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Commutators and generalized local Morrey spaces ☆

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ABSTRACT

In this paper we study the behavior of Hardy–Littlewood maximal operator and the action of commutators in generalized local Morrey spaces $LM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ and generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$.

1. Introduction

We study in generalized local Morrey spaces $LM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ and generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ the boundedness of Hardy–Littlewood maximal operator in terms of Sharp Maximal Function and, as consequence, the boundedness of Commutators of the type

$$[a,K](f) = a\,(K,f) - K\,(a,f),$$

where K is a Calderón–Zygmund singular integral operator, f is in a Generalized Local Morrey Space $LM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ and the function a belongs to the Bounded Mean Oscillation class (B.M.O.) at first defined by John–Nirenberg.

The Generalized Morrey Spaces $M^{p,\varphi}(\mathbb{R}^n)$ are obtained by replacing in the classical Morrey Space $M^{p,\lambda}(\mathbb{R}^n)$, r^λ by a function φ.

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The classical Morrey spaces were introduced by Morrey [19] to study the local behavior of solutions to second order elliptic partial differential equations (see e.g. [17,22]). For the properties and applications of classical Morrey spaces, we refer the readers to [8,9,14,19]. Mizuhara [18] and Nakai [21] introduced generalized Morrey spaces. Later, Guliyev [14] defined the generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ with normalized norm.

We point out that φ is a measurable non-negative function and no monotonicity type condition is imposed on it. Commutators in Generalized Morrey Spaces have not been studied up to now and this paper seems to be the first in this direction.

We observe that in this paper we extend results contained in [7], basic tool in the subsequent study of regularity results of solutions of partial differential equations of elliptic and parabolic type and systems (see e.g. [8,6,23,24] and others). Also, Corollary 4.3 can be viewed as a generalization of a well known inequality by Fefferman and Stein, see [10] p. 153, and Theorem 4.5, is true under more general hypotheses that can be found in literature, see [28] pp. 417–418. Aim of the authors is to continue the study of this kind of operators and apply the new results contained in the present paper to partial differential equations of different type.

2. Definitions and useful tools

We set, throughout the paper,

$$B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$$

a generic ball in $\mathbb{R}^n$ centered at x with radius r.

We find it convenient to define the generalized Morrey spaces in the following form.

Definition 2.1. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $M^{p,\varphi} \equiv M^{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L^p_{\rm loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M^{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{L^p(B(x,r))}.$$

Also by $WM^{p,\varphi} \equiv WM^{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL^p{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM^{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{WL^p(B(x,r))} < \infty,$$

where $WL^p(B(x,r))$ denotes the weak L^p-space consisting of all measurable functions f for which

$$\|f\|_{WL^p(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL^p(\mathbb{R}^n)} < \infty.$$

According to this definition we recover, for $0 \le \lambda < n$, the Morrey space $M^{p,\lambda}$ and weak Morrey space $WM^{p,\lambda}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$M^{p,\lambda} = M^{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}, \qquad WM^{p,\lambda} = WM^{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

The vanishing Morrey space $VL^{p,\lambda}(\mathbb{R}^n)$ in its classical version is introduced in [29], where applications to PDE were considered. We also refer to [5,27] for some properties of such spaces.

We are ready to give the following definition of Vanishing generalized Morrey spaces, inspired by the classical one of Vanishing Morrey spaces given by Vitanza and deeply treated in [29] and [30].

Definition 2.2 *(Vanishing generalized Morrey space).* The vanishing generalized Morrey space $VM^{p,\varphi}(\mathbb{R}^n)$ is defined as the space of functions $f \in M^{p,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{L^p(B(x,r))} = 0.$$

Definition 2.3 *(Vanishing weak generalized Morrey space).* The vanishing weak generalized Morrey space $VWM^{p,\varphi}(\mathbb{R}^n)$ is defined as the space of functions $f \in WM^{p,\varphi}(\mathbb{R}^n)$ such that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{WL^p(B(x,r))} = 0.$$

Everywhere in the sequel we assume that

$$\lim_{r\to 0} \frac{1}{\inf_{x\in\mathbb{R}^n} \varphi(x,r)} = 0 \tag{2.1}$$

and

$$\sup_{0<r<\infty} \frac{1}{\inf_{x\in\mathbb{R}^n} \varphi(x,r)} < \infty, \tag{2.2}$$

which makes the spaces $VM^{p,\varphi}(\mathbb{R}^n)$ and $VWM^{p,\varphi}(\mathbb{R}^n)$ non-trivial, because bounded functions with compact support belong then to this space.

The spaces $VM^{p,\varphi}(\mathbb{R}^n)$ and $WVM^{p,\varphi}(\mathbb{R}^n)$ are Banach spaces with respect to the norm

$$\|f\|_{VM^{p,\varphi}} \equiv \|f\|_{M^{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{L^p(B(x,r))},$$

$$\|f\|_{VWM^{p,\varphi}} \equiv \|f\|_{WM^{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{WL^p(B(x,r))},$$

respectively.

