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Preservice Middle and High School Mathematics Teachers' Strategies when Solving Proportion Problems

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Abstract The purpose of this study was to investigate eight preservice middle and high school mathematics teachers' solution strategies when solving single and multiple proportion problems. Real-world missing-value word problems were used in an interview setting to collect information about preservice teachers' (PSTs) reasoning about proportional relationships. An explanatory case study methodology with multiple cases was used to make comparisons within and across cases. Analysis of the semi-structured interviews with each PST revealed that using practical problems, in which plastic gears and a mini balance system were provided, and multiple proportion problems facilitated the PSTs' recognition of the proportional relationships in their solutions. Therefore, they avoided using cross-multiplication and erroneous strategies in those problems. Among the strategies that the PSTs used in solving single and multiple proportion problems, the ratio table strategy was the most frequent and effective strategy. The ratio table strategy enabled the PSTs to recognize the constant ratio and product relationships more than the other strategies. The results of this study illuminate how PSTs reason about proportional relationships when they cannot rely on computation methods like cross-multiplication.

Keywords Proportions · Proportional reasoning · Proportional relationships · Ratios

Introduction

Understanding ratios, proportions, and proportional reasoning constitutes an important area of school mathematics that is essential for students to learn but difficult for teachers to teach (Lobato & Ellis, 2010, p. 1). Proportional reasoning is also essential in understanding many situations in science and in everyday life

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(Cramer & Post, 1993). For instance, students need to understand and use ratios and proportions to represent quantitative relationships (National Council of Teachers of Mathematics, 2000), and they should be able to analyze proportional relationships and use them to solve real-world and mathematical problems (Common Core State Standards Initiative, 2010). The significance of proportional reasoning in students' mathematical development is also emphasized in the large-scale international assessments such as the Trends in International Mathematics and Science Study (TIMSS) and the Programme for International Student Assessment (PISA). These two assessments consider understanding ratios, proportions, and proportional relationships as a benchmark entity for students' mathematical proficiency. Therefore, proportional reasoning plays a key role in students' mathematical development, and it is an important concept in students' elementary school mathematics and in higher mathematics (Lesh, Post & Behr, 1988).

As stated in the Common Core State Standards for Mathematics, to be able to reason proportionally, students should be able to "Decide whether two quantities are in a proportional relationship (7.RP.2a)" (CCSSI, 2010, p. 48). There are two types of proportional relationships between quantities: *directly proportional relationships* and *inversely proportional relationships*. Lamon (2007) states that $y = k*x$ is the mathematical model for directly proportional relationships. In this model, the variables y and x represent the quantities that are in a directly proportional relationship, and the amount k represents the constant of proportionality. Since $y = k*x$ necessitates $y/x = k$, in a directly proportional relationship, the quotient of the two covarying quantities always remains constant. Whereas, the mathematical model for an inversely proportional relationship is $y*x = k$. Hence, in an inversely proportional relationship, the product of the values of inversely proportional quantities remains constant.

Despite the given importance and effort spent on teaching ratios and proportions, many studies (e.g. Clark, 2008; De Bock, Verschaffel & Janssens, 1998; Hart, 1984; Modestou & Gagatsis, 2007) have reported students' difficulties and poor performances on proportional relationships. A small number of studies on teachers' reasoning about proportional relationships (e.g. Izsák & Jacobson 2013) have reported that teachers' difficulties are often similar to those of students'. One of the problems with teaching and learning proportional relationships is that traditional proportion instruction places an emphasis on rule memorization and rote computations (Izsák & Jacobson, 2013). The most common textbook strategy for solving a missing-value problem is the cross-multiplication strategy (Fisher, 1988), which requires setting a proportion and cross-multiplying numbers within the proportion without thinking about proportional relationships between quantities. Although students and teachers may severely depend on using the cross-multiplication strategy when solving proportion problems, "*A robust understanding of proportional relationships includes understanding and using multiplicative relationships between two covarying quantities and recognizing whether or not two covarying quantities remain in the same constant ratio*" (Izsák & Jacobson, 2013, p. 2). Moreover, Stemn (2008) noted that although the cross-multiplication algorithm is effective in solving proportion problems, it hinders students' understanding of the multiplicative relationship.

A second problem with teaching and learning proportional relationships is that students and preservice teachers (PSTs) usually tend to judge nonproportional relationships to be proportional and apply proportional strategies for nonproportional situations (e.g. De Bock et al., 1998; Fisher, 1988; Izsák & Jacobson, 2013; Lim, 2009; Modestou & Gagatsis, 2007; Riley, 2010; Van Dooren, De Bock, Janssens & Verschaffel, 2007). For example, Fisher (1988) gave 20 secondary mathematics teachers the following inverse proportion problem:

If it takes nine workers 5 hours to mow a certain lawn, how long would it take six workers to mow the same lawn? (p. 160).

Fisher (1988) discussed that 12 out of 20 teachers solved this problem incorrectly, and 9 of them approached the problem as if it were a direct proportion problem. In addition, the following problems were also reported in the literature: Students and PSTs are likely to use additive strategies to solve proportion problems (Hart, 1984; Misailadou & Williams, 2003; Riley, 2010; Simon & Blume, 1994); they have difficulty creating suitable reciprocal multiplicative relationships for nonproportional problems (Izsák & Jacobson, 2013); and they have difficulty understanding ratio-as-measure and the invariance of a ratio (Simon & Blume, 1994).

In earlier research, researchers investigated teachers' proportional reasoning mostly using missing-value word problems, which usually involved a single proportional or nonproportional relationship. Moreover, as stated by Lamon (2007), mathematics education research has overlooked teachers' proportional reasoning. In particular, only a few studies have reported teachers' proportional reasoning regarding inverse proportions (e.g. Fisher, 1988; Izsák & Jacobson, 2013; Lim, 2009; Riley, 2010), and even fewer researchers have studied multiple proportions (e.g. Vergnaud, 1983, 1988). Hence, this study contributes to the current literature on proportions by investigating PSTs' reasoning on single and multiple proportions. In this study, I report results from semi-structured interviews during which preservice middle and high school mathematics teachers in the USA worked on real-world missing-value word problems that involved directly and inversely proportional relationships. The purpose of this study was to investigate PSTs' solution strategies and to report facilities and difficulties that they encountered when solving proportion problems. Therefore, the following research questions were investigated:

1. What strategies do preservice middle and high school mathematics teachers use to solve single and multiple proportion problems?
2. What difficulties and facilities do preservice middle and high school mathematics teachers encounter when solving proportion problems?

