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A comparative study on generating function relations for generalized hypergeometric functions via generalized fractional operators

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Abstract

In this paper, we present further generalizations of the beta function; Riemann–Liouville, Caputo and Kober–Erdelyi fractional operators by using confluent hypergeometric function with six parameters. We also define new generalizations of the Gauss F, Appell F_1, F_2 and Lauricella F_D^3 hypergeometric functions with the help of new beta function. Then we obtain some generating function relations for these generalized hypergeometric functions by using each generalized fractional operators, separately. One of the purposes of the present investigation is to give a chance to the reader to compare the results corresponding to each generalized fractional operators.

Keywords: Beta function; Hypergeometric functions; Fractional operators; Generating functions

1 Introduction and definitions

Many researchers used various generalizations of the beta function to introduce new generalizations of hypergeometric functions and fractional operators. They also used generalized fractional operators for obtaining some generating function relations of these hypergeometric functions [1–10]. Özarslan and Özergin [8] introduced the generalization of the Riemann–Liouville fractional operator with a one parameter exponential function and obtained the results for negative values of the order. Srivastava et al. [10] generalized the results which were obtained in [8]. Agarwal et al. [2] and Luo et al. [7] introduced the same generalization of Riemann–Liouville fractional operator with a five parameter confluent hypergeometric function and obtained the results for positive and negative values of the order, respectively. Agarwal and Agarwal [1] used the same five parameter confluent hypereometric function to define the generalization of Caputo fractional operator. Kıymaz et al. [5] and Agarwal et al. [3] introduced the generalizations of Caputo fractional derivatives with one and two parameter exponential functions, respectively. Baleanu et al. [4] introduced the generalization of Riemann–Liouville fractional operator with a different two parameter exponential function and obtained the results for negative values of the order. Kıymaz et al. [6] introduced the generalization of Caputo fractional operator with the same two parameter exponential function in [4]. In all the above works, the authors had to define different types of hypergeometric functions for the same type of fractional operators and none of them made a comparison between the corresponding results.

Motivated by the above works, we decided to define further generalizations of the hypergeometric functions which we can compare the results for different types of fractional operators by using the following generalization of the beta function.

Definition 1 The generalized beta function is defined by

$$B_{p,q}^{(\alpha,\beta;\kappa,\mu)}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;-\frac{p}{t^\kappa}-\frac{q}{(1-t)^\mu}\right)dt, \tag{1}$$
$$\left(\min\{\Re(p),\Re(q)\} \geq 0, \min\{\Re(x),\Re(y)\} > 0, \min\{\Re(\alpha),\Re(\beta),\Re(\kappa),\Re(\mu)\} > 0\right).$$

Note that this generalized beta function has the following symmetry property

$$B_{p,q}^{(\alpha,\beta;\kappa,\mu)}(x,y) = B_{q,p}^{(\alpha,\beta;\mu,\kappa)}(y,x),$$

and has the functional relation

$$B_{p,q}^{(\alpha,\beta;\kappa,\mu)}(x,y+1) + B_{p,q}^{(\alpha,\beta;\kappa,\mu)}(x+1,y) = B_{p,q}^{(\alpha,\beta;\kappa,\mu)}(x,y).$$

Clearly, the special case of (1) becomes the generalized beta function that has been discussed in [11] when $\kappa = \mu = 1$; in [12] when $p = q$ and $\kappa = \mu = 1$; in [13] when $\alpha = \beta$ and $\kappa = \mu = 1$; and in [14] when $\alpha = \beta$, $p = q$ and $\kappa = \mu = 1$. Also in the case of $p = q = 0$, it gives the original beta function [15].

Throughout this paper, we assume $\min\{\Re(\alpha),\Re(\beta),\Re(\kappa),\Re(\mu)\} > 0$, $\min\{\Re(p),\Re(q)\} > 0$, and we use the notation $\widehat{B}$ instead of $B_{p,q}^{(\alpha,\beta;\kappa,\mu)}$.

Definition 2 The generalizations of Gauss F, Appell F_1, F_2 and the Lauricella F_D^3 hypergeometric functions are as follows:

$$\begin{aligned}
&F^{(\alpha,\beta;\kappa,\mu;p,q)}(a,b;c;z;m_1,m_2) \\
&\quad := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{\widehat{B}(b-m_1+n,c-b+m_2)}{B(b-m_1+n,c-b+m_2)} \frac{z^n}{n!} \\
&\qquad \left(|z|<1\right),
\end{aligned} \tag{2}$$

$$\begin{aligned}
&F_1^{(\alpha,\beta;\kappa,\mu;p,q)}(a,b,c;d;x,y;m_1,m_2) \\
&\quad := \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}(b)_n(c)_k}{(d)_{n+k}} \frac{\widehat{B}(a-m_1+n+k,d-a+m_2)}{B(a-m_1+n+k,d-a+m_2)} \frac{x^n}{n!}\frac{y^k}{k!} \\
&\qquad \left(|x|<1,|y|<1\right),
\end{aligned} \tag{3}$$

