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Original article

On Riesz summability factors of Fourier series

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Abstract

In this paper, a main theorem dealing with $|\bar{N}, p_n|_k$ summability method has been generalized for $\varphi - |\bar{N}, p_n; \delta|_k$ summability by using different and general summability factors of Fourier series.
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Keywords: Summability factors; Absolute matrix summability; Fourier series; Infinite series; Hölder inequality; Minkowski inequality

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, \ldots. \tag{1.1}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \tag{1.2}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \quad (P_n \neq 0), \tag{1.3}$$

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defines the sequence (σ_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [1]). The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\Delta\sigma_{n-1}|^k < \infty. \tag{1.4}$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), then $\left|\bar{N}, p_n\right|_k$ summability is the same as $|C, 1|_k$ (resp. $\left|\bar{N}, p_n\right|$) summability.

The $\varphi - \left|\bar{N}, p_n; \delta\right|_k$ summability method is defined by Seyhan (see [3]). The series $\sum a_n$ is said to be summable $\varphi - \left|\bar{N}, p_n; \delta\right|_k$, $k \geq 1$ and $\delta \geq 0$, if

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} |\sigma_n - \sigma_{n-1}|^k < \infty. \tag{1.5}$$

If we take $\delta = 0$ and $\varphi_n = \frac{P_n}{p_n}$, then $\varphi - \left|\bar{N}, p_n; \delta\right|_k$ summability is the same as $\left|\bar{N}, p_n\right|_k$ summability.

Let f be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$.

Without loss of generality we may assume that the constant term in the Fourier series of f is zero, so that

$$\int_{-\pi}^{\pi} f(t)dt = 0, \tag{1.6}$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} C_n(t). \tag{1.7}$$

2. Known result

Many papers dealing with $\left|\bar{N}, p_n\right|_k$ summability factors and $\varphi - \left|\bar{N}, p_n; \delta\right|_k$ summability factors of Fourier series have been done (see [4–10]). Among them, Bor [5] has proved the following theorem.

Theorem A. *If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n\lambda_n < \infty$, where (p_n) is a sequence of positive numbers such that $P_n \to \infty$ as $n \to \infty$, and $\sum_{v=1}^{n} P_v C_v(t) = O(P_n)$, then the series $\sum C_n(t) P_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.*

3. Main result

The aim of this paper is to prove a more general theorem which includes the above mentioned result as special cases. Now, we shall prove the following theorem.

Theorem B. *Let (p_n) and (λ_n) be sequences satisfying the conditions of Theorem A and let (φ_n) be a sequence of positive real numbers such that*

$$\varphi_n p_n = O(P_n), \tag{3.1}$$

$$\sum_{n=v+1}^{\infty} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} = O\left(\varphi_v^{\delta k} \frac{1}{P_v}\right), \tag{3.2}$$

$$\sum_{n=1}^{m} \varphi_n^{\delta k} p_n \lambda_n = O(1) \quad as \quad m \to \infty, \tag{3.3}$$

$$\sum_{n=1}^{m} \varphi_n^{\delta k} P_n \Delta\lambda_n = O(1) \quad as \quad m \to \infty. \tag{3.4}$$

Then the series $\sum C_n(t) P_n \lambda_n$ is summable $\varphi - \left|\bar{N}, p_n; \delta\right|_k$, $k \geq 1$ and $0 \leq \delta k < 1$.

We need the following lemma for the proof of Theorem B.

Lemma 1 ([5]). *If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n\lambda_n$ is convergent, where (p_n) is a sequence of positive numbers such that $P_n \to \infty$ as $n \to \infty$, then $P_n\lambda_n = O(1)$ as $n \to \infty$ and $\sum P_n \Delta\lambda_n < \infty$.*

Remark 1. It should be noted that if we take $\delta = 0$ and $\varphi_n = \frac{P_n}{p_n}$ in this theorem, (3.4) is satisfied by Lemma 1. Condition (3.3) is satisfied by a hypothesis of Theorem A. Also in this case conditions (3.1) and (3.2) are obvious.