Theoretical Frameworks

The theoretical framework of this study is developed drawing on two frameworks. In particular, Vergnaud's (1983, 1988) the *multiplicative conceptual field theory* was used to develop the categories of proportions (see Fig. 1) and to interpret PSTs' reasoning on the problems presented in the mathematical tasks. Second, the solution strategies frameworks described by Fisher (1988), Inhelder and Piaget (1958), Karplus, Pulos and Stage (1983b), and (Lamon, 1993) were used as tools to classify and to explain the PSTs' solution strategies.

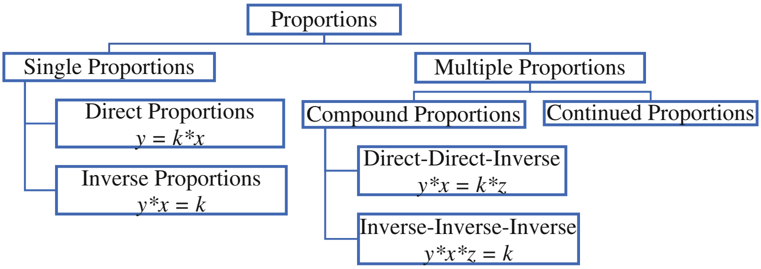


Fig. 1 Classification of proportions

The Multiplicative Conceptual Field Theory

Vergnaud (1983, 1988) emphasized that whole-number multiplication and division, fractions, ratios, rational numbers, and linear relationships are not mathematically independent of one another and placed these concepts within a larger context that he calls the *multiplicative conceptual field*. A *conceptual field* is a "set of problems and situations for the treatment of which concepts, procedures, and representations of different but narrowly interconnected types are necessary" (Vergnaud, 1983, p. 127). From this perspective, "a concept's meaning does not come from one situation only but from a variety of situations and that, reciprocally, a situation cannot be analyzed with one concept alone, but rather with several concepts, forming systems" (Vergnaud, 2009, p. 86). Hence, for Vergnaud (2009), a set of situations and a set of concepts are tied together, and "the development of a conceptual field requires children's meeting and being faced with contrasting situations" (p. 86). Therefore, as stated by Vergnaud (2009), he conceptualized the theory of *conceptual field* as a developmental theory.

Vergnaud (1983, 1988) discussed three types of multiplicative structures: *isomorphism of measures*, *product of measures*, and *multiple proportion other than product*. The isomorphism of measures structure "consists of a simple direct proportion between two measure-spaces *M1* and *M2*" (Vergnaud, 1983, p. 129) in which "Measure spaces usually refer to different sets of objects, different types of quantities, or different units of measure" (Lamon, 2007, p. 634). The *product of measures structure* "consists of the Cartesian composition of two measure-spaces, *M1* and *M2*, into a third, *M3*" (Vergnaud, 1983, p. 134). The problems in this structure are concerned with area, volume, Cartesian product, and work. For instance, multiplying the lengths of width and height of a rectangle yields its area. In the example, the area is *double proportional* to the lengths of width and height. In the *multiple proportion other than product* structure, "a measure-space *M3* is proportional to two different independent measure-spaces M1 and M2" (Vergnaud, 1983, p. 138). This type of proportional relationship is also called a *jointly proportional relationship* in the literature. For example, "The consumption of cereal in a scout camp is proportional to the number of persons and to the number of days" (Vergnaud, 1983, p. 138).

Vergnaud (1983) distinguished the second and third multiplicative structures as follows: In the *product of measures* structure, the constant of the proportionality is thought to be 1, but the same is not true for the *multiple proportion other than product* structure. For example, in the Speed task (see Table 1), the distance (*D*) a car traveled in some fixed time (*T*) at a constant speed (*V*) can be expressed by the equation $V*T = D$.

Table 1 Description of the mathematical tasks

Task and study used	Task descriptions	Problem characteristics
Gear study 1 and 2	This task included two parts: given that two gears were meshed, (a) the PSTs calculated either the number of notches or the size of a gear; (b) they calculated either the number of notches or the number of revolutions of a gear.	There was a single directly proportional relationship between the size of a gear and its number of notches and a single inversely proportional relationship between the number of revolutions that a gear made and its size. (Single proportion)
Bakery study 1 and 2	The PSTs calculated the number of cupcakes, number of minutes, and number of workers. In some problems, fixing one of the three quantities, the PSTs investigated a single relationship. In some other problems, there was not any fixed quantity, so the PSTs investigated multiple relationships.	The PSTs explored one inversely and two directly proportional relationships among the number of people, number of cupcakes, and number of minutes. Hence, the problems in this task could be classified within the multiple proportion category under the *direct-direct-inverse* proportions structure. (Multiple proportion)
Balance study 1 and 2	The PSTs were provided with a mini balance system. Given that some numbers of weights were hung on one side of the balance, they balanced the system on the other side by hanging weights at different distances.	The problems in this task could be classified within the single proportion category. The PSTs explored an inversely proportional relationship between the number of weights hung and distance (how far from the center a weight hung). (Single proportion)
Speed study 1 and 2	The PSTs calculated the distance, time, and speed of a given car. Similar to the Bakery task, single and multiple relationships were investigated.	The PSTs investigated one inversely and two directly proportional relationships among the distance, speed, and time. Hence, this task involved *direct-direct-inverse* proportional relationships. (Multiple proportion)
Fence study 1 and 2	In this task, the PSTs calculated the number of workers, number of days, and number of fences painted. Similar to the Bakery and Speed tasks, single and multiple relationships were investigated.	The PSTs identified one inversely and two directly proportional relationships among the number of workers, number of days, and number of fences painted. Therefore, this task involved *direct-direct-inverse* proportional relationships. (Multiple proportion)
Apartment study 1	The PSTs calculated the number of workers, number of hours they work each day, and number of days required to build an apartment.	This task involved *inverse-inverse-inverse* proportional relationships among the number of workers, number of hours they work each day, and number of days required to build an apartment. (Multiple proportion)
Painter study 1	Fixing the number of hours, the PSTs calculated either the number of bedrooms painted or the number of painters needed. Similarly, fixing	The PSTs explored a single directly proportional relationship between the number of painters and number of bedrooms and a single inversely