$$\begin{aligned}
&F_2^{(\alpha,\beta;\kappa,\mu;p,q)}(a,b,c;d,e;x,y;m_1,m_2) \\
&\quad := \sum_{n,k=0}^{\infty} \left[\frac{(a)_{n+k}(b)_n(c)_k}{(d)_n(e)_k} \frac{\widehat{B}(b-m_1+n,d-b+m_2)}{B(b-m_1+n,d-b+m_2)}\right. \\
&\qquad \left. \times \frac{\widehat{B}(c-m_1+k,e-c+m_2)}{B(c-m_1+k,e-c+m_2)} \frac{x^n}{n!}\frac{y^k}{k!}\right]
\end{aligned} \tag{4}$$

$$\begin{aligned}
&\bigl(|x| + |y| < 1\bigr), \\
&F_D^{3;(\alpha,\beta;\kappa,\mu;p,q)}(a,b,c,d;e;x,y,z;m_1,m_2) \\
&\quad := \sum_{n,k,r=0}^{\infty} \biggl[\frac{(a)_{n+k+r}(b)_n(c)_k(d)_r}{(e)_{n+k+r}} \\
&\qquad \times \frac{\widehat{B}(a-m_1+n+k+r,e-a+m_2)}{B(a-m_1+n+k+r,e-a+m_2)} \frac{x^n}{n!} \frac{y^k}{k!} \frac{z^r}{r!} \biggr] \\
&\qquad \bigl(|x| < 1, |y| < 1, |z| < 1\bigr),
\end{aligned} \tag{5}$$

where m_1 and m_2 are arbitrary parameters.

Remark 1 It is easy to see that when $p = q = 0$ these hypergeometric functions immediately reduce to their original forms. Also the reader can easily obtain various integral representations of these functions by using the integral representation of the new beta function (1) in the same way as in [1–6, 8–10].

We also use the notations $\widehat{F}$, $\widehat{F}_1$, $\widehat{F}_2$ and $\widehat{F}_D^3$ instead of $F^{(\alpha,\beta;\kappa,\mu;p,q)}$, $F_1^{(\alpha,\beta;\kappa,\mu;p,q)}$, $F_2^{(\alpha,\beta;\kappa,\mu;p,q)}$ and $F_D^{3;(\alpha,\beta;\kappa,\mu;p,q)}$, respectively.

Now we give the definitions of new generalized fractional operators.

Definition 3 Let $m \in \mathbb{N}$. The generalized Riemann–Liouville fractional operator is defined as

$$\begin{aligned}
&D^{\nu;(\alpha,\beta;\kappa,\mu;p,q)}\bigl[f(z)\bigr] \\
&\quad := \begin{cases} \frac{1}{\Gamma(-\nu)} \int_0^z (z-t)^{-\nu-1} f(t) {}_1F_1(\alpha;\beta;-p(\frac{z}{t})^\kappa - q(\frac{z}{z-t})^\mu)\,dt & (\Re(\nu) < 0), \\ \frac{d^m}{dz^m}(D^{\nu-m;(\alpha,\beta;\kappa,\mu;p,q)}[f(z)]) & (m-1 < \Re(\nu) < m). \end{cases}
\end{aligned} \tag{6}$$

The generalized Caputo fractional derivative operator is defined as

$$\begin{aligned}
\mathbf{D}^{\nu;(\alpha,\beta;\kappa,\mu;p,q)}\bigl[f(z)\bigr] &:= D^{\nu-m;(\alpha,\beta;\kappa,\mu;p,q)}\bigl[f^{(m)}(z)\bigr] \\
&= \frac{1}{\Gamma(m-\nu)} \int_0^z (z-t)^{m-\nu-1} f^{(m)}(t) \\
&\quad \times {}_1F_1\biggl(\alpha;\beta;-p\biggl(\frac{z}{t}\biggr)^\kappa - q\biggl(\frac{z}{z-t}\biggr)^\mu\biggr)\,dt,
\end{aligned} \tag{7}$$

where $m-1 < \Re(\nu) < m$.

The generalized Kober–Erdelyi fractional integral operator is defined as

$$\begin{aligned}
\mathbf{I}_\eta^{\nu;(\alpha,\beta;\kappa,\mu;p,q)}\bigl[f(z)\bigr] &:= z^{-\nu-\eta} D^{-\nu;(\alpha,\beta;\kappa,\mu;p,q)}\bigl[z^\eta f(z)\bigr] \\
&= \frac{z^{-\nu-\eta}}{\Gamma(\nu)} \int_0^z t^\eta (z-t)^{\nu-1} f(t) \\
&\quad \times {}_1F_1\biggl(\alpha;\beta;-p\biggl(\frac{z}{t}\biggr)^\kappa - q\biggl(\frac{z}{z-t}\biggr)^\mu\biggr)\,dt,
\end{aligned} \tag{8}$$

where $\Re(\nu) > 0$, $\eta \in \mathbb{C}$.