We will also use the notation

$$\mathfrak{M}^{p,\varphi}(f;x,r) := \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{L^p(B(x,r))}$$

and

$$\mathfrak{M}^{W}_{p,\varphi}(f;x,r) := \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{WL^p(B(x,r))}$$

for brevity, so that

$$VM^{p,\varphi}(\mathbb{R}^n) = \left\{ f \in M^{p,\varphi}(\mathbb{R}^n) : \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{M}^{p,\varphi}(f;x,r) = 0 \right\}$$

and similarly for $VWM^{p,\varphi}(\mathbb{R}^n)$.

Besides the modular $\mathfrak{M}^{p,\varphi}(f;x,r)$ we also use its least non-decreasing dominant

$$\widetilde{\mathfrak{M}}^{p,\varphi}(f;x,r) = \sup_{0<t<r} \mathfrak{M}^{p,\varphi}(f;x,t), \tag{2.3}$$

which may be equivalently used in the definition of the vanishing spaces, since

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{M}^{p,\varphi}(f;x,r) = 0 \not\Leftrightarrow \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \widetilde{\mathfrak{M}}^{p,\varphi}(f;x,r) = 0.$$

Let us consider, for $f \in L^1_{loc}(\mathbb{R}^n)$, the *Hardy Littlewood Maximal Operator* M as

$$M\,f\,(x) \;=\; \sup_{B(x,r)} \frac{1}{|\,B(x,r)\,|} \int\limits_{B(x,r)} |\,f(y)\,|\,d\,y,$$

where $B(x,r)$ is the ball centered at x of radius r (see [26], pp. 8–9).

Remark 2.4. We observe that the properties stated for M hold for the larger "uncentred" maximal function $\widetilde{M}f$ defined by

$$\widetilde{M}\,f\,(x) \;=\; \sup_{x\in B} \frac{1}{|\,B\,|} \int\limits_{B} |\,f(y)\,|\,d\,y,$$

where the supremum is taken, not just over all balls B centered in x but to all balls B containing x.

It is true because, for every x, we can write

$$(M\,f)(x) \le (\widetilde{M}f)(x)$$

and also exists a constant c greater than 1 such that

$$(\widetilde{M}f)(x) \le c\,(M\,f)(x).$$

For these observations see [26] p. 13 (also [28] p. 80).

Two variants of Hardy–Littlewood Maximal function M, are the following *Sharp Maximal Function*

$$f^{\#}(x) \;=\; \sup_{x\in B} \frac{1}{|\,B\,|} \int\limits_{B} |\,f(y) \,-\, f_B|\,d\,y,$$

where the supremum is taken over the balls B containing x (see [26], p. 146) and the *Fractional Maximal Function* $M_\eta\,f$ used, for instance, by Muchkenhoupt and Wheeden in their relevant results contained in [20]:

$$M_\eta\,f\,(x) \;=\; \sup_{x\in B} \frac{1}{|\,B\,|^{\,1\,-\,\eta}} \int\limits_{B} |\,f(y)\,|\,d\,y,$$

where $f \in L^1_{loc}(\mathbb{R}^n)$, $0 \,<\eta\, < 1$ and the supremum is taken over the balls B containing x.

Let K be a Calderón–Zygmund singular integral operator (see e.g. [22]). Useful in the sequel is the following *Commutator* between the operator K and the multiplication operator by a locally integrable function a on $\mathbb{R}^n$:

$$\big[a, K\big](f)\,x \;=\; a(x)\,\big(K\,f\big)(x) \,-\, K\,\big(a\,f\big)(x),$$

for suitable functions f. Later, is useful to consider the function a in the space BMO of Bounded Mean Oscillation functions (see [16]).

Lemma 2.5. *(See [7], Lemma 1.) Let K be a Calderón–Zygmund singular integral operator, $1 < q \,<\, s \,<\, p \,<\, +\infty$, $0 < \,\lambda < \,n$ and $a \in BMO(\mathbb{R}^n)$.*

Then there exists a constant $c \ge 0$ independent of a and f such that

$$\big(\big[\,a,\,K\,\big](\,f\,)\big)^{\#}\,(x) \;\le\; c\,\|a\|_*\,\Big\{\Big(\,M\,|Kf|^q\Big)^{\frac{1}{q}}\,(x) \;+\; \Big(\,M\,|f|^s\Big)^{\frac{1}{s}}\,(x)\Big\}$$

for a.a. $x \,\in\, \mathbb{R}^n$ and every $f \,\in\, M^{p,\lambda}(\mathbb{R}^n)$.

The proof of this Lemma is similar to that one contained in [28], pp. 418–419, due to J.-O. Strömberg, it could be generalized for a function $f \in M^{p,\varphi}(\mathbb{R}^n)$.

In the sequel we need the following supremal inequalities.

Let v be a weight. We denote by $L_v^\infty(0,\infty)$ the space of all functions $g(t)$, $t > 0$ with finite norm

$$\|g\|_{L_v^\infty(0,\infty)} = \sup_{t>0} v(t)|g(t)|$$

and $L^\infty(0,\infty) \equiv L_1^\infty(0,\infty)$. Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset of all nonnegative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and

$$\mathcal{A} = \left\{ \varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0 \right\}.$$

Let u be a continuous and non-negative function on $(0,\infty)$. We define the supremal operator $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(t,\infty)}, \quad t \in (0,\infty).$$

The following theorem was proved in [4].