Table 1 (continued)

Task and study used	Task descriptions	Problem characteristics
	the number of bedrooms, they calculated either the number of painters or the number of hours.	proportional relationship between the number of painters and number of hours. (Multiple proportion)
Cookie Fac. study 1	The PSTs calculated either the number of assembly lines to make boxes of cookies to fill a truck or the number of hours needed to fill this truck.	This task involved a single inversely proportional relationship between the number of assembly lines and number of hours. (Single proportion)
Scout Camp study 2	Given that some number of people went on a scout camp, the PSTs calculated the number of people, amount of cereal each person ate per day, and number of days they stayed in the camp.	There were *inverse-inverse-inverse* proportional relationships among the number of people, amount of cereal each person eats per day, and number of days they stayed in the camp. (Multiple proportion)

In the example, the distance equation demonstrates a *product of measures* structure because the constant of the proportionality is 1. On the other hand, in the Bakery task (Table 1), the multiplicative relationships among the number of people (P), number of minutes (M), and number of cupcakes (C) can be expressed by the equation $P*M = kC$, in which $k \neq 1$ represents the constant of proportionality and demonstrates "number of person-minutes per cupcake." Although these two structures include an inversely proportional relationship between quantities multiplied, Vergnaud (1983, 1988) did not focus on this inversely proportional relationship in detail. Vergnaud (1988) called the problems of the type of *isomorphism of measures* as simple proportion problems and called the problems of the last two structures as multiple proportion problems. As noted by Vergnaud (1988), multiple proportion problems have not been researched widely, and most teachers are unaware of students' difficulties with these problems.

In this study, I classify proportions into two main categories: *single proportions* and *multiple proportions* (Fig. 1). In Fig. 1, I classified multiple proportions for the case of the three measure spaces. My classification uses Vergnaud's (1983, 1988) classification of multiplicative structures and extends it in several ways. First, my definition of a single proportion structure combines *isomorphism of measures* (direct proportion) and *inverse proportion* structures. Because Vergnaud (1983, 1988) did not focus on the inverse proportion structure in detail and an inversely proportional relationship is also formed between two quantities, I placed inverse proportion structure in the single proportion category. Second, my definition of a multiple proportion structure contains two main types of proportions: *compound* and *continued*. A compound proportion consists of the *product of measures* and *multiple proportion other than product* structures, in which a combination of direct and inverse proportions are presented. For three measure spaces, the compound proportion structure involves *direct-direct-inverse* and *inverse-inverse-inverse* proportion structures. Vergnaud's (1983, 1988) *product of measures* and *multiple proportion other than product* structures are equivalent to a *direct-direct-inverse* proportion structure in my classification because both structures combine one inversely and two directly proportional relationships. Therefore, I placed them in the compound proportion category under the direct-direct-inverse proportion structure rather than discussing them in separate categories. On the other hand, if there are quantities that have a structure in which a:b = b:c = c:d and so on, then this structure is called *continued proportions*. In the study, continued proportions are not discussed further.

Strategy Frameworks

The literature on students' proportional reasoning indicate that strategies students use when solving proportion problems are usually classified in some developmental stages. For example, Inhelder and Piaget (1958) discussed four stages of development for proportional reasoning: concrete operational, additive, pre-proportional, and proportional (as cited in Lesh & Harel, 2003). In the concrete operational reasoning stage, students' reasoning tend to involve qualitative judgments (e.g. A > B). In the additive reasoning stage, students focus on the additive differences between quantities (e.g. A-B = C-D). In the pre-proportional reasoning stage, students think about proportions by noticing a pattern or a relationship between quantities (e.g. two apples for 1 dollar, so four apples for 2 dollars). Finally, in the proportional reasoning stage, students reason about multiplicative relationships between quantities (e.g. A/B = C/D).

Similar to Inhelder and Piaget (1958), several studies reported strategies used by students and teachers when solving proportions. Considering the four stages above as a basis for my discussion, in the following pages, I discuss some of the ways previous studies classified students and teachers' responses. Although there are several other studies (e.g. Ben-Chaim, Keret & Ilany, 2012; Canada, Gilbert & Adolphson, 2008; Cox, 2013; Harel & Behr, 1995; Tourniaire & Pulos, 1985) that describe students' or teachers' strategies when solving proportion problems, I report the solution strategies frameworks described by Karplus et al. (1983b), Fisher (1988), and Lamon (1993). I decided to report these three studies because the strategy categories discussed in these studies were consistent with the four stages above. In my analysis, I mainly followed the strategy categories described by Fisher (1988) because it is one of the very few examples that demonstrates teachers' solution strategies.

Karplus et al. (1983b) developed lemonade puzzles to explore proportional and other types of reasoning of 60 sixth graders and 60 eight graders. They collected students' strategies in a strategy scale that involved four categories: *Category I* (incomplete or illogical strategy); *Category Q* (qualitative strategy); *Category A* (additive strategy); and *Category P* (proportional strategy). The responses in *Category I* indicated that students did not know the answer, guessed, or used inappropriate quantitative operations. The responses in *Category Q* demonstrated that students compared given quantities qualitatively using words, such as *less* and *more*, or identical terms. The responses in *Category A* showed that students compared quantities by paying attention to the differences in the values of these quantities. Finally, the responses in *Category P* revealed that students compared quantities by paying attention to the proportional relationships.

Fisher (1988) used a list of nine strategies to classify secondary teachers' solution strategies to solve two direct and two inverse proportion problems. She treated the first five of the nine strategies as the incorrect strategies and the remaining four as the correct strategies. *No Answer*; *Intuitive* (guessing the answer or answering the question by just relying on feelings or intuition); *Additive* (the subject incorrectly focuses on the additive differences between the given quantities and does not consider multiplicative relationships); *Proportion Attempt* (the subject understands that proportion was involved but cannot express the relationship); *Incorrect Other* (an incorrect strategy that cannot be placed in categories 1 – 4); *Proportion Formula* (a correct strategy in which the subject solves a problem by showing the equivalence of two ratios or by generating an equation that expresses the equality of two products followed by an explicit statement noticing the inverse relationship); *Proportional Reasoning* (the subject solves the problem by using a correct proportion strategy other than the proportion formula); *Algebra* (the subject solves the problem by setting up an algebraic equation other than the proportion formula); and *Correct Other* (a correct strategy that cannot be placed in categories 6 – 8).