Remark 2 Note that the reduced forms of generalized fractional operators for $\alpha = \beta$, $\kappa = \mu = 1$ defined and studied in [4, 6] and for $\alpha = \beta$, $p = q$, $\kappa = \mu = 1$ defined and studied in [5, 8]. Also for $p = q = 0$ all the operators reduce to their original forms [16].

For the sake of shortness we use the notations $\widehat{D}^{\nu}$, $\widehat{\mathbf{D}}^{\nu}$ and $\widehat{\mathbf{I}}_{\eta}^{\nu}$ instead of $D^{\nu;(\alpha,\beta;\kappa,\mu;p,q)}$, $\mathbf{D}^{\nu;(\alpha,\beta;\kappa,\mu;p,q)}$ and $\mathbf{I}_{\eta}^{\nu;(\alpha,\beta;\kappa,\mu;p,q)}$, respectively. We also assume that $m \in \mathbb{N}$ in the following sections.

2 Fractional derivative and integral formulas

We start our examination by obtaining the fractional derivatives and integrals of some functions with each fractional operators.

Theorem 1

(a) *Let* $\Re(\nu) < 0$, $\Re(\lambda) > -1$, *then*

$$\widehat{D}^{\nu}\big[z^{\lambda}\big] = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\nu+1)}\frac{\widehat{B}(\lambda+1,-\nu)}{B(\lambda+1,-\nu)}z^{\lambda-\nu}. \tag{9}$$

(b) *Let* $\Re(\nu) > 0$, $\Re(\lambda+\eta) > -1$, *then*

$$\widehat{\mathbf{I}}_{\eta}^{\nu}\big[z^{\lambda}\big] = \frac{\Gamma(\lambda+\eta+1)}{\Gamma(\lambda+\eta+\nu+1)}\frac{\widehat{B}(\lambda+\eta+1,\nu)}{B(\lambda+\eta+1,\nu)}z^{\lambda}.$$

(c) *Let* $m-1 < \Re(\nu) < m$, $\Re(\lambda) > -1$, *then*

$$\widehat{D}^{\nu}\big[z^{\lambda}\big] = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\nu+1)}\frac{\widehat{B}(\lambda+1,m-\nu)}{B(\lambda+1,m-\nu)}z^{\lambda-\nu}.$$

(d) *Let* $m-1 < \Re(\nu) < m$, $\Re(\lambda) > m-1$, *then*

$$\widehat{\mathbf{D}}^{\nu}\big[z^{\lambda}\big] = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\nu+1)}\frac{\widehat{B}(\lambda-m+1,m-\nu)}{B(\lambda-m+1,m-\nu)}z^{\lambda-\nu}.$$

Proof For $\Re(\nu) < 0$ and $\Re(\lambda) > -1$, with direct calculations, we have

$$\begin{aligned}\widehat{D}^{\nu}\big[z^{\lambda}\big] &= \frac{1}{\Gamma(-\nu)}\int_0^z t^{\lambda}(z-t)^{-\nu-1}{}_1F_1\left(\alpha;\beta;-p\left(\frac{z}{t}\right)^{\kappa}-q\left(\frac{z}{z-t}\right)^{\mu}\right)dt\\ &= \frac{z^{\lambda-\nu}}{\Gamma(-\nu)}\int_0^1 u^{\lambda}(1-u)^{-\nu-1}{}_1F_1\left(\alpha;\beta;-\frac{p}{u^{\kappa}}-\frac{q}{(1-u)^{\mu}}\right)du\\ &= \frac{z^{\lambda-\nu}}{\Gamma(-\nu)}\widehat{B}(\lambda+1,-\nu)\\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\nu+1)}\frac{\widehat{B}(\lambda+1,-\nu)}{B(\lambda+1,-\nu)}z^{\lambda-\nu},\end{aligned}$$

which completes the proof of the first case.