Theorem 2.6. *Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L_\infty(t,\infty)} < \infty$ for any $t > 0$ and let u be a continuous non-negative function on $(0,\infty)$. Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(0,\infty)$ to $L_{\infty,v_2}(0,\infty)$ on the cone $\mathcal{A}$ if and only if*

$$\left\| v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)} \right) \right\|_{L_\infty(0,\infty)} < \infty. \tag{2.4}$$

3. Generalized local Morrey spaces and vanishing generalized local Morrey spaces

Definition 3.1. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $LM^{p,\varphi} \equiv LM^{p,\varphi}(\mathbb{R}^n)$ the local generalized Morrey space, the space of all functions $f \in L^p_{\rm loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{LM^{p,\varphi}} = \sup_{r>0} \varphi(0,r)^{-1} \, |B(0,r)|^{-\frac{1}{p}} \, \|f\|_{L^p(B(0,r))}.$$

Also by $WLM^{p,\varphi} \equiv WLM^{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL^p_{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM^{p,\varphi}} = \sup_{r>0} \varphi(0,r)^{-1} \, |B(0,r)|^{-\frac{1}{p}} \, \|f\|_{WL^p(B(0,r))} < \infty.$$

Definition 3.2. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $LM^{p,\varphi}_{\{x_0\}} \equiv LM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ the local generalized Morrey space, the space of all functions $f \in L^p_{\rm loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{LM^{p,\varphi}_{\{x_0\}}} = \|f(x_0 + \cdot)\|_{LM^{p,\varphi}}.$$

Also by $WLM^{p,\varphi}_{\{x_0\}} \equiv WLM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL^p_{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM^{p,\varphi}_{\{x_0\}}} = \|f(x_0+\cdot)\|_{WLM^{p,\varphi}} < \infty.$$

According to this definition we recover, for $0 \le \lambda < n$, the local Morrey space $LM^{p,\lambda}_{\{x_0\}}$ and weak local Morrey space $WLM^{p,\lambda}_{\{x_0\}}$ under the choice $\varphi(x_0,r) = r^{\frac{\lambda-n}{p}}$:

$$LM^{p,\lambda}_{\{x_0\}} = LM^{p,\varphi}_{\{x_0\}}\Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}, \qquad WLM^{p,\lambda}_{\{x_0\}} = WLM^{p,\varphi}_{\{x_0\}}\Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}.$$

Wiener [31,32] looked for a way to describe the behavior of a function at the infinity. The conditions he considered are related to appropriate weighted L^q spaces. Beurling [3] extended this idea and defined a pair of dual Banach spaces A_q and $B_{q'}$, where $1/q + 1/q' = 1$. To be precise, A_q is a Banach algebra with respect to the convolution, expressed as a union of certain weighted L^q spaces; the space $B_{q'}$ is expressed as the intersection of the corresponding weighted $L_{q'}$ spaces. Feichtinger [11] observed that the space B_q can be described by

$$\|f\|_{B_q} = \sup_{k\ge 0} 2^{-\frac{kn}{q}} \|f\chi_k\|_{L^q(\mathbb{R}^n)}, \tag{3.5}$$

where χ_0 is the characteristic function of the unit ball $\{x \in \mathbb{R}^n : |x| \le 1\}$, χ_k is the characteristic function of the annulus $\{x \in \mathbb{R}^n : 2^{k-1} < |x| \le 2^k\}$, $k = 1, 2, \dots$. By duality, the space $A_q(\mathbb{R}^n)$, called Beurling algebra now, can be described by

$$\|f\|_{A_q} = \sum_{k=0}^{\infty} 2^{-\frac{kn}{q'}} \|f\chi_k\|_{L^q(\mathbb{R}^n)}. \tag{3.6}$$

Let $\dot{B}_q(\mathbb{R}^n)$ and $\dot{A}_q(\mathbb{R}^n)$ be the homogeneous versions of $B_q(\mathbb{R}^n)$ and $A_q(\mathbb{R}^n)$ by taking $k \in \mathbb{Z}$ in (3.5) and (3.6) instead of $k \ge 0$ there.

If $\lambda < 0$, then $LM^{p,\lambda}_{\{x_0\}}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$. Note that $LM^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $LM^{p,n}(\mathbb{R}^n) = \dot{B}_p(\mathbb{R}^n)$.

Alvarez, Guzman-Partida and Lakey [2] in order to study the relationship between central BMO spaces and Morrey spaces, they introduced λ-central bounded mean oscillation spaces and central Morrey spaces $\dot{B}_{p,\lambda}(\mathbb{R}^n)$.

The following lemma, useful in itself, shows that the quasi-norm of the local Morrey space $LM^{p,\lambda}(\mathbb{R}^n)$, $\lambda \ge 0$ is equivalent to the quasi-norm $\dot{B}_{p,\lambda}(\mathbb{R}^n)$:

$$\|f\|_{\dot{B}_{p,\lambda}} = \sup_{k\in\mathbb{Z}} 2^{-\frac{k\lambda}{p}} \|f\chi_k\|_{L_p},$$

where χ_k is the characteristic function of the annulus $B(0,2^k) \setminus B(0,2^{k-1})$, $k \in \mathbb{Z}$.