Lamon (1993, p. 46) identified 6 strategies from 24 sixth-grade students' responses to a set of 40 ratio and proportion problems: *Avoiding* (students did not establish a genuine interaction with the problem); *Visual or Additive* (students solved problems by using the trial-and-error method, employed visual judgments, or they used incorrect additive strategies. They did not offer reasons for their responses); *Pattern Building* (students used verbal or written patterns without considering numerical relationships); *Pre-proportional Reasoning* (students used intuitive activities, such as charts, pictures, or models, to solve the problems. Some relative thinking was also involved in the solution processes); *Qualitative Proportional Reasoning* (students understood numerical relationships, used a ratio as a unit, and used relative thinking to solve the problems); and *Quantitative Proportional Reasoning* (students used algebraic symbols to show proportions and understood functional and scalar relationships in those symbols).

The strategy categories determined by Karplus et al. (1983b), Fisher (1988), and Lamon (1993) appeared to correspond and complete each other. The categories determined by Karplus et al. (1983b) were the relatively least extensive in comparison to the categories determined by Fisher (1988) and Lamon (1993). Lamon (1993) extended the categories of Karplus et al. (1983b) by considering the proportional reasoning category to include three separate categories: pre-proportional, qualitative proportional, and quantitative proportional. Furthermore, she considered visual and additive strategies in the same category and added pattern building as a new strategy category. Fisher (1988) had done relatively more extensive job in categorizing teachers' responses. Fisher's (1988) proportion formula, proportional reasoning, and algebra categories were consistent with Lamon's (1993) qualitative and quantitative proportional reasoning categories. Fisher (1988) grouped the strategies that fell out the categories that she determined by including incorrect other and correct other categories.

Methodology

Participants

An explanatory multiple-case study methodology (Yin, 1993, 2009) was used to design this study. Because one of the purposes of this study was to explore PSTs' reasoning, each individual PST constituted a case. Since there was more than one case, a multiple-case study methodology best suited the scope of this study. One of the most important strengths of case study is that it allows researchers to explore a real-life phenomenon in depth (Yin, 2009). Hence, conceptualizing this study as a multiple-case study methodology allowed me to explore PSTs' reasoning in depth. I recruited eight PSTs from one large public university in the Southeast USA. The university offers separate programs leading to certification for secondary grades (6 – 12) and middle grades (4 – 8) mathematics teachers. Because the focus of this study involved challenging mathematical problems, PSTs with some college level experience on direct and inverse proportions were preferred. The secondary grades program includes one content course with a focus on multiplicative relationships, ratios, and proportions; the middle grades program includes two such content courses. Therefore, PSTs who attended or were attending one of these courses were selected.

I designed a preliminary study (study 1) to learn PSTs' understanding of directly and inversely proportional relationships and to test the quality of the mathematical tasks and interview questions. Based on the responses collected in study 1, I reduced the number of tasks from eight to six and made small changes in the given tasks and follow-up questions; however, the mathematical structures of the tasks remained unchanged. In the spring semester of 2013, I recruited one female middle grade teacher (Abby) and one female (Sally) and two male secondary grade teachers (Jason and Robert) for study 1. In the fall semester of 2014, I recruited two female secondary grade (Kathy and Susan) and two female middle grade (Carol and Helen) PSTs for study 2. To maintain confidentiality, I replaced all real names with pseudonyms.

Data Collection and Analysis

I collected the data through semi-structured clinical interviews (Bernard, 1994). Conducting semi-structured interviews provided the flexibility to probe or modify the follow up questions depending on the PSTs' responses. Two video cameras were used during the interviews: One focused on the PST's written work, and the other focused on the PST and interviewer. I conducted all of the interviews, and one graduate student operated the video cameras. Each interview lasted between 45 and 90 min. In study 1, Robert, Jason, and Sally were interviewed for 3 h each, and Abby was interviewed for approximately 80 min. In study 2, each PST was interviewed approximately 4 to 5 h.

I used a thematic analysis approach (Boyatzis, 1998) to analyze the interview data. After transcribing the interviews verbatim, I open coded the interview transcripts line-by-line for the PSTs' strategies, and if necessary wrote memos about these strategies. Next, I created a code file for each task in Microsoft Excel. Then, I counted the number of occurrences of each code and entered that number in the record for the task. Later, I returned to the interview transcripts and recoded these to strengthen the reliability of the results. I then aggregated similar codes together to determine the connections among the codes and to identify relationships. In the last step of the data analysis, I wrote cross-task analyses of each case based on the strategies identified.

Mathematical Tasks

Nine mathematical tasks were used in this study (Table 1). Some combinations of the first eight tasks were used in study 1. Study 2 involved six tasks—Gear, Balance, Bakery, Balance, Speed, Fence, and Scout Camp—in which the Fence and Scout Camp tasks were used as extras. I developed the Gear, Painter, Apartment, and Balance System tasks and adapted the Bakery, Fence, Cookie Factory, and Speed tasks from *Mathematics for Elementary Teachers* (Beckmann, 2011), and adapted the Scout Camping task from Vergnaud's (1983) study. The mathematical tasks involved either single or multiple proportional relationships and all of the problems were missing-value type problems. In the Balance and Gear tasks, I provided the PSTs with a mini number balance system and plastic gears, respectively. The Gear task involved two parts in both study 1 and 2; the Bakery task involved two and three parts in study 1 and 2, respectively; and the Painter task involved two parts.

Mathematical Analysis of the Tasks. In the direct proportion problems, there was a constant ratio relationship between the two covarying quantities. For example, in the Gear task, there was a constant ratio relationship between the size of a gear and its number of notches. I provided the PSTs with two gears, which were thought to be meshed, and asked to calculate either the number of notches or the radius of one of the gears. In general, the multiplicative relationship between the number of notches (N) and radii (r) can be expressed with the following equation, where $\frac{N}{r}$ notches/cm represents the constant ratio or the unit rate.

$$(\mathrm{r\,cm})*\left(\frac{N}{r}\mathrm{notches/cm}\right) = N\,\mathrm{notches} \tag{1}$$

In the inverse proportion problems, there was a constant product relationship between the two covarying quantities. For example, in the Gear task, the product of the number of revolutions (R) and number of notches (N) yielded the total number of notches (T) revolved on a gear, which was constant. This relationship can be expressed with the following equation:

$$(R\,\mathrm{revolutions})*(N\,\mathrm{notches/revolution}) = T\ \mathrm{notches} \tag{2}$$

In the Balance task, to balance the system on two sides, the PSTs were required to have the same value, which was determined by multiplying the number of weights and distance from the center of the system, on both sides. Therefore, the contexts of the Gear and Balance tasks were appropriate for facilitating the PSTs' understanding of constant product relationships.