On the other hand, taking $f(z) = z^{\lambda}$ in (6), (7) and (8) we have the following relations:
For $\Re(\nu) > 0$, $\Re(\lambda+\eta) > -1$,

$$\widehat{\mathbf{I}}_{\eta}^{\nu}\big[z^{\lambda}\big] = z^{-\nu-\eta}\widehat{D}^{-\nu}\big[z^{\lambda+\eta}\big], \tag{10}$$

for $m-1 < \Re(\nu) < m$, $\Re(\lambda) > -1$,

$$\widehat{D}^{\nu}\big[z^{\lambda}\big] = \frac{d^m}{dz^m}\big(\widehat{D}^{\nu-m}\big[z^{\lambda}\big]\big), \tag{11}$$

for $m-1 < \Re(\nu) < m$ and $\Re(\lambda) > m-1$,

$$\widehat{\mathbf{D}}^{\nu}\big[z^{\lambda}\big] = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-m+1)}\widehat{D}^{\nu-m}\big[z^{\lambda-m}\big]. \tag{12}$$

Thus, the proofs of the other cases are straightforward from Eqs. (10), (11) and (12) by using (9). □

Theorem 2 *If $f(z)$ is an analytic function at the origin with its Maclaurin expansion given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < r$, then*

$$\widehat{D}^{\nu}\big[f(z)\big] = \sum_{n=0}^{\infty} a_n \widehat{D}^{\nu}\big[z^n\big],$$

$$\widehat{\mathbf{I}}_{\eta}^{\nu}\big[f(z)\big] = \sum_{n=0}^{\infty} a_n \widehat{\mathbf{I}}_{\eta}^{\nu}\big[z^n\big] \quad \big(\Re(\nu) > 0\big),$$

$$\widehat{\mathbf{D}}^{\nu}\big[f(z)\big] = \sum_{n=0}^{\infty} a_n \widehat{\mathbf{D}}^{\nu}\big[z^n\big] \quad \big(m-1 < \Re(\nu) < m\big).$$

Proof Under the hypothesis of the theorem, term-by-term integration is guaranteed. □

Theorem 3 *For $|az| < 1$, the following results hold true*:

(a) *Let $0 < \Re(\lambda) < \Re(\nu)$, then*

$$\widehat{D}^{\lambda-\nu}\big[z^{\lambda-1}(1-az)^{-\rho}\big] = \frac{\Gamma(\lambda)}{\Gamma(\nu)} z^{\nu-1}\widehat{F}(\rho,\lambda;\nu;az;0,0).$$

(b) *Let $\Re(\lambda-\nu) > 0$, $\Re(\lambda+\eta) > 0$, then*

$$\widehat{\mathbf{I}}_{\eta}^{\lambda-\nu}\big[z^{\lambda-1}(1-az)^{-\rho}\big] = \frac{\Gamma(\lambda+\eta)}{\Gamma(2\lambda+\eta-\nu)} z^{\lambda-1}\widehat{F}(\rho,\lambda+\eta;2\lambda+\eta-\nu;az;0,0).$$

(c) *Let $m-1 < \Re(\lambda-\nu) < m$, $\Re(\lambda) > 0$, then*

$$\widehat{D}^{\lambda-\nu}\big[z^{\lambda-1}(1-az)^{-\rho}\big] = \frac{\Gamma(\lambda)}{\Gamma(\nu)} z^{\nu-1}\widehat{F}(\rho,\lambda;\nu;az;0,m).$$

(d) *Let $m-1 < \Re(\lambda-\nu) < m < \Re(\lambda)$, then*

$$\widehat{\mathbf{D}}^{\lambda-\nu}\big[z^{\lambda-1}(1-az)^{-\rho}\big] = \frac{\Gamma(\lambda)}{\Gamma(\nu)} z^{\nu-1}\widehat{F}(\rho,\lambda;\nu;az;m,m).$$

Proof Using the power series expansion of

$$(1-az)^{-\rho} = \sum_{n=0}^{\infty} (\rho)_n \frac{(az)^n}{n!}, \quad |az| < 1,$$

and considering Theorem 1 and Theorem 2 together, we get the results by simple calculations. □

Due to the proofs of the following theorems in this section being similar to Theorem 3, we omit these proofs. The interested reader can find the detailed proofs of similar theorems in [1–10].

Theorem 4 *For* $|az| < 1$ *and* $|bz| < 1$, *the following results hold true*:

(a) *Let* $0 < \Re(\lambda) < \Re(\nu)$, *then*

$$\widehat{D}^{\lambda-\nu}\left[z^{\lambda-1}(1-az)^{-\rho}(1-bz)^{-\sigma}\right] = \frac{\Gamma(\lambda)}{\Gamma(\nu)}z^{\nu-1}\widehat{F}_1(\lambda,\rho,\sigma;\nu;az,bz;0,0).$$

(b) *Let* $\Re(\lambda-\nu) > 0$, $\Re(\lambda+\eta) > 0$, *then*

$$\begin{aligned}&\widehat{\mathbf{I}}_{\eta}^{\lambda-\nu}\left[z^{\lambda-1}(1-az)^{-\rho}(1-bz)^{-\sigma}\right]\\&\quad= \frac{\Gamma(\lambda+\eta)}{\Gamma(2\lambda+\eta-\nu)}z^{\lambda-1}\widehat{F}_1(\lambda+\eta,\rho,\sigma;2\lambda+\eta-\nu;az,bz;0,0).\end{aligned}$$