Lemma 3.3. *For $0 < p \le \infty$, $\lambda \ge 0$, the quasi-norm $\|f\|_{LM^{p,\lambda}}$ is equivalent to the quasi-norm $\|f\|_{\dot{B}_{p,\lambda}}$.*

Proof. Let $0 < p \le \infty$, $\lambda \ge 0$ and $f \in LM^{p,\lambda}(\mathbb{R}^n)$. Then

$$\|f\|_{\dot{B}_{p,\lambda}} \le \sup_{k\in\mathbb{Z}} (2^k)^{-\frac{\lambda}{p}} \|f\|_{L_p(B(0,2^k))} \le \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(0,r))} = \|f\|_{LM^{p,\lambda}}.$$

On the other hand, for $0 < p < \infty$,

$$\|f\|^p_{LM^{p,\lambda}} = \sup_{k\in\mathbb{Z}} \sup_{2^{k-1}<r\le 2^k} r^{-\lambda} \int\limits_{B(0,r)} |f(y)|^p dy$$

$$\le 2^{\lambda} \sup_{k\in\mathbb{Z}} (2^k)^{-\lambda} \int\limits_{B(0,2^k)} |f(y)|^p dy$$

$$= 2^{\lambda} \sup_{k\in\mathbb{Z}} 2^{-k\lambda} \sum_{m=-\infty}^{k} 2^{m\lambda} 2^{-m\lambda} \int\limits_{B(0,2^m)\setminus B(0,2^{m-1})} |f(y)|^p dy$$

$$\le 2^{\lambda} \Big(\sup_{m\in\mathbb{Z}} 2^{-m\lambda} \int\limits_{B(0,2^m)\setminus B(0,2^{m-1})} |f(y)|^p dy \Big) \sup_{k\in\mathbb{Z}} \Big(2^{-k\lambda} \sum_{m=-\infty}^{k} 2^{m\lambda} \Big)$$

$$= \frac{2^{\lambda}}{1-2^{-\lambda}} \|f\|^p_{\dot{B}_{p,\lambda}}.$$

So for $0 < p < \infty$

$$\|f\|_{LM^{p,\lambda}} \le 2^{\frac{\lambda}{p}} \left(1-2^{-\lambda}\right)^{-\frac{1}{p}} \|f\|_{\dot{B}_{p,\lambda}}.$$

A similar argument shows that

$$\|f\|_{LM_{\infty,\lambda}} \le \|f\|_{\dot{B}_{\infty,\lambda}}. \quad \square$$

The quasi-norms $\|f\|_{\dot{B}_{p,\lambda}}$ in the case $\lambda = n$ were investigated by Beurling [3], Feichtinger [11] and others. The following statement was proved in [12] (see also [13–15]).

Theorem 3.4. *Let $x_0 \in \mathbb{R}^n$, $1 \le p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\int\limits_r^{\infty} \varphi_1(x_0,t) \frac{dt}{t} \le C\, \varphi_2(x_0,r), \tag{3.7}$$

where C does not depend on r. Let also K be a Calderón–Zygmund singular integral operator. Then the operator K is bounded from $LM^{p,\varphi_1}_{\{x_0\}}$ to $LM^{p,\varphi_2}_{\{x_0\}}$ for $p > 1$ and from $LM^{1,\varphi_1}_{\{x_0\}}$ to $WLM^{1,\varphi_2}_{\{x_0\}}$ for $p = 1$.

The following statement, containing results obtained in [18,21] was proved in [12] (see also [13,14]).

Corollary 3.5. *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\int\limits_r^{\infty} \varphi_1(x,t) \frac{dt}{t} \le C\, \varphi_2(x,r), \tag{3.8}$$

where C does not depend on x and r. Let also K be a Calderón–Zygmund singular integral operator. Then the operator K is bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2} for $p = 1$.

4. Results

Theorem 4.1. *For any fixed $x_0 \in \mathbb{R}^n$, $r > 0$, $f \in L^q_{\rm loc}(\mathbb{R}^n)$ and $1 < q < +\infty$*

$$\begin{aligned} \|Mf\|_{L^q(B(x_0,r))} &\le c\, r^{\frac{n}{q}} \sup_{t>2r} t^{-\frac{n}{q}} \|f\|_{L^q(B(x_0,t))} \\ &\le c\, r^{\frac{n}{q}} \sup_{t>2r} t^{-\frac{n}{q}} \|f^{\sharp}\|_{L^q(B(x_0,t))}, \end{aligned} \tag{4.9}$$

and for all $x_0 \in \mathbb{R}^n$*,* $r > 0$ *and* $f \in L^1_{\rm loc}(\mathbb{R}^n)$

$$\|Mf\|_{WL^1(B(x_0,r))} \le c\,r^n \sup_{t>2r} t^{-n}\, \|f\|_{L^1(B(x_0,t))}$$
$$\le c\,r^n \sup_{t>2r} t^{-n}\, \|f^\sharp\|_{L^1(B(x_0,t))}, \tag{4.10}$$

where c *is independent of* f*,* x_0 *and* r*.*

Proof. Inequalities (4.9) and (4.10) are consequence of Lemma 3.3 in [1] and the following inequality