Because the PSTs compared three quantities in the multiple proportion tasks, identifying multiplicative relationships were expected to be more difficult than single proportion tasks. For instance, in the Bakery task, there was a *direct-direct-inverse* proportional relationships structure. The multiplicative relationships among the number of people (P), the number of minutes (M), and the number of cupcakes (C) can be expressed with the following equation. In the equation, k represents the *number of person-minutes per cupcake*.

$$(P\,\mathrm{people})*(M\,\mathrm{minutes}) = (C\,\mathrm{cupcakes})*k \tag{3}$$

Results

Based on the analysis of the PSTs' responses to the proportion tasks in study 1 and 2, I provided the PSTs' solution strategies in Table 2. Table 2 aligns the PSTs and mathematical tasks that they worked on. Moreover, it shows the strategies that they used for each task and the counts of these strategies appeared in each task. In the table, I entered strategy names in abbreviated forms and provided extended forms of these abbreviations at the end of the table. The table suggests that many of the PSTs' strategies could be classified as proportional reasoning, proportion formula, and algebra strategies (e.g. Fisher, 1988). I classified the ratio table, unit ratio, and double number line strategies within the proportional reasoning category because in these strategies, the PSTs considered proportionality in their reasoning and solved the given problems without necessarily using a proportion formula. If the PSTs used a proportion formula (i.e. a/b = c/d) to solve a given problem, then I classified this strategy under the proportion formula strategy. Following Fisher (1988) study, if the PSTs solved a problem by forming an algebraic equation other than a proportion formula, then I classified the strategy as an algebra strategy. Additive, computation (i.e. unit conversion method), intuitive (i.e. double counting strategy), and visual strategies were also observed but occurred in very few instances.

As Table 2 shows, the ratio table appeared to be the most preferred strategy. Three different variations of this strategy were observed. In the most common usage, the PSTs entered the given information side by side—without necessarily having rows and columns—and either multiplied or divided within and/or between measure spaces (Fig. 2a). In the second type of usage, they entered the information and separated the values of the quantities from different measure spaces by rows and columns, and again either multiplied or divided within and/or between measure spaces (Fig. 2b). In the last type, which was only used by Sally in the Bakery task, the information was entered into a parenthesis rather than into a table (Fig. 2c).

The ratio table strategy usually yielded the correct results if the PSTs inferred the correct relationships between quantities, and it was effective in facilitating the PSTs' recognition of the constant ratio and product relationships. For example, in the Gear task, Kathy calculated the number of revolutions of Gear L, with eight notches, given that Gear M had 14 notches and revolved four times. She used the idea "two gears go through the same number of notches" and calculated the answer to be seven revolutions. She calculated this answer by determining the total notches revolved on Gear M, in which she multiplied 14 notches and 4 revolutions, and dividing the total notches by 8 notches. The remaining PSTs also recognized this constant product relationship in the Gear task. Hence, the context of this task facilitated the PSTs' determination of the constant product, the total notches moved, by coordinating the number of revolutions and number of notches. When asked if she could use a ratio table strategy to solve the same problem, Kathy was able to generate one for the relationship between the number of notches and number of revolutions depicted in the problem (see Fig. 2b). She recognized that the product of all rows (notches and revolutions) was equal to 56:

Table 2 Preservice teachers' solution strategies

Study 1				
Task	Abby	Sally	Jason	Robert
Gear I		PF (11); RT (2)	PF (8); Additive (1)	PF (7); UR (2)
Gear II		PF (5)	Algebra (6); PF (1)	Algebra (6) Additive (2)
Bakery I		PF (3); RT (1)	RT (3)	UR (2) Algebra (1)
Bakery II		RT (3); Algebra (2)	RT (2), Algebra (1)	RT (3); Algebra (1)
Painter I				RT (2)
Painter II				RT (1); Algebra (1)
Fence		RT (4)	RT (3)	RT (1)
Apartment		RT (2)	RT (2); Algebra (1)	Algebra (3)
Cookie Factory				Algebra (2); RT (1)
Speed	UR (4); RT (4)	RT (4); UC (2)	RT (2)	
Balance	Algebra (5); RT (1)			
Study 2				
Task	Kathy	Susan	Carol	Helen
Gear I	RT (9); DNL (2) UR (2)	PF (11); RT (1)	PF (10); UR (4) SD (1); RT (1) DC (1)	RT (5); UR (4) PF (2); DC (1)
Gear II	Algebra (7) Additive (1) DNL (1); RT (1)	Algebra (9); RT (1) DNL (1); UR (1)	PF (3) Algebra (3)	RT (5) DNL (2)
Bakery I	DNL (3); RT (2)	RT (2); PF (2) UR (1); DC (1)	RT (3)	RT (5) UR (2)
Bakery II	RT (4)	RT (2); UR (1)	RT (2); Visual (1)	RT (2); Visual (2)
Bakery III	RT (3)	RT (6); DNL (6) UR (1)	RT (2); UR (2)	RT (3)
Balance	Algebra (3) RT (1)	Algebra (4); RT (1)	Algebra (3)	Algebra (4) RT (2)
Speed	RT (6); UC (4) DC (1)	RT (5); UC (4) Algebra (3); PF (1) DNL (1); UR (1)	RT (7); PF (1) UR (3); DNL (1) Additive (2)	UC (4); RT (3) PF (1)
Fence	RT (1)			RT (2)
Scout Camp	RT (3)	Algebra (2)	Visual (1)	Visual (1); RT (2)

RT ratio table, *PF* proportion formula, *UR* unit ratio, *DNL* double number line, *SD* strip diagram, *UC* unit conversion, *DC* double counting

Kathy: Okay, so well that has to be 56, I mean this is 56 here. I just know, I just kind of know that like all of these, like these two [pointed at notches and revolutions] have to multiply to give me 56 like every single time. So, I am saying what times two is 56 and that is, I do not know, 28. And then $\frac{56}{3}$, I do not know what that is.

Int: You can leave like that. Knowing that 56, you said 56 is the?

