpha+\beta}{n}}}\, \|M_{b,\alpha}(\chi_B)\|_{L^\Psi(B)} \\
&\lesssim \frac{\eta(r_B)}{r_B^{\alpha+\beta}}\, \|M_{b,\alpha}(\chi_B)\|_{\mathcal{M}^{\Psi,\eta}} \lesssim \frac{\eta(r_B)}{r_B^{\alpha+\beta}}\, \|\chi_B\|_{\mathcal{M}^{\Phi,\varphi}} \lesssim \frac{\eta(r_B)}{r_B^{\alpha+\beta}\varphi(r_B)} \lesssim 1.
\end{aligned}$$

Thus, by Lemma 4.1 we get $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

(3) The third statement of the theorem follows from the first and second parts of the theorem. □

Remark 4.2. We assume $\varphi \in \mathcal{G}_\Phi$, in particular, φ is an almost decreasing function in the third part of the theorem. Note that, the monotonicity of φ and the condition (4.2) imply the condition (4.7). As a results of this fact and Remark 4.1 we do not need the hypothesis that φ satisfies (3.1) and (4.7) in the third part of the Theorem 4.3.

The following theorem is valid.

Theorem 4.4 (Spanne type result). *Let $0 < \beta < 1$, $0 \le \alpha < n$, $0 < \alpha+\beta < n$, $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\Phi, \Psi \in \mathcal{Y}$.*

1. *Let $\Phi \in \nabla_2$, (3.7) holds and (Φ, Ψ) satisfy the condition*

$$r^{\alpha+\beta}\Phi^{-1}\big(r^{-n}\big) \le C\Psi^{-1}\big(r^{-n}\big) \tag{4.8}$$

for all $r > 0$, where $C > 0$ does not depend on r, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is sufficient for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

2. *If $\varphi_1 \in \mathcal{G}_\Phi$ and the condition (4.5) holds, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.*

3. *Let $\Phi \in \nabla_2$, condition (4.8) holds and $\varphi_2(t) \approx t^{\alpha+\beta}\varphi_1(t)$. If $\varphi_1 \in \mathcal{G}_\Phi$ satisfies the condition*

$$\sup_{t<r<\infty} \frac{\Psi^{-1}\big(r^{-n}\big)}{\Phi^{-1}\big(r^{-n}\big)}\varphi_1(r) \le Ct^{\alpha+\beta}\varphi_1(t) \tag{4.9}$$

for all $t > 0$, where $C > 0$ does not depend on t, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

Proof. Similar to the proof of Theorem 4.3, we can obtain Theorem 4.4. □

Remark 4.3. Note that the statements of the Theorems 4.1 and 4.2 is also true for the operator

$$|b, I_\alpha| f(x) := \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|}{|x-y|^{n-\alpha}} f(y)\mathrm{d}y$$

under same conditions. It is easy to see the following inequality for $0 < \alpha < n$:

$$M_{b,\alpha}(f)(x) \lesssim |b, I_\alpha|(|f|)(x).$$

Although $M_{b,\alpha}$ is pointwise dominated by $|b, I_\alpha|$, and consequently, the results for the former could be derived from the results for the latter, we consider them separately, because we are able to study the boundedness of $M_{b,\alpha}$ under weaker assumptions than it derived from the results for the operator $|b, I_\alpha|$.

Let us first compare characterizations given in Theorems 4.1 and 4.3. We do not need regularity condition (4.1) in Theorem 4.3 unlike Theorem 4.1.

Now we will compare characterizations given in Theorems 4.2 and 4.4. The condition (4.9) in Theorem 4.4 is weaker than the condition (4.6) in Theorem 4.2. More precisely, for a function φ such that $t \in (0, \infty) \mapsto \frac{1}{\Phi^{-1}(t^{-n})}\varphi(t)$

is almost increasing, integral condition (4.6) implies the supremal condition (4.9). Indeed, by (2.1) we have

$$\Psi^{-1}\big(s^{-n}\big) \approx \Psi^{-1}\big(s^{-n}\big)s^n \int_s^\infty \frac{\mathrm{d}t}{t^{n+1}} \lesssim \int_s^\infty \Psi^{-1}\big(t^{-n}\big)\frac{\mathrm{d}t}{t}.$$