(c) *Let* $m-1 < \Re(\lambda-\nu) < m$, $\Re(\lambda) > 0$, *then*

$$\widehat{D}^{\lambda-\nu}\left[z^{\lambda-1}(1-az)^{-\rho}(1-bz)^{-\sigma}\right] = \frac{\Gamma(\lambda)}{\Gamma(\nu)}z^{\nu-1}\widehat{F}_1(\lambda,\rho,\sigma;\nu;az,bz;0,m).$$

(d) *Let* $m-1 < \Re(\lambda-\nu) < m < \Re(\lambda)$, *then*

$$\widehat{\mathbf{D}}^{\lambda-\nu}\left[z^{\lambda-1}(1-az)^{-\rho}(1-bz)^{-\sigma}\right] = \frac{\Gamma(\lambda)}{\Gamma(\nu)}z^{\nu-1}\widehat{F}_1(\lambda,\rho,\sigma;\nu;az,bz;m,m).$$

Theorem 5 *For* $|az| < 1$, $|bz| < 1$ *and* $|cz| < 1$, *the following results hold true*:

(a) *Let* $0 < \Re(\lambda) < \Re(\nu)$, *then*

$$\begin{aligned}&\widehat{D}^{\lambda-\nu}\left[z^{\lambda-1}(1-az)^{-\rho}(1-bz)^{-\sigma}(1-cz)^{-\tau}\right]\\&\quad= \frac{\Gamma(\lambda)}{\Gamma(\nu)}z^{\nu-1}\widehat{F}_D^3(\lambda,\rho,\sigma,\tau;\nu;az,bz,cz;0,0).\end{aligned}$$

(b) *Let* $\Re(\lambda-\nu) > 0$, $\Re(\lambda+\eta) > 0$, *then*

$$\begin{aligned}&\widehat{\mathbf{I}}_{\eta}^{\lambda-\nu}\left[z^{\lambda-1}(1-az)^{-\rho}(1-bz)^{-\sigma}(1-cz)^{-\tau}\right]\\&\quad= \frac{\Gamma(\lambda+\eta)}{\Gamma(2\lambda+\eta-\nu)}z^{\lambda-1}\widehat{F}_D^3(\lambda+\eta,\rho,\sigma,\tau;2\lambda+\eta-\nu;az,bz,cz;0,0).\end{aligned}$$

(c) *Let* $m-1 < \Re(\lambda-\nu) < m$, $\Re(\lambda) > 0$, *then*

$$\begin{aligned}&\widehat{D}^{\lambda-\nu}\left[z^{\lambda-1}(1-az)^{-\rho}(1-bz)^{-\sigma}(1-cz)^{-\tau}\right]\\&\quad= \frac{\Gamma(\lambda)}{\Gamma(\nu)}z^{\nu-1}\widehat{F}_D^3(\lambda,\rho,\sigma,\tau;\nu;az,bz,cz;0,m).\end{aligned}$$

(d) *Let* $m-1<\Re(\lambda-\nu)<m<\Re(\lambda)$, *then*

$$\widehat{\mathbf{D}}^{\lambda-\nu}\left[z^{\lambda-1}(1-az)^{-\rho}(1-bz)^{-\sigma}(1-cz)^{-\tau}\right] = \frac{\Gamma(\lambda)}{\Gamma(\nu)}z^{\nu-1}\widehat{F}_D^3(\lambda,\rho,\sigma,\tau;\nu;az,bz,cz;m,m).$$

Theorem 6 *For* $|x|+|az|<1$, *the following results hold true*:

(a) *Let* $0<\Re(\lambda)<\Re(\nu)$, *then*

$$\widehat{D}^{\lambda-\nu}\left[z^{\lambda-1}(1-az)^{-\rho}\widehat{F}\left(\rho,\sigma;\tau;\frac{x}{1-az};0,0\right)\right] = \frac{\Gamma(\lambda)}{\Gamma(\nu)}z^{\nu-1}\widehat{F}_2(\rho,\sigma,\lambda;\tau,\nu;x,az;0,0).$$

(b) *Let* $\Re(\lambda-\nu)>0$, $\Re(\lambda+\eta)>0$, *then*

$$\widehat{\mathbf{I}}_\eta^{\lambda-\nu}\left[z^{\lambda-1}(1-az)^{-\rho}\widehat{F}\left(\rho,\sigma;\tau;\frac{x}{1-az};0,0\right)\right] = \frac{\Gamma(\lambda+\eta)}{\Gamma(2\lambda+\eta-\nu)}z^{\lambda-1}\widehat{F}_2(\rho,\sigma,\lambda+\eta;\tau,2\lambda+\eta-\nu;x,az;0,0).$$