Kathy: is the product of notches and revolutions.

In the exchange, Kathy explicitly stated that 56 was the product of notches and revolutions. In her explanation, she attended to the multiplicative relationships between the number of notches and number of revolutions to discuss the constant product relationship. Therefore, the exchange and Fig. 2b suggested that the ratio table strategy facilitated Kathy's recognition of the constant product relationship between the number of notches and number of revolutions.

The use of the ratio table also appeared to be effective in the multiple proportion tasks (i.e. Bakery, Speed, Fence, Apartment, and Scout Camp) because it allowed the PSTs to fix one quantity at a time and then to compare the remaining two quantities. As a general strategy, the PSTs usually fixed the value of one of the three quantities and then compared the remaining two quantities. If the values of the two quantities were both increasing or decreasing together, then the PSTs concluded the relationship to be directly proportional. On the other hand, if the value of one quantity was increasing and the value of the second quantity was decreasing or vice versa, then they decided the relationship to be inversely proportional. For example, in the Bakery task, when I asked Susan to calculate the time required by one person to frost 2N cupcakes, given that three people frosted N cupcakes in T minutes, she immediately recognized that there was not a constant value and stated this by saying "Nothing's constant now." In her ratio table strategy (see Fig. 2a), Susan first fixed N cupcakes to a constant number, and by reasoning within measure spaces, she decided that one person could frost N cupcakes in 3T minutes. Next, she fixed the number of people to one, and by multiplying within measure spaces, Susan decided that one person could frost 2N cupcakes in 6T minutes. As shown in Fig. 2a, the PSTs

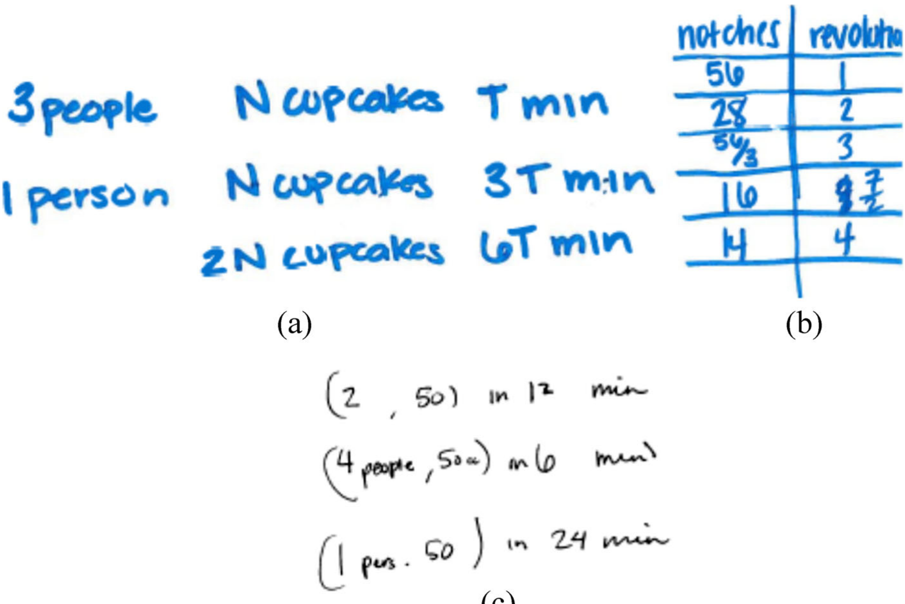


Fig. 2 **a** Susan's ratio table strategy in the Bakery task, **b** Kathy's ratio table strategy in the Gear task, and **c** Sally's ratio table strategy in the Bakery task

occasionally performed the multiplication and division operations without writing on the paper.

In the Speed task, by fixing a quantity at a time and describing qualitative relationships between the remaining two quantities, Abby determined a directly proportional relationship between distance and time and an inversely proportional relationship between time and speed:

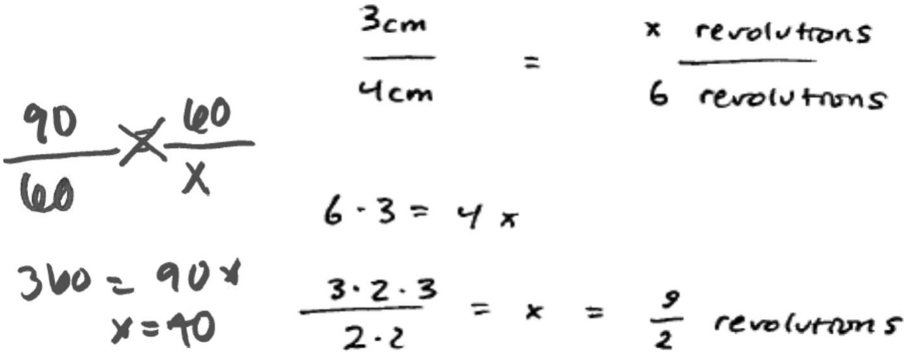
> Abby: The amount of miles and the amount of seconds is the proportional relationship if the miles per hour stay the same. But here if you are looking at the seconds and the miles per hours, inversely proportional because uhmmm the amount of miles that you are traveling stay the same, then as the seconds…as the amount of time you take increases the miles per hour that you are driving decreases if you are traveling the same distance. So, it depends on what uhmmm what variables you look at or what two variables you are comparing.

Abby's reasoning indicated that she inferred the directly and inversely proportional relationships by considering speed and distance as constants one at a time and describing the coordinated increases and decreases in the values of the remaining two quantities. In the multiple proportion problems, the PSTs successfully coordinated the need for fixing a quantity at a time with the presence of a directly or inversely proportional relationship between the other two quantities.

The second most common approach to solve a proportion problem was the proportion formula strategy. In this strategy, the PSTs formed a direct or an inverse proportion, showing the equivalence of two ratios, and they calculated the missing value by multiplying (or dividing) within or between measure spaces or by cross-multiplying values within the proportion (see Fig. 3a, b). The commonly observed mistake the PSTs had within this strategy was forming a direct proportion to solve an inverse proportion problem. In this study, Carol and Jason used a direct proportion to solve some of the inverse proportion problems. For instance, when I asked Carol to calculate the speed of a car that covered a certain distance in 60 s, given that another car covered the same distance in 90 s at 60 mph, she incorrectly formed a direct proportion. Next, she cross-multiplied the values within this proportion and calculated the speed of the car to be 40 mph (Fig. 3a). Carol immediately recognized that the answer she found was not correct and explained:

> Carol: Yeah, I cross multiply by, but it's not going to equals 90x and x equals 40 but that doesn't make sense because if you are traveling the same distance in the shorter amount of time, your speed will be…will increase, not decrease. So, that's not right, that's why I x'd it out…let me think, hold on…So, if you traveled X distance in 90 seconds at 60 mph and you want to travel X in 60 seconds and you want to know how much…how fast that would take. Gosh, I don't know…It would be I think it's…well if your time is decreasing, your speed is increasing because your distance's staying the same, right?