It follows from this inequality

$$t^{\alpha+\beta}\varphi(t) \gtrsim \int_t^\infty \frac{\Psi^{-1}\big(r^{-n}\big)}{\Phi^{-1}\big(r^{-n}\big)}\varphi(r)\frac{\mathrm{d}r}{r} \gtrsim \int_s^\infty \frac{\Psi^{-1}\big(r^{-n}\big)}{\Phi^{-1}\big(r^{-n}\big)}\varphi(r)\frac{\mathrm{d}r}{r}$$
$$\gtrsim \frac{\varphi(s)}{\Phi^{-1}\big(s^{-n}\big)}\int_s^\infty \Psi^{-1}\big(r^{-n}\big)\frac{\mathrm{d}r}{r} \gtrsim \frac{\Psi^{-1}\big(s^{-n}\big)}{\Phi^{-1}\big(s^{-n}\big)}\varphi(s)$$

where we took $s \in (t,\infty)$ arbitrarily, so that

$$\sup_{s>t} \frac{\Psi^{-1}\big(s^{-n}\big)}{\Phi^{-1}\big(s^{-n}\big)}\varphi(s) \lesssim t^{\alpha+\beta}\varphi(t).$$

For example, the function $\varphi(r) = \frac{\Phi^{-1}\big(r^{-n}\big)}{\Psi^{-1}\big(r^{-n}\big)}$ satisfies the condition (4.9), but does not satisfy the condition (4.6).

To state our results for the operator $[b, M_\alpha]$, we recall the definition of the fractional maximal operator with respect to a ball. For a fixed ball B_0, the fractional maximal function with respect to B_0 of a function f is given by

$$M_{\alpha,B_0}(f)(x) = \sup_{B_0\supseteq B\ni x} \frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B |f(y)|\mathrm{d}y, \qquad 0 \le \alpha < n,$$

where the supremum is taken over all the balls B with $B \subseteq B_0$ and $x \in B$. If $\alpha = 0$, then $M_{B_0} \equiv M_{0,B_0}$ is the Hardy–Littlewood maximal operator relative to B_0.

Lemma 4.4 [15]. *Let $0 < \beta < 1$, $0 \le \alpha < n$, Ψ be a Young function and b be a locally integrable function. If there exists a constant $C > 0$ such that*

$$\sup_B |B|^{-\beta/n}\Psi^{-1}\big(|B|^{-1}\big)\|b(\cdot) - |B|^{-\alpha/n}M_{\alpha,B}(b)(\cdot)\|_{L^\Psi(B)} \le C,$$

then $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

The following theorem is valid.

Theorem 4.5 (Adams type result). *Let $0 < \beta, \gamma < 1$, $0 < \alpha$, $\alpha + \beta < n$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $\Phi \in \mathcal{Y} \cap \nabla_2$, $\eta(t) \equiv \varphi(t)^\gamma$ and $\Psi(t) \equiv \Phi(t^{1/\gamma})$.*

1. *If conditions (3.1), (4.2) and (4.7) be satisfied then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \ge 0$ is sufficient for the boundedness of $[b, M_\alpha]$ from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.*
2. *If $\varphi \in \mathcal{G}_\Phi$ and the condition $\varphi(t)^\gamma \approx t^{\alpha+\beta}\varphi(t)$ holds, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \ge 0$ is necessary for the boundedness of $[b, M_\alpha]$ from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.*
3. *If $\varphi \in \mathcal{G}_\Phi$ and $\varphi(t)^\gamma \approx t^{\alpha+\beta}\varphi(t)$, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \ge 0$ is necessary and sufficient for the boundedness of $[b, M_\alpha]$ from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.*

Proof. (1) The first statement of the theorem follows from Theorem 4.3 and (1.2).