$$\|f\|_{L^q(B(x_0,t))} \le \|f^\sharp\|_{L^q(B(x_0,t))}$$

is contained in [10]. □

Theorem 4.2. *Let* $x_0 \in \mathbb{R}^n$*,* $1 \le q < \infty$ *and the functions* φ_1*,* φ_2 *satisfy the condition*

$$\sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x_0,\tau)\,\tau^{\frac{n}{q}}}{t^{\frac{n}{q}}} \le C\,\varphi_2(x_0,r), \tag{4.11}$$

where C *does not depend on* r*. Then for* $1 < q < \infty$ *the maximal operator* M *is bounded from* $LM^{q,\varphi_1}_{\{x_0\}}(\mathbb{R}^n)$ *to* $LM^{q,\varphi_2}_{\{x_0\}}(\mathbb{R}^n)$ *and for* $1 \le q < \infty$ *the operator* M *is bounded from* $LM^{q,\varphi_1}_{\{x_0\}}(\mathbb{R}^n)$ *to* $WLM^{q,\varphi_2}_{\{x_0\}}(\mathbb{R}^n)$*. Moreover, for* $1 < q < \infty$

$$\|Mf\|_{LM^{q,\varphi_2}_{\{x_0\}}} \le c\,\|f\|_{LM^{q,\varphi_1}_{\{x_0\}}} \le c\,\|f^\sharp\|_{LM^{q,\varphi_1}_{\{x_0\}}},$$

where c *does not depend on* x_0 *and* f *and for* $1 \le q < \infty$

$$\|Mf\|_{WLM^{q,\varphi_2}_{\{x_0\}}} \le c\,\|f\|_{LM^{q,\varphi_1}_{\{x_0\}}} \le c\,\|f^\sharp\|_{LM^{q,\varphi_1}_{\{x_0\}}},$$

where c *does not depend on* x_0 *and* f*.*

Proof. By Theorems 4.1 and 2.6 we get

$$\|Mf\|_{LM^{q,\varphi_2}_{\{x_0\}}} \le c \sup_{r>0} \varphi_2(x_0,r)^{-1} \sup_{t>2r} t^{-\frac{n}{q}}\, \|f\|_{L^q(B(x_0,t))}$$
$$\le c \sup_{r>0} \varphi_1(x_0,r)^{-1}\, r^{-\frac{n}{q}}\, \|f\|_{L^q(B(x_0,t))}$$
$$= c\,\|f\|_{LM^{q,\varphi_1}_{\{x_0\}}} \le c\,\|f^\sharp\|_{LM^{q,\varphi_1}_{\{x_0\}}},$$

where c does not depend on x_0 and f, if $1 \le q < \infty$ and

$$\|Mf\|_{WLM^{q,\varphi_2}_{\{x_0\}}} \le c \sup_{r>0} \varphi_2(x_0,r)^{-1} \sup_{t>2r} t^{-\frac{n}{q}}\, \|f\|_{L^q(B(x_0,t))}$$
$$\lesssim \sup_{r>0} \varphi_1(x_0,r)^{-1}\, r^{-\frac{n}{q}}\, \|f\|_{L^q(B(x_0,t))}$$
$$= \|f\|_{LM^{q,\varphi_1}_{\{x_0\}}} \le c\,\|f^\sharp\|_{LM^{q,\varphi_1}_{\{x_0\}}},$$

where c does not depend on x_0 and f, if $1 \le q < \infty$. □

Remark 4.3. Let $1 \le q < \infty$ and the functions φ_1, φ_2 satisfy the condition

$$\sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau)\, \tau^{\frac{n}{q}}}{t^{\frac{n}{q}}} \le C\, \varphi_2(x,r), \tag{4.12}$$

where C does not depend on x and r. Then for $1 < q < \infty$ the maximal operator M is bounded from $M^{q,\varphi_1}(\mathbb{R}^n)$ to $M^{q,\varphi_2}(\mathbb{R}^n)$ and for $1 \le q < \infty$ the operator M is bounded from $M^{q,\varphi_1}(\mathbb{R}^n)$ to $WM^{q,\varphi_2}(\mathbb{R}^n)$. Moreover, for $1 < q < \infty$

$$\|Mf\|_{M^{q,\varphi_2}} \le c\, \|f\|_{M^{q,\varphi_1}} \le c\, \|f^\sharp\|_{M^{q,\varphi_1}},$$

where c does not depend on f and for $1 \le q < \infty$

$$\|Mf\|_{WM^{q,\varphi_2}} \le c\, \|f\|_{M^{q,\varphi_1}} \le c\, \|f^\sharp\|_{M^{q,\varphi_1}},$$

where c does not depend on f.

Remark 4.4. Let us consider $x_0 \in \mathbb{R}^n$, $1 < p < +\infty$, $0 < \lambda < n$.

Then, there exists a nonnegative constant c independent of x_0 and f such that

$$\|M\, f\|_{LM^{p,\lambda}_{\{x_0\}}} \le c\, \|f\|_{LM^{p,\lambda}_{\{x_0\}}} \le c\, \|f^\sharp\|_{LM^{p,\lambda}_{\{x_0\}}}$$

for every $f \in LM^{p,\lambda}_{\{x_0\}}(\mathbb{R}^n)$.

