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A General Matrix Application of Convex Sequences to Fourier Series

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Abstract. By using a convex sequence Bor [H. Bor, Local properties of factored Fourier series, Appl. Math. Comp. 212 (2009) 82-85] has obtained a result dealing with local property of factored Fourier series for weighted mean summability. The purpose of this paper is to extend this result to more general cases by taking normal matrices in place of weighted mean matrices.

1. Introduction

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ... \tag{1}$$

The series $\sum a_n$ is said to be summable $|A, \theta_n|_k$, $k \geq 1$, if (see [14])

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \left| \bar{\Delta} A_n(s) \right|^k < \infty, \tag{2}$$

where (θ_n) is any sequence of positive constants and

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s). \tag{3}$$

(see also [11]). Note that in the special case when A is the matrix of weighted mean, i.e.,

$$a_{nv} = \begin{cases} \frac{p_v}{P_n}, & 0 \leq v \leq n \\ 0, & n > v, \end{cases}$$

then the summability $|A, \theta_n|_k$, reduces to the summability $\left|\bar{N}, p_n, \theta_n\right|_k$, $k \geq 1$, which also includes the summabilities $|\bar{N}, p_n|_k$ and $|R, p_n|_k$ for $\theta_n = \frac{P_n}{p_n}$ and $\theta_n = n$, respectively, (see [2], [3]). Furthermore, if A is the matrix of Cesaro mean of order α, with $\alpha > -1$ and $\theta_n = n$, then it is the same as the summability $|C, \alpha|_k$ (see [6]), which is one of ancestor summability methods.

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2. The Known Results

Let f be function with period 2π, and Lebesgue integrable over $(-\pi, \pi)$. Without loss of generality, we may assume that the constant term of the Fourier series of f is zero, that is

$$\int_{-\pi}^{\pi} f(t)dt = 0.$$

Write

$$f(t) \sim \sum_{n=1}^{\infty}(a_n cosnt + b_n sinnt) = \sum_{n=1}^{\infty} C_n(t), \tag{4}$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)cos(nt)dt, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)sin(nt)dt.$$

It is well known that convergence of a Fourier series at any point $t = x$ is a local property of the generating function f, that is to say, depends only on the values of the function f in the interval $(x - \delta, x + \delta)$ for arbitrarily small $\delta > 0$ and it is not affected by the values it takes outside the interval (see [16]).
The Fourier series play an important role in many areas of applied mathematics and mechanics. Since the convergence of such a series at any point $t = x$ is a local property of the generating function f, therefore the summability of this series at the point by any regular linear summability method is also a local property of f. Some known results have been proved dealing with local property of Fourier series (see [9]-[10], [12]-[13]). Furthermore, Bhatt [1] has proved the following result.

Theorem 2.1. *If (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then the summability $|R, \log n, 1|$ of the series $\sum C_n(t)\lambda_n \log n$ at a point can be ensured by a local property.*

Bor has proved the following theorems in a more general form which includes of the above result as special cases.

Theorem 2.2. *[4] Let $k \geq 1$. If (λ_n) is a convex sequence such that $\sum p_n\lambda_n$ is convergent, then the summability $|\bar{N}, p_n|_k$ of the series $\sum_{n=1}^{\infty} C_n(t)\lambda_n P_n$ at a point is a local property of the generating function $f(t)$.*

Theorem 2.3. *[5] Let $k \geq 1$. If (λ_n) is a convex sequence such that $\sum p_n\lambda_n$ is convergent and (θ_n) is any sequence of positive constants such that*

$$\sum_{v=1}^{m}\left(\frac{\theta_v p_v}{P_v}\right)^{k-1} P_v\Delta\lambda_v = O(1), \tag{5}$$

$$\sum_{v=1}^{m}\left(\frac{\theta_v p_v}{P_v}\right)^{k-1} p_v\lambda_v = O(1), \tag{6}$$

$$\sum_{v=1}^{m}\left(\frac{\theta_v p_v}{P_v}\right)^{k-1} p_{v+1}\lambda_{v+1} = O(1), \tag{7}$$

and

$$\sum_{n=v+1}^{m+1}\left(\frac{\theta_n p_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} = O\left\{\left(\frac{\theta_v p_v}{P_v}\right)^{k-1}\frac{1}{P_v}\right\}, \tag{8}$$

then the summability $|\bar{N}, p_n, \theta_n|_k$ of the series $\sum_{n=1}^{\infty} C_n(t)\lambda_n P_n$ at a point is a local property of the generating function $f(t)$.