(2) We shall now prove the second part. Suppose that $\varphi(t)^\gamma \approx t^{\alpha+\beta}\varphi(t)$ and $[b, M_\alpha]$ is bounded from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$. For any fixed ball $B \subset \mathbb{R}^n$ and all $x \in B$, we have (see the proof of Proposition 4.1 in [3])

$$M(\chi_B)(x) = \chi_B(x) \qquad \text{and} \qquad M(b\chi_B)(x) = M_B(b)(x) \tag{4.10}$$

and (see (2.4) in [21])

$$M_\alpha(\chi_B)(x) = |B|^{\alpha/n} \qquad \text{and} \qquad M_\alpha(b\chi_B)(x) = M_{\alpha,B}(b)(x). \tag{4.11}$$

Then by Lemma 2.2,

$$\begin{aligned}
&|B|^{-\beta/n}\Psi^{-1}\big(|B|^{-1}\big)\|b - |B|^{-\alpha/n}M_{\alpha,B}(b)\|_{L^\Psi(B)} \\
&\quad = |B|^{-\frac{\alpha+\beta}{n}}\Psi^{-1}\big(|B|^{-1}\big)\|bM_\alpha(\chi_B) - M_\alpha(b\chi_B)\|_{L^\Psi(B)} \\
&\quad = |B|^{-\frac{\alpha+\beta}{n}}\Psi^{-1}\big(|B|^{-1}\big)\|[b, M_\alpha](\chi_B)\|_{L^\Psi(B)} \\
&\quad \lesssim \frac{\eta(r_B)}{r_B^{\alpha+\beta}}\,\|[b, M_\alpha]\big(\chi_B\big)\|_{\mathcal{M}^{\Psi,\eta}} \lesssim \frac{\eta(r_B)}{r_B^{\alpha+\beta}}\,\|\chi_B\|_{\mathcal{M}^{\Phi,\varphi}} \lesssim \frac{\eta(r_B)}{r_B^{\alpha+\beta}\varphi(r_B)} \lesssim 1.
\end{aligned}$$

Consequently, we have

$$\sup_B |B|^{-\beta/n}\Psi^{-1}\big(|B|^{-1}\big)\|b - |B|^{-\alpha/n}M_{\alpha,B}(b)\|_{L^\Psi(B)} \lesssim 1. \tag{4.12}$$

Thus, by Lemma 4.4 we get $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

Noting that $0 \leq |b| \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ when $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, it follows from Lemma 4.3, Theorem 3.3 and (1.2) that the operator $[|b|, M]$ is bounded from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\nu}(\mathbb{R}^n)$ under the assumptions of statement 2, where $\nu(t) = \eta(t)t^{-\alpha}$.

By (4.10) and (4.11) we have (for details see [21, page 1238])

$$|B|^{\alpha/n}M_B(b)(x) - M_{\alpha,B}(b)(x) = [|b|, M_\alpha](\chi_B)(x) - |B|^{\alpha/n}[|b|, M](\chi_B)(x). \tag{4.13}$$

Hence, by (4.12), (4.13), boundedness of $[|b|, M]$ and the hypotheses of statement 2 we get