4. Proof of Theorem B

Let $I_n(t)$ be the sequence of $(\bar{N}, p_n)$ means of the series $\sum C_n(t)P_n\lambda_n$. Then, by definition, we have

$$I_n(t) = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{i=0}^{v} C_i(t)P_i\lambda_i = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})C_v(t)P_v\lambda_v.$$

Then, for $n \geq 1$, we have

$$I_n(t) - I_{n-1}(t) = \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n} P_{v-1}C_v(t)P_v\lambda_v.$$

By Abel's transformation, we have

$$\begin{aligned}
I_n(t) - I_{n-1}(t) &= \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1} \Delta(P_{v-1}\lambda_v)\sum_{r=1}^{v} P_r C_r(t) + \frac{p_n}{P_n}\lambda_n \sum_{v=1}^{n} P_v C_v(t)\\
&= O(1)\left\{\frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1}(P_v\lambda_v - p_v\lambda_v - P_v\lambda_{v+1})P_v\right\} + O(1)p_n\lambda_n\\
&= O(1)\left\{\frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1} P_v P_v \Delta\lambda_v - \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1} P_v p_v \lambda_v + p_n\lambda_n\right\}\\
&= O(1)\left\{I_{n,1}(t) + I_{n,2}(t) + I_{n,3}(t)\right\}.
\end{aligned}$$

To prove Theorem B, by Minkowski's inequality it is sufficient to show that

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k + k - 1}|I_{n,r}(t)|^k < \infty, \quad for \quad r = 1, 2, 3.$$

First, using the hypotheses of Theorem B, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1}|I_{n,1}(t)|^k &= \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|\frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1} P_v P_v \Delta\lambda_v\right|^k\\
&\leq \sum_{n=2}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1} P_v P_v\Delta\lambda_v\right\} \times \left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_v P_v \Delta\lambda_v\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_v P_v \Delta\lambda_v\\
&= O(1)\sum_{v=1}^{m} P_v P_v \Delta\lambda_v \sum_{n=v+1}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}}\\
&= O(1)\sum_{v=1}^{m}\varphi_v^{\delta k} P_v \Delta\lambda_v = O(1) \quad as \quad m \to \infty.
\end{aligned}$$

Now, when $k > 1$, applying Hölder's inequality with indices k and k' where $\frac{1}{k} + \frac{1}{k'} = 1$, we have that

$$\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}|I_{n,2}(t)|^k = \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|\frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1} P_v p_v \lambda_v\right|^k$$

$$\leq \sum_{n=2}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} P_v^k p_v \lambda_v^k \right\} \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$
$$= O(1) \sum_{v=1}^{m} P_v^k \lambda_v^k p_v \sum_{n=v+1}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}}$$
$$= O(1) \sum_{v=1}^{m} P_v^k \lambda_v^k p_v \varphi_v^{\delta k} \frac{1}{P_v}$$
$$= O(1) \sum_{v=1}^{m} \varphi_v^{\delta k} (P_v \lambda_v)^{k-1} p_v \lambda_v$$
$$= O(1) \sum_{v=1}^{m} \varphi_v^{\delta k} p_v \lambda_v = O(1) \quad as \quad m \to \infty,$$

by virtue of the hypotheses of Theorem B and Lemma 1. Finally, using the fact that $P_n \lambda_n = O(1)$, by Lemma 1, we obtain that

$$\sum_{n=1}^{m} \varphi_n^{\delta k+k-1} |I_{n,3}(t)|^k = \sum_{n=1}^{m} \varphi_n^{\delta k+k-1} |p_n \lambda_n|^k$$
$$\leq \sum_{n=1}^{m} \varphi_n^{\delta k} \varphi_n^{k-1} (p_n \lambda_n)^{k-1} (p_n \lambda_n)$$
$$= \sum_{n=1}^{m} \varphi_n^{\delta k} (\varphi_n p_n)^{k-1} \lambda_n^{k-1} (p_n \lambda_n)$$
$$= O(1) \sum_{n=1}^{m} \varphi_n^{\delta k} (P_n \lambda_n)^{k-1} (p_n \lambda_n)$$
$$= O(1) \sum_{n=1}^{m} \varphi_n^{\delta k} (p_n \lambda_n) = O(1) \quad as \quad m \to \infty,$$

by virtue of the hypotheses of Theorem B. This completes the proof of Theorem B.

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