Acta Math. Univ. Comenianae
Vol. LXXXVII, 2 (2018), pp. 191–198

SOME RESULTS FOR MAX-PRODUCT OPERATORS VIA POWER SERIES METHOD

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ABSTRACT. In this paper, we obtain an approximation theorem by max-product operators with the use of power series method which is more effective than ordinary convergence and includes both Abel and Borel methods. We also estimate the error in this approximation. Finally, we provide an example which satisfies our theorem.

1. INTRODUCTION

The classical Korovkin theorem states the uniform convergence of a sequence of positive linear operators in $C[a,b]$, the space of all continuous real valued functions defined on $[a,b]$ by providing the convergence only on three test functions $\{1,x,x^2\}$. There are also trigonometric versions of this theorem with the test functions $\{1,\cos x,\sin x\}$, and abstract Korovkin type given in [**12, 15**]. These type of results let us to say the convergence with minimum calculations have also important applications in the polynomial approximation theory, in various areas of functional analysis, in numerical solutions of differential and integral equations [**1, 2**]. Recently it has been asked: Do all the approximation operators need to be linear? It was shown by Bede and et. al. [**3**]–[**10**] that the linear structure is not the only one which allows us to construct particular approximation operators. Then following this idea, these type of approximation results were extended with the use of statistical convergence and summation process by max-product operators [**13, 14**].

In this paper, we obtain new approximation results for these max-product operators by using power series method which is not only more effective than ordinary convergence but also includes Abel and Borel methods. We also estimate the error in this approximation. Finally we provide an example which satisfies our theorem.

Let us begin with recalling some basic definitions and notations used in the paper.

Let (p_j) be a real sequence with $p_1 > 0$ and $p_2, p_3, \cdots \geq 0$ such that the corresponding power series $p(t) := \sum_{n=1}^{\infty} p_n t^{n-1}$ has radius of convergence R with

Received June 8, 2017; revised March 7, 2018.
2010 *Mathematics Subject Classification.* Primary 40G10; Secondary 40C15, 41A36.
Key words and phrases. power series method; max-product operators; approximation theory.

$0 < R \leq \infty$. If the limit

$$\lim_{t\to R^-} \frac{1}{p(t)} \sum_{n=1}^{\infty} x_n p_n t^{n-1} = L$$

exists, then we say that $x = (x_n)$ is convergent in the sense of power series method [**16, 18**]. Power series method includes many well known summability methods such as Abel and Borel. Both methods have in common that their definitions are based on power series and that they are not matrix methods (See [**11, 19**] for details). In order to see that power series method is more effective than ordinary convergence, let $x = (1, 0, 1, 0, \dots)$, $R = \infty$, $p(t) = \mathrm{e}^t$ and for $n \in \mathbb{N}$, $p_n = \frac{1}{(n-1)!}$. Then it is easy to see that

$$\lim_{t\to\infty} \frac{1}{\mathrm{e}^t} \sum_{n=1}^{\infty} \frac{x_n t^{n-1}}{(n-1)!} = \lim_{t\to\infty} \frac{1}{\mathrm{e}^t} \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} = \lim_{t\to\infty} \left\{ \frac{\mathrm{e}^t + \mathrm{e}^{-t}}{2} \right\} = \frac{1}{2}.$$

So the sequence $x = (x_n)$ is convergent to $\frac{1}{2}$ in the sense of power series method but it is not convergent in the ordinary sense. Note that the power series method is regular if and only if

$$\lim_{t\to R^-} \frac{p_n t^{n-1}}{p(t)} = 0 \qquad \text{for each } n \in \mathbb{N},$$

holds [**11, 19**]. Throughout the paper, we assume that power series method is regular.