$$\begin{aligned}
&\|b - M_B(b)\|_{L^\Psi(B)} \lesssim \|b - |B|^{-\alpha/n}M_{\alpha,B}(b)\|_{L^\Psi(B)} \\
&\quad + \|M_B(b) - |B|^{-\alpha/n}M_{\alpha,B}(b)\|_{L^\Psi(B)} \\
&\lesssim \frac{|B|^{\beta/n}}{\Psi^{-1}\big(|B|^{-1}\big)} + |B|^{-\alpha/n}\||B|^{\alpha/n}M_B(b) - M_{\alpha,B}(b)\|_{L^\Psi(B)} \\
&\lesssim \frac{|B|^{\beta/n}}{\Psi^{-1}\big(|B|^{-1}\big)} + |B|^{-\alpha/n}\|[|b|, M_\alpha](\chi_B) - |B|^{\alpha/n}[|b|, M](\chi_B)\|_{L^\Psi(B)} \\
&\lesssim \frac{|B|^{\beta/n}}{\Psi^{-1}\big(|B|^{-1}\big)} + |B|^{-\alpha/n}\|[|b|, M_\alpha](\chi_B)\|_{L^\Psi(B)} + \|[|b|, M](\chi_B)\|_{L^\Psi(B)} \\
&\lesssim \frac{|B|^{\beta/n}}{\Psi^{-1}\big(|B|^{-1}\big)} + \frac{|B|^{-\alpha/n}\eta(r_B)}{\Psi^{-1}\big(|B|^{-1}\big)}\|[|b|, M_\alpha](\chi_B)\|_{\mathcal{M}^{\Psi,\eta}}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\nu(r_B)}{\Psi^{-1}\big(|B|^{-1}\big)}\|[|b|,M](\chi_B)\|_{\mathcal{M}^{\Psi,\nu}}\\
&\lesssim \frac{|B|^{\beta/n}}{\Psi^{-1}\big(|B|^{-1}\big)}+\frac{|B|^{-\alpha/n}\eta(r_B)}{\Psi^{-1}\big(|B|^{-1}\big)}\|\chi_B\|_{\mathcal{M}^{\Phi,\varphi}}+\frac{\nu(r_B)}{\Psi^{-1}\big(|B|^{-1}\big)}\|\chi_B\|_{\mathcal{M}^{\Phi,\varphi}}\\
&\lesssim \frac{|B|^{\beta/n}}{\Psi^{-1}\big(|B|^{-1}\big)}+\frac{|B|^{-\alpha/n}\eta(r_B)}{\Psi^{-1}\big(|B|^{-1}\big)\varphi(r_B)}+\frac{|B|^{-\alpha/n}\eta(r_B)}{\Psi^{-1}\big(|B|^{-1}\big)\varphi(r_B)}\\
&\lesssim \frac{|B|^{\beta/n}}{\Psi^{-1}\big(|B|^{-1}\big)}.
\end{aligned}
$$

Consequently, we have for any ball B

$$\|b-M_B(b)\|_{L^\Psi(B)}\lesssim \frac{|B|^{\beta/n}}{\Psi^{-1}\big(|B|^{-1}\big)}. \tag{4.14}$$

To prove $b\geq 0$, it suffices to show that $b^-=0$. For any fixed ball B, observe that (for details see [3])

$$0\leq b^-(x)\leq M_B(b)(x)-b(x),\qquad x\in B.$$

Hence, it follows from (4.14) and Lemma 2.1 that, for any ball B,

$$
\begin{aligned}
\frac{1}{|B|}\int_B b^-(x)\mathrm{d}x &\leq \frac{1}{|B|}\int_B |M_B(b)(x)-b(x)|\mathrm{d}x\\
&\lesssim \Psi^{-1}\left(|B|^{-1}\right)\|b-M_B(b)\|_{L^\Psi(B)}\\
&\lesssim |B|^{\beta/n}.
\end{aligned}
$$

Thus, $b^-=0$ follows from Lebesgue differentiation theorem.

(3) The third statement of the theorem follows from the first and second parts of the theorem. □

The following theorem is valid.

Theorem 4.6 (Spanne type result). *Let $0<\beta<1$, $0\leq\alpha<n$, $0<\alpha+\beta<n$, $b\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\Phi,\Psi\in\mathcal{Y}$.*

1. *If $\Phi\in\nabla_2$, conditions (3.7) and (4.8) be satisfied, then $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b\geq 0$ is sufficient for the boundedness of $[b,M_\alpha]$ from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.*

2. *If $\varphi_1\in\mathcal{G}_\Phi$ and the condition (4.5) holds, then the condition $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b\geq 0$ is necessary for the boundedness of $[b,M_\alpha]$ from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.*

3. *Let $\Phi\in\nabla_2$, condition (4.8) holds and $\varphi_2(t)\approx t^{\alpha+\beta}\varphi_1(t)$. If $\varphi_1\in\mathcal{G}_\Phi$ satisfies the condition (4.9), then the condition $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b\geq 0$ is necessary and sufficient for the boundedness of $[b,M_\alpha]$ from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.*

Proof. Similar to the proof of Theorem 4.5, we can obtain Theorem 4.6. □

The following theorem is valid.