Carol's determination of the inverse qualitative relationship, which she stated by saying "if your time is decreasing, your speed is increasing," seemed to help her understand that her claim of the speed of the car as 40 mph was incorrect. Although Carol recognized that



(a) (b)

Fig. 3 **a** Carol's incorrect proportion formula strategy in the Speed task and **b** Jason's incorrect proportion formula strategy in the Gear task

forming a direct proportion was an inappropriate method, she had difficulty finding a better strategy to solve this problem. Similarly, in study 1, Jason also erroneously formed a direct proportion to solve an inverse proportion problem (Fig. 3b) in the Gear task. Nevertheless, overall, the use of incorrect proportion formula strategy was limited.

In the Speed task, Helen and Sally solved a direct proportion problem as if it were an inverse proportion problem. When I asked Helen to calculate the time needed by a car to travel 16 mi, given that it covered 40 mi in 60 min, she used a ratio table strategy and justified her reasoning:

> Helen: So, I can just do the table. So if I was traveling 1 mile it would take me more, like it would take me longer to travel 1 mile at the same rate.
>
> Int: Yeah at the same rate, sorry I forgot to mention that. You are driving at the same rate, same speed.
>
> Helen: So, that means it would take 2000 minutes for 1 mile.

Continuing the same strategy, Helen calculated the answer to be 125 min but then she immediately recognized that her answer was incorrect:

> Helen: …Because four times 5 is 20, yeah that is right yeah that time I have two zeros that is 2000 yeah yeah that is right. Okay, so then to get to 16 miles, that would times by 16, so that means I would divide this [pointed 2000 minutes] by 16. So, it would be 2000 over 16, which is [obtained 125 minutes]. So, 16 miles ohhh that does not make sense.
>
> Int: What happened?
>
> Helen: If I travel 40 miles in 50 minutes uhmm then to travel 1 mile it is, if I am just traveling wait if I am just going to travel just 1 mile it is going to take less time.

Helen's attention to the qualitative relationship, *less miles less time*, appeared to help her recognize the mistake. Considering Sally also reasoned in a similar fashion, the involvement of the time concept might have affected Helen and Sally's incorrect strategy decisions, or it is possible that they might confused the units, *miles* and *miles per hour*.

The PSTs preferred the unit ratio strategy after the ratio table and proportion formula strategies. In this strategy, the PSTs usually inferred a unit ratio relationship between two quantities and used this relationship to calculate a missing value in the problem. The PSTs usually used this strategy with the other strategies (Fig. 4a, b). For example, as displayed in Fig. 4a, Carol used this strategy with a proportion formula strategy, where she obtained a 1-cm-to-4-notches unit ratio relationship from 3 cm and 12 notches information. Similarly, Helen used the unit ratio strategy with a double number line strategy to obtain a 1-notch-24-revolutions relationship (see Fig. 4b). In study 1, Robert frequently stated the unit ratio relationships between any two quantities in the form of equations (e.g. 1 cm = 4 notches, 1 person = 25 cupcakes) and used this equation to calculate the missing-value asked in the problem.

The PSTs also used the algebra strategy to solve the given problems. In this strategy, the PSTs determined an algebraic expression or formula to indicate the multiplicative relationships between quantities and used this expression to solve the given problems. For instance, the PSTs easily recognized the constant product relationships in the Gear and Balance tasks, but they usually had difficulty recognizing constant products in the remaining inverse proportion tasks. In the Gear task, the PSTs generally considered the equality of the total number of notches revolved on two meshed gears after some number of revolutions. The PSTs' ideas about total number of notches revolved on a given gear could be expressed with the multiplication statement in Eq. 1 (see "Mathematical Analysis of the Tasks" section).

In the Balance task, the PSTs performed experiments to balance the system. Hence, they empirically determined the constant product relationship, $W1*D1 = W2*D2$, and used this relationship to solve the given problems. In this equation, $D1$ and $D2$ expressed the distances from the center in the first and second sides, and $W1$ and $W2$ expressed the number of weights hung in the first and second sides. Therefore, the contexts of the Gear and Balance tasks might have facilitated the PSTs' understanding of the constant product relationships between quantities. On the other hand, in the multiple proportion problems, the PSTs had difficulty determining algebraic equations or formulas in expressing the multiplicative relationships presented in those problems.

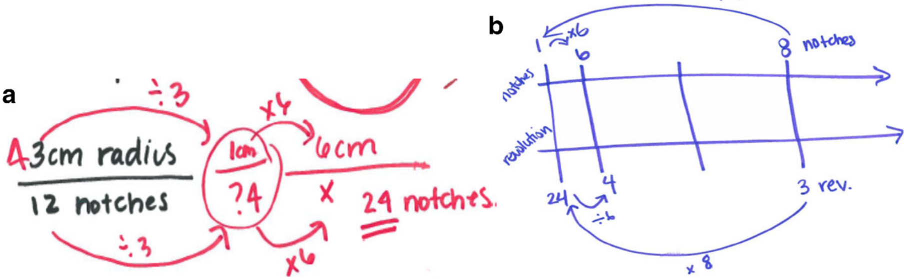


Fig. 4 **a** Carol's unit ratio strategy in the Gear task and;**b** Helen's double number line strategy in the Gear task

It is possible that their difficulties were related with inability to coordinate multiple relationships. Therefore, the PSTs' difficulty expressing multiplicative relationships in the multiple proportion tasks suggested possible constraints in their understanding of the multiplicative relationships when more than two quantities were present.