Let (X, d) be an arbitrary compact metric space and $C(X, [0, \infty))$ denote the space of all nonnegative continuous functions on X. Let also the symbol $\bigvee$ denotes the maximum operation. Then we consider the following max-product operators defined by

$$L_n(f; x) := \bigvee_{k=0}^{n} K_{n,k}(x) \cdot f(x_{n,k}),$$

where $x \in X$, $n \in \mathbb{N}$, $x_{n,k} \in X$, $k = 0, 1, 2, \dots, n$, $f \in C(X, [0, \infty))$ and $K_{n,k}(\cdot)$ is a nonnegative continuous function on X. Note that the max-product operators are positive but not linear. Indeed they satisfy the property of the pseudo-linearity as follows:

$$L_n(\alpha f \bigvee \beta g) = \alpha L_n(f) \bigvee \beta L_n(g)$$

holds for all $f, g \in C(X, [0, \infty))$ and for nonnegative numbers α, β.

It is important to recall the following Lemma of [**4**] which plays an useful role in our results.

Lemma 1. *For any a_k, $b_k \in [0, \infty)$, $k = 0, 1, 2, \dots, n$, we have*

$$\left| \bigvee_{k=0}^{n} a_k - \bigvee_{k=0}^{n} b_k \right| \leq \bigvee_{k=0}^{n} |a_k - b_k|.$$

Since the power series method is regular, the operator defined by

$$V_t := \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} L_n$$

is acting from $C(X, [0, \infty))$ into itself. For $y \in X$, consider the test function $e_0(y) := 1$ and the moment function $\varphi_x(y) := d^2(y, x)$ for each fixed $x \in X$. Then we can give the following theorem

Theorem 1. *If the following conditions*

$$\lim_{t \to R^-} \left\| \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} L_n(e_0) - e_0 \right\| = 0$$

and

$$\lim_{t \to R^-} \left\| \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} L_n(\varphi_x) \right\| = 0$$

hold, then for all $f \in C(X, [0, \infty))$, we have

$$\lim_{t \to R^-} \left\| \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} L_n(f) - f \right\| = 0.$$

Proof. Let $x \in X$ and $f \in C(X, [0, \infty))$ be given. Then we have

$$\begin{aligned}
&\left| \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} L_n(f; x) - f(x) \right| \\
&\quad \leq \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} \left| \bigvee_{k=0}^{n} K_{n,k}(x) \cdot f(x_{n,k}) - \bigvee_{k=0}^{n} K_{n,k}(x) \cdot f(x) \right| \\
&\qquad + |f(x)| \left| \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} \bigvee_{k=0}^{n} K_{n,k}(x) - 1 \right| \\
&\quad \leq \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} \bigvee_{k=0}^{n} K_{n,k}(x) |f(x_{n,k}) - f(x)| \\
&\qquad + |f(x)| \left| \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} L_n(e_0; x) - e_0(x) \right|.
\end{aligned}$$

Since the function f is uniform continuous on the compact set X, for a given $\varepsilon > 0$, one can choose $\delta > 0$ such that the following inequality

$$|f(x_{n,k}) - f(x)| \leq \varepsilon + \frac{2\|f\|}{\delta^2} \varphi_x(x_{n,k})$$

holds for all $x, x_{n,k} \in X$. By combining the above inequalities, we get

$$\begin{aligned}
&\Big|\frac{1}{p(t)}\sum_{n=1}^{\infty} p_n t^{n-1} L_n(f;x) - f(x)\Big| \\
&\quad \le \varepsilon \frac{1}{p(t)}\sum_{n=1}^{\infty} p_n t^{n-1} L_n(e_0;x) + \frac{2\|f\|}{\delta^2}\frac{1}{p(t)}\sum_{n=1}^{\infty} p_n t^{n-1} L_n(\varphi_x;x) \\
&\qquad + |f(x)|\Big|\frac{1}{p(t)}\sum_{n=1}^{\infty} p_n t^{n-1} L_n(e_0;x) - e_0(x)\Big| \\
&\quad \le \varepsilon + (\varepsilon + \|f\|)\Big|\frac{1}{p(t)}\sum_{n=1}^{\infty} p_n t^{n-1} L_n(e_0;x) - e_0(x)\Big| \\
&\qquad + \frac{2\|f\|}{\delta^2}\frac{1}{p(t)}\sum_{n=1}^{\infty} p_n t^{n-1} L_n(\varphi_x;x).
\end{aligned}$$