Theorem 4.7 (Adams type result). *Let $0 < \beta, \gamma < 1$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $\Phi \in \mathcal{Y}$, $\eta(t) \equiv \varphi(t)^\gamma$ and $\Psi(t) \equiv \Phi(t^{1/\gamma})$.*

1. *If $\Phi \in \nabla_2$ and $\varphi(t)$ satisfies (3.1) and*

$$\int_t^\infty r^\beta \varphi(r) \frac{\mathrm{d}r}{r} \le C t^\beta \varphi(t), \tag{4.15}$$

$$t^\beta \varphi(t) \le C \varphi(t)^\gamma$$

hold for all $t > 0$, where $C > 0$ does not depend on t, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is sufficient for the boundedness of $[b, T]$ from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

2. *If $\varphi \in \mathcal{G}_\Phi$ and the condition*

$$\varphi(t)^\gamma \le C t^\beta \varphi(t)$$

holds for all $t > 0$, where $C > 0$ does not depend on t, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary for the boundedness of $[b, T]$ from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

3. *If $\Phi \in \nabla_2$, $\varphi \in \mathcal{G}_\Phi$, condition (4.15) holds and $\varphi(t)^\gamma \approx t^\beta \varphi(t)$, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of $[b, T]$ from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.*

Proof. (1) The first statement of the theorem follows from the following pointwise estimate and Theorem 3.1.

$$\begin{aligned} |[b,T]f(x)| &\le \int_{\mathbb{R}^n} |b(x) - b(y)||K(x-y)||f(y)| \mathrm{d}y \\ &\lesssim \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\beta}} \mathrm{d}y = \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} I_\beta(|f(x)|). \end{aligned}$$

(2) We shall now prove the second part. We use the idea given in [16] (see also [18,20]). By choosing $z_0 \ne 0$ and $\delta > 0$ such that $K(z)^{-1}$ can be expressed in the neighborhood $|z - z_0| < \sqrt{n}\delta$ as the absolute convergent Fourier series

$$\frac{1}{K(z)} = \sum_{n=0}^\infty a_n e^{iv_n \cdot z},$$

where the exact form of the vectors v_n is irrelevant. Set $z_1 = \delta^{-1} z_0$. If $|z - z_1| < \sqrt{n}$, it follows from (1.1) that

$$\frac{1}{K(z)} = \frac{\delta^{-n}}{K(\delta z)} = \delta^{-n} \sum_{n=0}^\infty a_n e^{iv_n \cdot \delta z}. \tag{4.16}$$

Choose now any ball $B = B(x_0, r)$. Set $y_0 = x_0 - 2rz_1$ and $B' = B(y_0, r)$. Thus, if $x \in B$ and $y \in B'$,

$$\left| \frac{x-y}{2r} - z_1 \right| \le \left| \frac{x - x_0}{r} - \frac{y - y_0}{r} \right| \le 1.$$

Denoting $s(x) = sgn(b(x) - b_{B'})$, by (4.16) we have

$$\begin{aligned}
\int_B |b(x) - b_{B'}|\mathrm{d}x &= \int_B (b(x) - b_{B'})s(x)dx \\
&= \frac{1}{|B'|} \int_B \int_{B'} (b(x) - b(y))s(x)\mathrm{d}y\mathrm{d}x \\
&\approx r^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (b(x) - b(y))s(x) \frac{2^n r^n K(x-y)}{K((x-y)/2r)} \chi_B(x)\chi_{B'}(y)\mathrm{d}y\mathrm{d}x \\
&\approx \sum_{n=0}^{\infty} a_n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (b(x) - b(y))K(x-y)e^{i(\delta/2r)v_n\cdot x} s(x)\chi_B(x) \\
&\quad \times\ e^{-i(\delta/2r)v_n\cdot y}\chi_{B'}(y)\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