3. The Main Result

In this paper, taking a normal matrix instead of a weighted mean matrix, we extend Theorem 2.3 to $|A,\theta_n|_k$ summability. Before stating the main theorem we must first introduce some further notation. Given a normal matrix $A=(a_{nv})$, we associate two lower semimatrices $\bar{A}=(\bar{a}_{nv})$ and $\hat{A}=(\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv}=\sum_{i=v}^{n}a_{ni},\quad n,v=0,1,\ldots\quad \bar{\Delta}a_{nv}=a_{nv}-a_{n-1,v}\quad a_{-1,0}=0 \tag{9}$$

and

$$\hat{a}_{00}=\bar{a}_{00}=a_{00}, \tag{10}$$

$$\hat{a}_{nv}=\bar{\Delta}\bar{a}_{nv}=\bar{a}_{nv}-\bar{a}_{n-1,v},\quad n=1,2,\ldots \tag{11}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s)=\sum_{v=0}^{n}a_{nv}s_v=\sum_{v=0}^{n}\bar{a}_{nv}a_v\quad and\quad \bar{\Delta}A_n(s)=\sum_{v=0}^{n}\hat{a}_{nv}a_v. \tag{12}$$

With this notation we have the following theorem.

Theorem 3.1. *Suppose that $A=(a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0}=1,\ n=0,1,\ldots, \tag{13}$$

$$a_{n-1,v}\geq a_{nv},\ for\quad n\geq v+1, \tag{14}$$

$$a_{nn}=O\left(\frac{p_n}{P_n}\right). \tag{15}$$

If the conditions (5)-(7) of Theorem 2.3 are satisfied and, if (θ_n) is any sequence of positive constants holds for the following conditions,

$$\sum_{n=v+1}^{\infty}(\theta_n a_{nn})^{k-1}\hat{a}_{n,v+1}=O\left\{(\theta_v a_{vv})^{k-1}\right\}, \tag{16}$$

and

$$\sum_{n=v+1}^{\infty}(\theta_n a_{nn})^{k-1}|\bar{\Delta}a_{nv}|=O\left\{(\theta_v a_{vv})^{k-1}a_{vv}\right\}, \tag{17}$$

then the series $\sum C_n(t)\lambda_n P_n$ is summable $|A,\theta_n|_k$, $k\geq 1$, where (λ_n) is as in Theorem 2.3.

We need the following lemma for the proof of Theorem 3.1.

Lemma 3.2. *[8] If (λ_n) is a convex sequence such that $\sum p_n\lambda_n$ is convergent, then (λ_n) is a non-negative monotonic decreasing sequence tending to zero,*

$$P_n\lambda_n=O(1)\quad as\quad n\to\infty\quad \text{and}\quad \sum P_n\Delta\lambda_n<\infty.$$

4. Proof of Theorem 3.1

Proof. Since the behaviour of the Fourier series, as far as convergence is concerned, for a particular value of x depends on the behaviour of the function in the immediate neighbourhood of this point only. To complete the proof of Theorem 3.1 it is sufficient to prove that if (s_n) is bounded, then under the conditions of Theorem 3.1 $\sum a_n\lambda_n P_n$ is summable $|A,\theta_n|_k$, $k\geq 1$.
Let (I_n) denotes the A-transform of the series $\sum_{n=1}^{\infty} a_nP_n\lambda_n$. Then, by definition, we have

$$\bar{\Delta} I_n = \sum_{v=1}^{n} \hat{a}_{nv} a_v P_v \lambda_v .$$

Applying Abel's transformation to this sum, we have that

$$\begin{aligned}\bar{\Delta} I_n &= \sum_{v=1}^{n} \hat{a}_{nv} a_v P_v \lambda_v = \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv}\lambda_v P_v)\sum_{r=1}^{v} a_r + \hat{a}_{nn}\lambda_n P_n \sum_{v=1}^{n} a_v\\ &= \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv}\lambda_v P_v)s_v + a_{nn}\lambda_n P_n s_n\\ &= \sum_{v=1}^{n-1} \bar{\Delta} a_{nv}\lambda_v P_v s_v + \sum_{v=1}^{n-1}\hat{a}_{n,v+1}\Delta\lambda_v P_v s_v - \sum_{v=1}^{n-1}\hat{a}_{n,v+1} p_{v+1}\lambda_{v+1}s_v + a_{nn}\lambda_n P_n s_n\\ &= I_{n,1}+I_{n,2}+I_{n,3}+I_{n,4}.\end{aligned}$$