(c) *Let* $m-1<\Re(\lambda-\nu)<m$, $\Re(\lambda)>0$, *then*

$$\widehat{D}^{\lambda-\nu}\left[z^{\lambda-1}(1-az)^{-\rho}\widehat{F}\left(\rho,\sigma;\tau;\frac{x}{1-az};0,m\right)\right] = \frac{\Gamma(\lambda)}{\Gamma(\nu)}z^{\nu-1}\widehat{F}_2(\rho,\sigma,\lambda;\tau,\nu;x,az;0,m).$$

(d) *Let* $m-1<\Re(\lambda-\nu)<m<\Re(\lambda)$, *then*

$$\widehat{\mathbf{D}}^{\lambda-\nu}\left[z^{\lambda-1}(1-az)^{-\rho}\widehat{F}\left(\rho,\sigma;\tau;\frac{x}{1-az};m,m\right)\right] = \frac{\Gamma(\lambda)}{\Gamma(\nu)}z^{\nu-1}\widehat{F}_2(\rho,\sigma,\lambda;\tau,\nu;x,az;m,m).$$

3 Generating function relations

The proofs of following theorems can be given similar to the proofs of the corresponding results in [15, Sects. 5.2 and 5.3]. Besides, the proofs of similar theorems can be found in many papers, such as [1–6, 8–10]. That is why we omit the proofs.

Theorem 7 *For* $|z|<\min\{1,|1-t|\}$ *and* $|t|<|1-z|$, *the following relations hold true*:

(a) *Let* $0<\Re(\lambda)<\Re(\nu)$, *then*

$$\sum_{n=0}^{\infty}\frac{(\rho)_n}{n!}\widehat{F}(\rho+n,\lambda;\nu;z;0,0)t^n = (1-t)^{-\rho}\widehat{F}\left(\rho,\lambda;\nu;\frac{z}{1-t};0,0\right).$$

(b) *Let* $\Re(\lambda-\nu)>0$, $\Re(\lambda+\eta)>0$, *then*

$$\sum_{n=0}^{\infty}\frac{(\rho)_n}{n!}\widehat{F}(\rho+n,\lambda+\eta;2\lambda+\eta-\nu;z;0,0)t^n$$
$$=(1-t)^{-\rho}\widehat{F}\left(\rho,\lambda+\eta;2\lambda+\eta-\nu;\frac{z}{1-t};0,0\right).$$

(c) *Let* $m-1<\Re(\lambda-\nu)<m$, $\Re(\lambda)>0$, *then*

$$\sum_{n=0}^{\infty}\frac{(\rho)_n}{n!}\widehat{F}(\rho+n,\lambda;\nu;z;0,m)t^n=(1-t)^{-\rho}\widehat{F}\left(\rho,\lambda;\nu;\frac{z}{1-t};0,m\right).$$

(d) *Let* $m-1<\Re(\lambda-\nu)<m<\Re(\lambda)$, *then*

$$\sum_{n=0}^{\infty}\frac{(\rho)_n}{n!}\widehat{F}(\rho+n,\lambda;\nu;z;m,m)t^n=(1-t)^{-\rho}\widehat{F}\left(\rho,\lambda;\nu;\frac{z}{1-t};m,m\right).$$

Theorem 8 *For* $|z|<\min\{1,|\frac{1-t}{t}|\}$ *and* $|t|<|1-z|^{-1}$, *the following relations hold true*:

(a) *Let* $0<\Re(\lambda)<\Re(\nu)$, *then*

$$\sum_{n=0}^{\infty}\frac{(\rho)_n}{n!}\widehat{F}(\sigma-n,\lambda;\nu;z;0,0)t^n=(1-t)^{-\rho}\widehat{F}_1\left(\lambda,\rho,\sigma;\nu;\frac{-zt}{1-t},z;0,0\right).$$

(b) *Let* $\Re(\lambda-\nu)>0$, $\Re(\lambda+\eta)>0$, *then*

$$\sum_{n=0}^{\infty}\frac{(\rho)_n}{n!}\widehat{F}(\sigma-n,\lambda+\eta;2\lambda+\eta-\nu;z;0,0)t^n$$
$$=(1-t)^{-\rho}\widehat{F}_1\left(\lambda+\eta,\rho,\sigma;2\lambda+\eta-\nu;\frac{-zt}{1-t},z;0,0\right).$$

(c) *Let* $m-1<\Re(\lambda-\nu)<m$, $\Re(\lambda)>0$, *then*

$$\sum_{n=0}^{\infty}\frac{(\rho)_n}{n!}\widehat{F}(\sigma-n,\lambda;\nu;z;0,m)t^n=(1-t)^{-\rho}\widehat{F}_1\left(\lambda,\rho,\sigma;\nu;\frac{-zt}{1-t},z;0,m\right).$$