An improvement of the above theorem is the next result in the Vanishing Generalized Morrey Spaces.

Theorem 4.5. *Let us consider $1 \le q < +\infty$, φ_2 satisfy the condition (2.1), the functions φ_1, φ_2 satisfy the conditions*

$$c_\delta := \sup_{\delta<t<\infty} \sup_{x\in\mathbb{R}^n} \varphi_1(x,t) < \infty \tag{4.13}$$

for every $\delta > 0$ and

$$\frac{\sup_{r<t<\infty} \varphi_1(x,t)}{\varphi_2(x,r)} \le C_0, \tag{4.14}$$

where C_0 does not depend on $x \in \mathbb{R}^n$ and $r > 0$. Then, for $1 < q < \infty$ the maximal operator M is bounded from $VM^{q,\varphi_1}(\mathbb{R}^n)$ to $VM^{q,\varphi_2}(\mathbb{R}^n)$ and, for $1 \le q < \infty$, from $VM^{q,\varphi_1}(\mathbb{R}^n)$ to $VWM^{q,\varphi_2}(\mathbb{R}^n)$.

Proof. The norm inequalities follow from Remark 4.3, so we only have to prove that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{M}^{q,\varphi_1}(f;x,r) = 0 \implies \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{M}^{q,\varphi_2}(Mf;x,r) = 0, \tag{4.15}$$

when $1 < q < \infty$, and

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{M}^{q,\varphi_1}(f;x,r) = 0 \implies \lim_{r\to 0} \sup_{x\in\mathbb{R}^n} \mathfrak{M}^{q,\varphi_2}_W(Mf;x,r) = 0, \tag{4.16}$$

when $1 \le q < \infty$. In this estimation we follow some ideas of [25], but base ourselves on Theorem 4.1.

We start with (4.15). We rewrite the inequality (4.9) in the form

$$\mathfrak{M}^{q,\varphi_2}(Mf;x,r) \le C \frac{\sup_{t>r} t^{-\frac{n}{q}} \|f\|_{L^q(B(x,t))}}{\varphi_2(x,r)}. \tag{4.17}$$

To show that $\sup\limits_{x\in\mathbb{R}^n} \mathfrak{M}^{q,\varphi_2}(Mf;x,r) < \varepsilon$ for small r, we split the right-hand side of (4.17):

$$\mathfrak{M}^{q,\varphi_2}(Mf;x,r) \le C\,[I_{\delta_0}(x,r) + J_{\delta_0}(x,r)], \tag{4.18}$$

where $\delta_0 > 0$ will be chosen as shown below (we may take $\delta_0 < 1$) and

$$I_{\delta_0}(x,r) := \frac{\sup_{r<t<\delta_0} t^{-\frac{n}{q}} \|f\|_{L^q(B(x,t))}}{\varphi_2(x,r)},$$

$$J_{\delta_0}(x,r) := \frac{\sup_{t>\delta_0} t^{-\frac{n}{q}} \|f\|_{L^q(B(x,t))}}{\varphi_2(x,r)}$$

and it is supposed that $r < \delta_0$. Now we choose any fixed $\delta_0 > 0$ such that

$$\sup_{x\in\mathbb{R}^n} \mathfrak{M}^{q,\varphi_1}(f;x,t) < \frac{\varepsilon}{2CC_0}, \quad \text{for all} \quad 0<t<\delta_0,$$

where C and C_0 are constants from (4.18) and (4.14), which is possible since $f \in VM^{q,\varphi_1}(\mathbb{R}^n)$. Then $\|f\|_{L^q(B(x,t))} < \frac{\varepsilon}{2CC_0}\,\varphi_1(x,t)$ and we obtain the estimate of the first term uniform in $r \in (0,\delta_0)$:

$$\sup_{x\in\mathbb{R}^n} CI_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \quad 0<r<\delta_0$$

by (4.14).

The estimation of the second term now may be made already by the choice of r sufficiently small thanks to the condition (2.1). We have

$$J_\delta(x,r) \le \frac{c_{\delta_0}\|f\|_{M^{q,\varphi_1}}}{\varphi_2(x,r)},$$

where c_{δ_0} is the constant of (4.13) for $\delta = \delta_0$.

Then, by (2.1) it suffices to choose r small enough such that

$$\sup_{x\in\mathbb{R}^n} \frac{1}{\varphi_1(x,r)} \le \frac{\varepsilon}{2c_{\delta_0}\|f\|_{M^{q,\varphi_1}}},$$

which completes the proof of (4.15).

The proof of (4.16) is, line by line, similar to the proof of (4.15). □

The following theorem was proved by Guliyev in [15].