Because the PSTs who participated in study 2 were given instruction on double number line and strip diagram on proportions, they sometimes used the double number line and strip diagram strategies in solving given problems. In a double number line, a pair of quantities from two separate measure spaces generates a batch, and one can iterate this batch by a factor to get multiple batches. Hence, the quantities that generate a batch covary directly. Therefore, the double number line strategy was naturally more suitable in expressing directly proportional relationships. In the inverse proportion problems, the product of the inversely proportional quantities was directly proportional to each inversely proportional quantity. Hence, inverse proportion problems could also be solved by the double number line strategy if the intention was to express a directly proportional relationship. For instance, as shown in Fig. 4b, using the double number line strategy, Helen calculated the number of revolutions of a gear with six notches to be four, given that another gear with eight notches revolved three times. Because quantities inversely covary in an inverse proportion, using double number line was not an appropriate strategy; however, Helen just showed how to use a double number line to solve an inverse proportion problem without attending to the operations used.

In the Gear task, Kathy drew two separate double number lines (Fig. 5) to calculate the number of revolutions of Gear L, with 8 notches, given that Gear M had 14 notches and revolved four times. Because Kathy successfully inferred 56 notches to be the total notches moved on both gears by multiplying 14 notches per revolution and 4 revolutions, this information suggested her understanding of the constant product relationship between the number of notches and number of revolutions. Therefore, Kathy's understanding of the constant product relationship facilitated her comprehension of using two separate double number lines to solve this problem.

The PSTs also used additive, computation (i.e. unit conversion method), intuitive (i.e. double counting strategy), and visual strategies. Although the additive strategy is the most frequently reported erroneous method in the literature regarding ratios and proportions (Misailadou & Williams, 2003), in this study, the PSTs used additive,

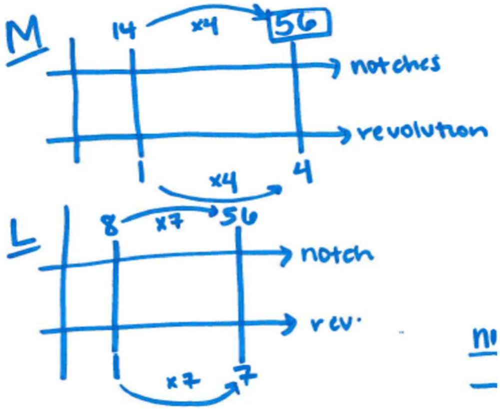


Fig. 5 Kathy's two double number lines strategy in the Gear task

computation, intuitive, and visual strategies to verify their calculations. Because the PSTs used these strategies in very few instances, they seemed to prefer reasoning multiplicatively rather than additively, and this was supported by the inclusion of Gear and Balance tasks and multiple proportion problems.

Discussion and Implications

In this study, I investigated how PSTs reason about proportional relationships. This is a critical topic because existing mathematics education research documents numerous difficulties that students and teachers have with this topic, some of which I discussed in the previous pages. In earlier studies on proportions, researchers generally used word problems with a single proportional or a nonproportional relationship to investigate how students or teachers reason about proportions. In this study, the PSTs examined single and multiple proportional relationships that were presented through real-world missing-value word problems. The purpose of this study was to report PSTs' strategies when solving these problems.

The PSTs usually persisted using a strategy if they observed that the strategy yielded the correct results. Therefore, in many instances, I encouraged them to use other strategies than the ones they generally employed. Based on the PSTs' responses, the ratio table strategy appeared to be the most frequent and convenient strategy in solving single and multiple proportion problems. In the single proportion problems, the use of the ratio table strategy facilitated the PSTs' recognition of the constant ratio and product relationships between quantities. The PSTs easily recognized the constancy of the quotients and products when they entered the given data into a ratio table. Otherwise, they had difficulty recognizing these constant relationships, especially the constant product relationships. Furthermore, in the multiple proportion problems, the ratio table strategy allowed the PSTs to fix one quantity at a time and then to compare the remaining two quantities. Therefore, the ratio table strategy was effective in solving multiple proportion problems. These results may inspire teachers and teacher educators to use the ratio table strategy in teaching direct and inverse proportions.

Besides the ratio table strategy, the PSTs also used the proportion formula very often. The common mistake of the PSTs with the proportion formula strategy was that because some of them tended to judge inversely proportional relationships to be directly proportional, which was consistent with the findings from previous studies (e.g. Cramer, Post & Currier, 1993; Fisher, 1988; Lim, 2009; Riley, 2010), they formed a direct proportion to solve an inverse proportion problem. In addition, in this study, two PSTs solved a direct proportion problem as if it were an inverse proportion problem. Therefore, these results suggested that they might not have had well-developed strategies for inferring directly and inversely proportional relationships and for successfully distinguishing these relationships from each other.

The use of physical materials (i.e. balance and plastic gears) provided practical experiences and helped the PSTs have a robust understanding of directly and inversely proportional relationships. The PSTs easily recognized the constancy of the products in the Gear and Balance tasks but they usually had difficulty recognizing constant products in the remaining inverse proportion tasks. In the Gear task, the context helped the PSTs in determining the constant product—the total notches moved—by coordinating the number of groups (where a group corresponded to one rotation) and the size of groups (where the size was the number of notches). Similarly, in the Balance task, the PSTs made experiments to balance the system on both sides, and so they empirically determined the constant product in the balance. Hence, the contexts of the Gear and Balance tasks facilitated the PSTs' understanding of the constant product relationships between quantities. This result encourages using concrete materials in teaching inversely proportional relationships.

Because multiple proportion problems were formed by three quantities, it was not easy to solve those problems by simply forming a proportion and applying the cross-multiplication strategy. The PSTs avoided using cross-multiplication and additive strategies in those problems. Hence, the inclusion of multiple proportion problems precipitated the use of ratio table, proportion formula, and unit ratio strategies. Because solving proportion problems using mechanical knowledge or cross-multiplication does not guarantee proportionality taking place in students' reasoning, and students often engage in more sophisticated reasoning when not using the cross-multiplication strategy (Avcu & Dogan, 2014; Karplus, Pulos and Stage, 1983a; Lamon, 2007), the use of multiple proportion tasks was effective in revealing the PSTs' reasoning about proportional relationships. Therefore, the results of this study illuminate how PSTs reason about proportional relationships when they cannot rely on computation methods like cross-multiplication. Moreover, the PSTs' difficulty expressing multiplicative relationships between quantities in the multiple proportion problems suggested possible constraints in their understanding of the multiplicative relationships when more than two quantities were present. Thus, there may be important differences in how students reason about multiplicative relationships among the tasks.

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