(d) *Let* $m-1<\Re(\lambda-\nu)<m<\Re(\lambda)$, *then*

$$\sum_{n=0}^{\infty}\frac{(\rho)_n}{n!}\widehat{F}(\sigma-n,\lambda;\nu;z;m,m)t^n=(1-t)^{-\rho}\widehat{F}_1\left(\lambda,\rho,\sigma;\nu;\frac{-zt}{1-t},z;m,m\right).$$

Theorem 9 *For* $|z|<1$, $|\frac{1-u}{1-z}t|<1$ *and* $|\frac{z}{1-t}|+|\frac{ut}{1-t}|<1$, *the following relations hold true*:

(a) *Let* $0<\Re(\lambda)<\Re(\nu)$, $0<\Re(\sigma)<\Re(\tau)$, *then*

$$\sum_{n=0}^{\infty}\frac{(\rho)_n}{n!}\widehat{F}(\rho+n,\lambda;\nu;z;0,0)\widehat{F}(-n,\sigma;\tau;u;0,0)t^n$$
$$=(1-t)^{-\rho}\widehat{F}_2\left(\rho,\lambda,\sigma;\nu,\tau;\frac{z}{1-t},\frac{-ut}{1-t};0,0\right).$$

(b) *Let* $\Re(\lambda-\nu)>0$, $\Re(\lambda+\eta)>0$, $\Re(\sigma-\tau)>0$, $\Re(\sigma+\xi)>0$, *then*

$$\begin{aligned}\sum_{n=0}^{\infty}\bigg[&\frac{(\rho)_n}{n!}\widehat{F}(\rho+n,\lambda+\eta;2\lambda+\eta-\nu;z;0,0)\\&\times\widehat{F}(-n,\sigma+\xi;2\sigma+\xi-\tau;u;0,0)t^n\bigg]\\&=(1-t)^{-\rho}\widehat{F}_2\left(\rho,\lambda+\eta,\sigma+\xi;2\lambda+\eta-\nu,2\sigma+\xi-\tau;\frac{z}{1-t},\frac{-ut}{1-t};0,0\right).\end{aligned}$$

(c) *Let* $m-1<\Re(\lambda-\nu)<m$, $\Re(\lambda)>0$, $m-1<\Re(\sigma-\tau)<m$, $\Re(\sigma)>0$, *then*

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(\rho)_n}{n!}\widehat{F}(\rho+n,\lambda;\nu;z;0,m)\widehat{F}(-n,\sigma;\tau;u;0,m)t^n\\&=(1-t)^{-\rho}\widehat{F}_2\left(\rho,\lambda,\sigma;\nu,\tau;\frac{z}{1-t},\frac{-ut}{1-t};0,m\right).\end{aligned}$$

(d) *Let* $m-1<\Re(\lambda-\nu)<m<\Re(\lambda)$, $m-1<\Re(\sigma-\tau)<m<\Re(\sigma)$, *then*

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(\rho)_n}{n!}\widehat{F}(\rho+n,\lambda;\nu;z;m,m)\widehat{F}(-n,\sigma;\tau;u;m,m)t^n\\&=(1-t)^{-\rho}\widehat{F}_2\left(\rho,\lambda,\sigma;\nu,\tau;\frac{z}{1-t},\frac{-ut}{1-t};m,m\right).\end{aligned}$$

Theorem 10 *For* $|az|<\min\{1,|1-t|\}$, $|t|<|1-az|$, $|bz|<1$, $|cz|<1$, *the following relations hold true*:

(a) *Let* $0<\Re(\lambda)<\Re(\nu)$, *then*

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(\rho)_n}{n!}\widehat{F}_D^3(\lambda,\rho+n,\sigma,\tau;\nu;az,bz,cz;0,0)t^n\\&=(1-t)^{-\rho}\widehat{F}_D^3\left(\lambda,\rho,\sigma,\tau;\nu;\frac{az}{1-t},bz,cz;0,0\right).\end{aligned}$$

(b) *Let* $\Re(\lambda-\nu)>0$, $\Re(\lambda+\eta)>0$, *then*

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(\rho)_n}{n!}\widehat{F}_D^3(\lambda+\eta,\rho+n,\sigma,\tau;2\lambda+\eta-\nu;az,bz,cz;0,0)t^n\\&=(1-t)^{-\rho}\widehat{F}_D^3\left(\lambda+\eta,\rho,\sigma,\tau;2\lambda+\eta-\nu;\frac{az}{1-t},bz,cz;0,0\right).\end{aligned}$$