Theorem 4.6. *Let $x_0 \in \mathbb{R}^n$, $1 \le q < \infty$, K be a Calderón–Zygmund singular integral operator and the functions φ_1, φ_2 satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}\limits_{t<\tau<\infty} \varphi_1(x_0,\tau)\,\tau^{\frac{n}{q}}}{t^{\frac{n}{q}+1}}\,dt \le C\,\varphi_2(x_0,r), \tag{4.19}$$

where C does not depend on r. Then for $1 < q < \infty$ the operator K is bounded from $LM^{q,\varphi_1}_{\{x_0\}}(\mathbb{R}^n)$ to $LM^{q,\varphi_2}_{\{x_0\}}(\mathbb{R}^n)$ and for $1 \le q < \infty$ the operator K is bounded from $LM^{q,\varphi_1}_{\{x_0\}}(\mathbb{R}^n)$ to $WLM^{q,\varphi_2}_{\{x_0\}}(\mathbb{R}^n)$. Moreover, for $1 < q < \infty$

$$\|Kf\|_{LM^{q,\varphi_2}_{\{x_0\}}} \le c\,\|f\|_{LM^{q,\varphi_1}_{\{x_0\}}} \le c\,\|f^\sharp\|_{LM^{q,\varphi_1}_{\{x_0\}}},$$

where c does not depend on x_0 and f and for $1 \le q < \infty$

$$\|Kf\|_{WLM^{q,\varphi_2}_{\{x_0\}}} \le c\,\|f\|_{LM^{q,\varphi_1}_{\{x_0\}}} \le c\,\|f^\sharp\|_{LM^{q,\varphi_1}_{\{x_0\}}},$$

where c does not depend on x_0 and f.

The following theorem is valid.

Theorem 4.7. *Let $x_0 \in \mathbb{R}^n$, $1 < q < s < p < +\infty$, K be a Calderón–Zygmund singular integral operator and the function φ satisfy the condition*

$$\sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x_0,\tau)\,\tau^{\frac{nq}{p}}}{t^{\frac{nq}{p}}} \le C\,\varphi(x_0,r), \tag{4.20}$$

$$\sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x_0,\tau)\,\tau^{\frac{ns}{p}}}{t^{\frac{ns}{p}}} \le C\,\varphi(x_0,r) \tag{4.21}$$

and

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x_0,\tau)\,\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}\,dt \le C\,\varphi(x_0,r), \tag{4.22}$$

where C does not depend on r.

If $a \in BMO(\mathbb{R}^n)$ then, the commutator

$$[a,K](f) \,=\, a\,Kf \,-\, K\,(af)$$

is a bounded operator from $LM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ in itself. Precisely, $\forall f \in LM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$, we have

$$\|[a,K](f)\|_{LM^{p,\varphi}_{\{x_0\}}} \;\le\; c\,\|a\|_*\,\|f\|_{LM^{p,\varphi}_{\{x_0\}}} \le c\;\|a\|_*\,\|f^\sharp\|_{LM^{q,\varphi_1}_{\{x_0\}}},$$

for some constant $c \ge 0$ independent of a and f.

Proof. Using Lemma 1 in [7] and Theorem 4.2 we get, for $1 < q < s < p < \infty$,

$$\begin{aligned}\|[a,K](f)\|_{LM^{p,\varphi}_{\{x_0\}}} &\le c\cdot\|M([a,K])\|_{LM^{p,\varphi}_{\{x_0\}}}\\ &\le c\cdot\|\;[a,\,K]^\sharp\;\|_{LM^{p,\varphi}_{\{x_0\}}}\\ &\le c\cdot\|a\|_*\cdot\|\Big(M\,|Kf|^q\Big)^{\frac{1}{q}} \;+\; \Big(M\,|f|^s\Big)^{\frac{1}{s}}\|_{LM^{p,\varphi}_{\{x_0\}}}.\end{aligned}$$

Note that from the boundedness of the maximal operator M from $LM^{\frac{p}{q},\varphi}_{\{x_0\}}(\mathbb{R}^n)$ in itself and from $LM^{\frac{p}{s},\varphi}_{\{x_0\}}(\mathbb{R}^n)$ in itself, $1 < q < s < p < \infty$ the sufficient conditions are (4.20) and (4.21), consequently (see, Theorem 4.2).

Also, from the boundedness of the Calderón–Zygmund singular integral operator K from $LM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ in itself the sufficient condition is (4.19) (see, Theorem 4.6).

Then, we have

$$\begin{aligned}
\|\Big(M\,|Kf|^q\Big)^{\frac{1}{q}}\|_{LM^{p,\varphi}_{\{x_0\}}} &\le \Big(\|M(|K\,f|^q)\|_{LM^{\frac{p}{q},\varphi}_{\{x_0\}}}\Big)^{\frac{1}{q}} \\
&\le c\cdot \Big(\||K\,f|^q\|_{LM^{\frac{p}{q},\varphi}_{\{x_0\}}}\Big)^{\frac{1}{q}} \\
&\le c\||K\,f|\|_{LM^{p,\varphi}_{\{x_0\}}} \\
&\le c\|\,f\|_{LM^{p,\varphi}_{\{x_0\}}}
\end{aligned}$$

and

$$\|(M(|K\,f|^q)^{\frac{1}{q}}\|_{LM^{p,\varphi}_{\{x_0\}}} \le c\,\|f\|_{LM^{p,\varphi}_{\{x_0\}}}.$$

In the same way one can easily see that

$$\|(M(|\,f|^s)^{\frac{1}{s}}\|_{LM^{p,\varphi}_{\{x_0\}}} \le c\,\|f\|_{LM^{p,\varphi}_{\{x_0\}}},$$

we get

$$\|[a,K](f)\|_{LM^{q,\varphi}_{\{x_0\}}} \le c\,\|a\|_*\,\|f\|_{LM^{q,\varphi}_{\{x_0\}}}.$$