To complete the proof of Theorem 3.1, by Minkowski inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \theta_n^{k-1} \mid I_{n,r}\mid^k < \infty, \quad for \quad r=1,2,3,4. \tag{18}$$

It is noted that by using the conditions (13) and (14) we have

$$\begin{aligned}\sum_{v=1}^{n-1} |\bar{\Delta} a_{nv}| &= \sum_{v=1}^{n-1} |a_{nv}-a_{n-1,v}| = \sum_{v=1}^{n-1} (a_{n-1,v}-a_{nv}) \\ &= \sum_{v=0}^{n-1} a_{n-1,v} - a_{n-1,0} - \sum_{v=0}^{n} a_{nv} + a_{n0} + a_{nn}\\ &= 1 - a_{n-1,0} - 1 + a_{n0} + a_{nn} \leq a_{nn}.\end{aligned} \tag{19}$$

First, by applying Hölder's inequality with indices k and k', where $k>1$ and $\frac{1}{k}+\frac{1}{k'}=1$, and from (19), we have

$$\begin{aligned}\sum_{n=2}^{m+1} \theta_n^{k-1} \mid I_{n,1}\mid^k &\leq \sum_{n=2}^{m+1} \theta_n^{k-1} \left\{\sum_{v=1}^{n-1} |\bar{\Delta} a_{nv}||\lambda_v|P_v|s_v|\right\}^k\\ &\leq \sum_{n=2}^{m+1} \theta_n^{k-1} \left\{\sum_{v=1}^{n-1} |\bar{\Delta} a_{nv}|\lambda_v^k P_v^k|s_v|^k\right\} \times \left\{\sum_{v=1}^{n-1} |\bar{\Delta} a_{nv}|\right\}^{k-1}\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}|\bar{\Delta}a_{nv}|\lambda_v^kP_v^k\\
&= O(1)\sum_{v=1}^{m}\lambda_v^kP_v^k\sum_{n=v+1}^{m+1}(\theta_na_{nn})^{k-1}|\bar{\Delta}a_{nv}|\\
&= O(1)\sum_{v=1}^{m}(\theta_va_{vv})^{k-1}\lambda_v^kP_v^ka_{vv} = O(1)\sum_{v=1}^{m}(\theta_va_{vv})^{k-1}\lambda_v^kP_v^k\frac{p_v}{P_v}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{\theta_vp_v}{P_v}\right)^{k-1}(\lambda_vP_v)^{k-1}p_v\lambda_v = O(1)\sum_{v=1}^{m}\left(\frac{\theta_vp_v}{P_v}\right)^{k-1}p_v\lambda_v\\
&= O(1)\quad as\quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.2.
It should be noted that the elements $\hat{a}_{nv}\geq 0$ for each v,n. In fact, it is easily seen from the conditions (13) and (14), that $\hat{a}_{00}=1$,

$$
\begin{aligned}
\hat{a}_{nv} &= \bar{a}_{n0}-\bar{a}_{n-1,0}+\sum_{i=0}^{v-1}(a_{n-1,i}-a_{ni}) \qquad (20)\\
&= \sum_{i=0}^{v-1}(a_{n-1,i}-a_{ni})\geq 0,\quad \text{for}\quad 1\leq v\leq n,\quad \text{and equal to zero otherwise, and also}\\
\hat{a}_{n,v+1} &= \sum_{i=v+1}^{n}(a_{ni}-a_{n-1,i}) = \sum_{i=0}^{v}(a_{n-1,i}-a_{ni}) \qquad (21)\\
&\leq \sum_{i=0}^{n-1}(a_{n-1,i}-a_{ni}) = \bar{a}_{n-1,0}-\bar{a}_{n0}+a_{nn}=a_{nn}.
\end{aligned}
$$