(c) *Let* $m-1<\Re(\lambda-\nu)<m$, $\Re(\lambda)>0$, *then*

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(\rho)_n}{n!}\widehat{F}_D^3(\lambda,\rho+n,\sigma,\tau;\nu;az,bz,cz;0,m)t^n\\&=(1-t)^{-\rho}\widehat{F}_D^3\left(\lambda,\rho,\sigma,\tau;\nu;\frac{az}{1-t},bz,cz;0,m\right).\end{aligned}$$

(d) *Let* $m-1<\Re(\lambda-\nu)<m<\Re(\lambda)$, *then*

$$\sum_{n=0}^{\infty}\frac{(\rho)_n}{n!}\widehat{F}_D^3(\lambda,\rho+n,\sigma,\tau;\nu;az,bz,cz;m,m)t^n$$
$$=(1-t)^{-\rho}\widehat{F}_D^3\left(\lambda,\rho,\sigma,\tau;\nu;\frac{az}{1-t},bz,cz;m,m\right).$$

4 Concluding remarks

Recent developments in the theory of fractional calculus show its importance; therefore, the generalized Riemann–Liouville fractional operator $\widehat{D}^{\nu}$, the Caputo fractional operator $\widehat{\mathbf{D}}^{\nu}$ and the Kober–Erdelyi fractional operator $\widehat{\mathbf{I}}_{\eta}^{\nu}$ will be useful for investigators in various disciplines of applied sciences and engineering physics. We also try to find certain possible applications of these results presented here to some other research areas due to the presence of the generalized beta function $\widehat{B}$ possessing the advantage that a number of special functions happen to be particular cases of these functions.

We conclude this investigation by noting that the results deduced above are significant and can lead to numerous other fractional integral and derivative formulas and integral transforms involving various special functions by suitable specializations of arbitrary parameters in the main findings. More importantly, they are expected to find some applications in probability theory and to the solutions of differential equations.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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References

1. Agarwal, R.P., Agarwal, P.: Extended Caputo fractional derivative operator. Adv. Stud. Contemp. Math. **25**, 301–316 (2015)
2. Agarwal, P., Choi, J., Paris, R.B.: Extended Riemann–Liouville fractional derivative operator and its applications. J. Nonlinear Sci. Appl. **8**, 451–466 (2015)
3. Agarwal, P., Jain, S., Mansour, T.: Further extended Caputo fractional derivative operator and its applications. Russ. J. Math. Phys. **24**(4), 415–425 (2017)
4. Baleanu, D., Agarwal, P., Parmar, R.K., Alqurashi, M.M., Salahshour, S.: Extension of the fractional derivative operator of the Riemann–Liouville. J. Nonlinear Sci. Appl. **10**, 2914–2924 (2017)
5. Kıymaz, İ.O., Çetinkaya, A., Agarwal, P.: An extension of Caputo fractional derivative operator and its applications. J. Nonlinear Sci. Appl. **9**, 3611–3621 (2016)
6. Kıymaz, İ.O., Çetinkaya, A., Agarwal, P., Jain, S.: On a new extension of Caputo fractional derivative operator. In: Advances in Real and Complex Analysis with Applications, pp. 261–275. Birkhäuser, Basel (2017)
7. Luo, M.-J., Milovanovic, G.V., Agarwal, P.: Some results on the extended beta and extended hypergeometric functions. Appl. Math. Comput. **248**, 631–651 (2014)
8. Özarslan, M.A., Özergin, E.: Some generating relations for extended hypergeometric functions via generalized fractional derivative operator. Math. Comput. Model. **52**, 1825–1833 (2010)

9. Parmar, R.K.: Some generating relations for generalized extended hypergeometric functions involving generalized fractional derivative operator. J. Concr. Appl. Math. **12**, 217–228 (2014)
10. Srivastava, H.M., Parmar, R.K., Chopra, P.: A class of extended fractional derivative operators and associated generating relations involving hypergeometric functions. Axioms **1**, 238–258 (2012)
11. Goswami, A., Jain, S., Agarwal, P., Araci, S.: A note on the new extended beta and Gauss hypergeometric functions. Appl. Math. Inf. Sci. **12**(1), 139–144 (2018)
12. Özergin, E., Özarslan, M.A., Altın, A.: Extension of gamma, beta and hypergeometric functions. J. Comput. Appl. Math. **235**, 4601–4610 (2011)
13. Choi, J., Rathie, A.K., Parmar, R.K.: Extension of extended beta, hypergeometric and confluent hypergeometric functions. Honam Math. J. **36**(2), 357–385 (2014)
14. Chaudhry, M.A., Qadir, A., Rafique, M., Zubair, S.M.: Extension of Euler's beta function. J. Comput. Appl. Math. **78**, 19–32 (1997)
15. Srivastava, H.M., Manocha, H.L.: A Treatise on Generating Functions. Ellis Horwood Limited, Chichester (1984)
16. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)