So, the theorem was proved. □

Corollary 4.8. *Let $x_0 \in \mathbb{R}^n$, $1 < p < +\infty$, K be a Calderón–Zygmund singular integral operator and the function $\varphi(x_0,\cdot) : (0,\infty) \to (0,\infty)$ be an decreasing function. Assume that the mapping $r \mapsto \varphi(x_0,r)\,r^{\frac{n}{p}}$ is almost increasing (there exists a constant c such that for $s < r$ we have $\varphi(x_0,s)\,s^{\frac{n}{p}} \le c\varphi(x_0,r)\,r^{\frac{n}{p}}$). Let also*

$$\int_r^\infty \varphi(x_0,t)\,\frac{dt}{t} \le C\,\varphi(x_0,r), \tag{4.23}$$

where C does not depend on r.

If $a \in BMO(\mathbb{R}^n)$, then the commutator $[a,K]$ is a bounded operator from $LM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ in itself.

From Theorem 4.7 we get the following corollary.

Corollary 4.9. *Let $1 < q < s < p < +\infty$, K be a Calderón–Zygmund singular integral operator and the function φ satisfy the condition*

$$\sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x,\tau)\,\tau^{\frac{nq}{p}}}{t^{\frac{nq}{p}}} \le C\,\varphi(x,r),$$
$$\sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x,\tau)\,\tau^{\frac{ns}{p}}}{t^{\frac{ns}{p}}} \le C\,\varphi(x,r)$$

and

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x,\tau)\,\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}\,dt \le C\,\varphi(x,r),$$

where C does not depend on x and r.

If $a \in BMO(\mathbb{R}^n)$, then the commutator $[a,K]$ is a bounded operator from $M^{p,\varphi}(\mathbb{R}^n)$ in itself. Precisely, $\forall f \in M^{p,\varphi}(\mathbb{R}^n)$, we have

$$\|[a,K](f)\|_{M^{p,\varphi}} \le c\,\|a\|_*\,\|f\|_{M^{p,\varphi}} \le c\,\|a\|_*\,\|f^\sharp\|_{M^{p,\varphi}},$$

for some constant $c \ge 0$ independent of a and f.

Corollary 4.10. *Let $1 < p < +\infty$, K be a Calderón–Zygmund singular integral operator and the function $\varphi(x,r) : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be an decreasing function on r. Assume that the mapping $r \mapsto \varphi(x,r)\,r^{\frac{n}{p}}$ is almost increasing on r (there exists a constant c such that for $s < r$ we have $\varphi(x,s)\,s^{\frac{n}{p}} \le c\varphi(x,r)\,r^{\frac{n}{p}}$). Let also*

$$\int_r^\infty \varphi(x,t)\,\frac{dt}{t} \le C\,\varphi(x,r), \tag{4.24}$$

where C does not depend on x and r.

If $a \in BMO(\mathbb{R}^n)$, then the commutator $[a,K]$ is a bounded operator from $M^{p,\varphi}(\mathbb{R}^n)$ in itself.

Remark 4.11. Note that, the Corollaries 4.8, 4.9 and 4.10 are news.

Remark 4.12. Note that the condition (4.12) in Theorem 4.3 is weaker than the condition (4.19) in Theorem 4.6 and the condition (4.19) in Theorem 4.6 is weaker than the condition (4.23) in Corollary 4.8. Indeed, if condition (4.23) holds, then

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}dt \le \int_r^\infty \varphi_1(x_0,t)\frac{dt}{t},$$

so condition (4.19) holds.

Also, if condition (4.19) holds, then for any $\tau \in (r,\infty)$

$$\begin{aligned} C\varphi_2(x_0,r) &\ge \int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}dt \ge \int_\tau^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}dt \\ &\ge \operatorname*{ess\,inf}_{\tau<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}} \int_\tau^\infty \frac{dt}{t^{\frac{n}{p}+1}} \approx \frac{\operatorname*{ess\,inf}_{\tau<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}}}{\tau^{\frac{n}{p}}}, \end{aligned}$$

so that

$$\sup_{r<\tau<\infty} \frac{\operatorname*{ess\,inf}_{\tau<s<\infty} \varphi_1(x_0,s)\,s^{\frac{n}{p}}}{\tau^{\frac{n}{p}}} \le C \int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}dt \le C\,\varphi_2(x_0,r),$$

so condition (4.12) holds.

On the other hand, the functions

$$\varphi_1(r) = \frac{1}{\chi_{(1,\infty)}(r) r^{\frac{n}{p}-\beta}}, \quad \varphi_2(r) = r^{-\frac{n}{p}}\left(1 + r^{\beta}\right) \tag{4.25}$$

for $0 < \beta \le \frac{n}{p}$ satisfy condition (4.12), for $0 < \beta < \frac{n}{p}$ satisfy condition (4.19), but for $0 < \beta < \frac{n}{p}$ do not satisfy condition (4.23). Also for $\beta = \frac{n}{p}$ the pair function (4.25) satisfies condition (4.12), but does not satisfy condition (4.19).

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