Now, taking supremun over $x \in X$ on the both sides of the last inequality, one can see that

$$\begin{aligned}
\Big\|\frac{1}{p(t)}\sum_{n=1}^{\infty} p_n t^{n-1} L_n(f) - f\Big\| &\le \varepsilon + (\varepsilon + \|f\|)\Big\|\frac{1}{p(t)}\sum_{n=1}^{\infty} p_n t^{n-1} L_n(e_0) - e_0\Big\| \\
&\quad + \frac{2\|f\|}{\delta^2}\Big\|\frac{1}{p(t)}\sum_{n=1}^{\infty} p_n t^{n-1} L_n(\varphi_x)\Big\|.
\end{aligned}$$

Finally by taking limit as $t \to R^-$ and also using the hypothesis, we complete the proof. □

2. Error Estimation in the Approximation by Power Series Method

In this section, we give the error estimation in our theorem with the use of classical modulus of continuity. First recall the following Lemma of [**13**]

Lemma 2. *For every $a_k, b_k \ge 0$, $k = 0, 1, 2, \ldots, n$, we have*

$$\bigvee_{k=0}^{n} a_k b_k \le \sqrt{\bigvee_{k=0}^{n} a_k^2}\sqrt{\bigvee_{k=0}^{n} b_k^2}.$$

For the classical modulus of continuity which is defined as follows;

$$w(f;\delta) = \sup_{d(x,y)\le\delta} |f(y) - f(x)|,$$

where δ is a positive constant and $f \in C(X, [0,\infty))$. Also if (X, d) is compact convex metric space, then it is well known from [**17**] that

$$w(f, \lambda\delta) \le (\lambda + 1) w(f, \delta) \qquad \text{for any } \lambda, \delta \in [0, \infty).$$

Theorem 2. *Let (X,d) be compact convex metric space. For all $f\in C(X,[0,\infty))$, we have*

$$\begin{aligned}
&\Big\|\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}L_n(f)-f\Big\|\\
&\quad\le w(f,\delta_t)\Big\|\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}L_n(e_0)\Big\|+w(f,\delta_t)\sqrt{\Big\|\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}L_n(e_0)\Big\|}\\
&\qquad+\|f\|\Big\|\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}L_n(e_0)-e_0\Big\|,
\end{aligned}$$

where $\delta_t := \sqrt{\|\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}L_n(\varphi_x)\|}$.

Proof. Let $x\in X$ and $f\in C(X,[0,\infty))$ be given. Then

$$\begin{aligned}
&\Big|\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}L_n(f;x)-f(x)\Big|\\
&\quad\le\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}\bigvee_{k=0}^{n}K_{n,k}(x)|f(x_{n,k})-f(x)|\\
&\qquad+|f(x)|\Big|\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}L_n(e_0;x)-e_0(x)\Big|\\
&\quad\le w(f,\delta)\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}\bigvee_{k=0}^{n}K_{n,k}(x)(1+\frac{d(x_{n,k},x)}{\delta})\\
&\qquad+|f(x)|\Big|\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}L_n(e_0;x)-e_0(x)\Big|
\end{aligned}$$

holds and we get

$$\begin{aligned}
&\Big|\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}L_n(f;x)-f(x)\Big|\\
&\quad\le w(f,\delta)\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}L_n(e_0;x)\\
&\qquad+\frac{w(f,\delta)}{\delta}\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}\bigvee_{k=0}^{n}K_{n,k}(x)d(x_{n,k},x)\\
&\qquad+|f(x)|\Big|\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}L_n(e_0;x)-e_0(x)\Big|.
\end{aligned}$$

By the above lemma and from Cauchy-Schwarz inequality, we also have

$$\begin{aligned}
&\Big|\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}L_n(f;x)-f(x)\Big|\\
&\quad\le w(f,\delta)\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}L_n(e_0;x)\\
&\qquad+\frac{w(f,\delta)}{\delta}\sqrt{\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}L_n(e_0;x)}\sqrt{\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}L_n(\varphi_x;x)}\\
&\qquad+|f(x)|\Big|\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}L_n(e_0;x)-e_0(x)\Big|.
\end{aligned}$$