Now, again using Hölder's inequality, and (20)-(21) we have that

$$
\begin{aligned}
\sum_{n=2}^{m+1}\theta_n^{k-1}\mid I_{n,2}\mid^k &\leq \sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v|P_v|s_v|\right\}^k\\
&\leq \sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}\hat{a}_{n,v+1}^k\Delta\lambda_vP_v|s_v|^k\right\}\times\left\{\sum_{v=1}^{n-1}\Delta\lambda_vP_v\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}\left\{\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\hat{a}_{n,v+1}^{k-1}\Delta\lambda_vP_v\right\}\\
&= O(1)\sum_{n=2}^{m+1}\theta_n^{k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\Delta\lambda_vP_v\\
&= O(1)\sum_{v=1}^{m}\Delta\lambda_vP_v\sum_{n=v+1}^{m+1}(\theta_na_{nn})^{k-1}\hat{a}_{n,v+1}\\
&= O(1)\sum_{v=1}^{m}(\theta_va_{vv})^{k-1}\Delta\lambda_vP_v\\
&= O(1)\sum_{v=1}^{m}\left(\frac{\theta_vp_v}{P_v}\right)^{k-1}\Delta\lambda_vP_v\\
&= O(1)\quad as\quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.2.
Using the fact that $P_v < P_{v+1}$ and (20)-(21), we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \theta_n^{k-1} \mid I_{n,3} \mid^k &\leq \sum_{n=2}^{m+1} \theta_n^{k-1} \left\{ \sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| p_{v+1} \lambda_{v+1} |s_v| \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left\{ \sum_{v=1}^{n-1} \hat{a}_{n,v+1}^k p_{v+1} \lambda_{v+1} \right\} \times \left\{ \sum_{v=1}^{n-1} p_{v+1} \lambda_{v+1} \right\}^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} \left\{ \sum_{v=1}^{n-1} \hat{a}_{n,v+1}^{k-1} \hat{a}_{n,v+1} p_{v+1} \lambda_{v+1} \right\} \\
&= O(1) \sum_{n=2}^{m+1} \theta_n^{k-1} a_{nn}^{k-1} \left\{ \sum_{v=1}^{n-1} \hat{a}_{n,v+1} p_{v+1} \lambda_{v+1} \right\} \\
&= O(1) \sum_{v=1}^{m} p_{v+1} \lambda_{v+1} \sum_{n=v+1}^{m+1} (\theta_n a_{nn})^{k-1} \hat{a}_{n,v+1} \\
&= O(1) \sum_{v=1}^{m} (\theta_v a_{vv})^{k-1} p_{v+1} \lambda_{v+1} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{\theta_v p_v}{P_v} \right)^{k-1} p_{v+1} \lambda_{v+1} = O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.2.
Finally, since $P_n \lambda_n = O(1)$ as $n \to \infty$, we have that

$$\begin{aligned}
\sum_{n=1}^{m} \theta_n^{k-1} \mid I_{n,4} \mid^k &= \sum_{n=1}^{m} \theta_n^{k-1} a_{nn}^k \lambda_n^k P_n^k |s_n|^k \\
&= O(1) \sum_{n=1}^{m} \theta_n^{k-1} a_{nn}^{k-1} \lambda_n^{k-1} \lambda_n P_n^k \frac{p_n}{P_n} \\
&= O(1) \sum_{n=1}^{m} (\theta_n a_{nn})^{k-1} p_n \lambda_n \\
&= O(1) \sum_{n=1}^{m} \left(\frac{\theta_n p_n}{P_n} \right)^{k-1} p_n \lambda_n = O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.2. □

This completes the proof of Theorem 3.1.

5. Conclusions

We can apply Theorem 3.1 to weighted mean $A = (a_{nv})$ is defined as $a_{nv} = \frac{p_v}{P_n}$ when $0 \leq v \leq n$, where $P_n = p_0 + p_1 + ... + p_n$. We have that,

$$\bar{a}_{nv} = \frac{P_n - P_{v-1}}{P_n} \qquad \hat{a}_{n,v+1} = \frac{p_n P_v}{P_n P_{n-1}}.$$

It may be noted that if we take $A = (\bar{N}, p_n)$, then the conditions (13)-(15) are satisfied automatically and the conditions (16) and (17) are reduced to (8). The following results can be easily verified.

1. If we take $\theta_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 3.1, then we have Theorem 2.3.
2. If we take $\theta_n = n$ and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 3.1, then we obtain a theorem dealing with $|R, p_n|_k$ summability.
3. If we put $\theta_n = n$, $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n in Theorem 3.1, then we have a result for $|C, 1|_k$ summability.

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