Taking

$$g_n(y) = e^{-i(\delta/2r)v_n\cdot y}\chi_{B'}(y) \text{ and } h_n(x) = e^{i(\delta/2r)v_n\cdot x}s(x)\chi_B(x),$$

we obtain

$$\begin{aligned}
\int_B |b(x) - b_{B'}|\mathrm{d}x &\approx \sum_{n=0}^{\infty} a_n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (b(x) - b(y))K(x-y)g_n(y)h_n(x)\mathrm{d}y\mathrm{d}x \\
&\lesssim \sum_{n=0}^{\infty} a_n \int_{\mathbb{R}^n} [b,T]g_n(x)h_n(x)\mathrm{d}x \\
&\lesssim \sum_{n=0}^{\infty} |a_n| \int_{\mathbb{R}^n} |[b,T]g_n(x)||h_n(x)|\mathrm{d}x \\
&\lesssim \sum_{n=0}^{\infty} |a_n| \int_B |[b,T]g_n(x)|\mathrm{d}x.
\end{aligned}$$

Applying Lemma 2.1, we have

$$\begin{aligned}
&\int_B |[b,T]g_n(x)|dx \le 2|B|\Psi^{-1}\left(|B|^{-1}\right)\|[b,T]g_n\|_{L^{\Psi}(B)} \\
&\lesssim |B|\eta(r)\|[b,T]g_n\|_{\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)} \lesssim r^n\eta(r)\|g_n\|_{\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)} \lesssim r^n\varphi(r)^{\gamma-1} \\
&\lesssim r^{n+\beta} \lesssim |B|^{1+\beta/n},
\end{aligned}$$

since $\|g_n\|_{\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)} \lesssim \varphi(r)^{-1}$ by Lemma 2.2.

Thus, we have obtained

$$\frac{1}{|B|^{1+\frac{\beta}{n}}} \int_B |b(x) - b_B|\mathrm{d}x \le \frac{2}{|B|^{1+\frac{\beta}{n}}} \int_B |b(x) - b_{B'}|\mathrm{d}x \lesssim 1$$

which completes the proof of Theorem by Lemma 4.1.

(3) The third statement of the theorem follows from the first and second parts of the theorem. □

The following theorem is valid.

Theorem 4.8 (Spanne type result). *Let* $0 < \beta < 1$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *and* $\Phi, \Psi \in \mathcal{Y}$.

1. *Let* $\Phi \in \nabla_2$, (3.4) *holds and* (Φ, Ψ) *satisfy the condition*

$$r^{\beta}\Phi^{-1}\big(r^{-n}\big) + \int_r^{\infty} \Phi^{-1}\big(t^{-n}\big)t^{\beta}\frac{\mathrm{d}t}{t} \le C\Psi^{-1}\big(r^{-n}\big) \tag{4.17}$$

holds for all $r > 0$, *where* $C > 0$ *does not depend on* r, *then the condition* $b \in \dot{\Lambda}_{\beta}(\mathbb{R}^n)$ *is sufficient for the boundedness of* $[b, T]$ *from* $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

2. *If* $\varphi_1 \in \mathcal{G}_{\Phi}$ *and the condition*

$$\varphi_2(t) \le Ct^{\beta}\varphi_1(t)$$

holds for all $t > 0$, *where* $C > 0$ *does not depend on* t, *then the condition* $b \in \dot{\Lambda}_{\beta}(\mathbb{R}^n)$ *is necessary for the boundedness of* $[b, T]$ *from* $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

3. *Let* $\Phi \in \nabla_2$, *condition* (4.17) *holds and* $\varphi_2(t) \approx t^{\alpha+\beta}\varphi_1(t)$. *If* $\varphi_1 \in \mathcal{G}_{\Phi}$ *satisfies the regularity type condition*

$$\int_t^{\infty} \frac{\Psi^{-1}\big(r^{-n}\big)}{\Phi^{-1}\big(r^{-n}\big)}\varphi_1(r)\frac{\mathrm{d}r}{r} \le Ct^{\beta}\varphi_1(t)$$

for all $t > 0$, *where* $C > 0$ *does not depend on* t, *then the condition* $b \in \dot{\Lambda}_{\beta}(\mathbb{R}^n)$ *is necessary and sufficient for the boundedness of* $[b, T]$ *from* $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ *to* $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

Proof. Similar to the proof of Theorem 4.7, we can obtain Theorem 4.8. □

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