Taking supremum over $x\in X$ and also taking

$$\delta=\delta_t=\sqrt{\Big\|\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}L_n(\varphi_x;x)\Big\|}.$$

we get

$$\begin{aligned}
&\Big\|\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}L_n(f)-f\Big\|\\
&\quad\le w(f,\delta_t)\Big\|\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}L_n(e_0)\Big\|\\
&\qquad+w(f,\delta_t)\sqrt{\Big\|\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}L_n(e_0)\Big\|}\\
&\qquad+\|f\|\Big\|\frac{1}{p(t)}\sum_{n=1}^{\infty}p_nt^{n-1}L_n(e_0)-e_0\Big\|,
\end{aligned}$$

which completes the proof. □

3. Concluding Remarks

As a concluding remark, we present an example which makes impossible to approximate f by means of $L_n(f)$ with the given approximation theorems above. For this purpose, consider the sequence defined as $s_n=0$, n is square and 1 otherwise. Also let $R=1$, $p(t)=\dfrac{1}{1-t}$ and for $n\in\mathbb{N}$, $p_n=1$. In this case the power series method coincides with Abel method. Note that $\{s_n\}$ is convergent to 1 in the sense of power series method. Now take $X=[0,1]$, $n\in\mathbb{N}$, $x_{n,k}=\frac{k}{n}\in[0,1]$, $k=0,1,2\dots,n$, and

$$K_{n,k}(x):=\frac{\binom{n}{k}x^k(1-x)^{n-k}}{\bigvee_{m=0}^{n}\binom{n}{m}x^m(1-x)^{n-m}}.$$

Bede and Gal introduced the max-product Bernstein operators as follows [**3**].

$$B_n(f;x) := \bigvee_{k=0}^{n} K_{n,k}(x) \cdot f\Big(\frac{k}{n}\Big) = \frac{\bigvee_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(\frac{k}{n})}{\bigvee_{m=0}^{n} \binom{n}{m} x^m (1-x)^{n-m}}.$$

It is known that

$$\lim_{n\to\infty} \|B_n(f) - f\| = 0.$$

With the use of these operators, define L_n by $L_n(f;x) := s_n B_n(f;x)$. Note that it is impossible to approximate f by means of $L_n(f)$ since the sequence $\{s_n\}$ is nonconvergent in the ordinary sense. Furthermore, for any $f \in (C[0,1],[0,\infty))$, we can write that

$$\begin{aligned}
&\Big\| \frac{1}{1-t} \sum_{n=1}^{\infty} t^{n-1} L_n(f) - f \Big\| \\
&\quad = \Big\| \frac{1}{1-t} \sum_{n=1}^{\infty} t^{n-1} s_n B_n(f) - f \Big\| \\
&\quad \le \frac{1}{1-t} \sum_{n=1}^{\infty} t^{n-1} \|B_n(f) - f\| + \|f\| \Big| \frac{1}{1-t} \sum_{n=1}^{\infty} t^{n-1} s_n - 1 \Big|.
\end{aligned}$$

Finally by taking limit as $t \to R^-$ and also using the regularity of power series method, we get

$$\Big\| \frac{1}{1-t} \sum_{n=1}^{\infty} t^{n-1} L_n(f) - f \Big\| \to 0,$$

which provides an example for our theorem.

It is also noteworthy that

- in the case of $R = 1$, $p(t) = \frac{1}{1-t}$ and for $j \ge 1$, $p_j = 1$, the power series method coincides with Abel method which is a sequence-to-function transformation,
- in the case of $R = \infty$, $p(t) = \mathrm{e}^t$ and for $j \ge 1$, $p_j = \frac{1}{(j-1)!}$, the power series method coincides with Borel method.

We can therefore give all of the theorems of this paper for Abel and Borel convergences.

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