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Approximation by positive linear operators in modular spaces by power series method

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Abstract In the present paper, we study the problem of approximation to a function by means of positive linear operators in modular spaces in the sense of power series method. Indeed, in order to get stronger results than the classical cases, we use the power series method which also includes both Abel and Borel methods. An application that satisfies our theorem is also provided.

Keywords Power series method · Modular spaces · Positive linear operators

Mathematics Subject Classification 40G10 · 41A36 · 40C15

1 Introduction

The main theorems in classical approximation theory are the Weierstrass-type approximation theorems which state a continuous function can be uniformly approximated by some approximations and error estimates which are obtained in terms of the modulus of continuity [13,17]. These type of results let us to say the convergence with minimum calculations and also have important applications in the polynomial approximation theory, in various of functional analysis, in numerical solutions of differential and integral equations [1,2]. Recently some versions of Korovkin type theorems have been given in modular spaces that include as particular cases L^p, Orlicz and Musielak-

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Orlicz spaces [8,19] with the use of statistical convergence, filter convergence and convergences generated by summability methods [9–11,14–16,20,24].

An outline of the paper is as follows: The next section contains basic notation and definitions. In Sect. 3, we give a Korovkin type theorem in modular spaces by power series method which includes both Abel and Borel methods. In the final section, we provide an example which is an application of our theorem.

2 Notations and definitions

Let us begin with recalling some basic definitions and notations which will be used throughout the paper.

Let (p_j) be real sequence with $p_0 > 0$ and $p_1, p_2, p_3, \cdots \geq 0$, and such that the corresponding power series $p(t) := \sum_{j=0}^{\infty} p_j t^j$ has radius of convergence R with $0 < R \leq \infty$. If, for all $t \in (0, R)$,

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} x_j p_j t^j = L$$

then we say that $x = (x_j)$ is convergent in the sense of power series method [18, 22]. Power series method includes many well known summability methods such as Abel and Borel. Both methods have in common that their definitions are based on power series and they are not matrix methods (see [12,23] for details). In order to see that power series method is more effective than ordinary convergence, let $x = (1, 0, 1, 0, \ldots)$, $R = \infty$, $p\,(t) = e^t$ and for $j \geq 0$, $p_j = \dfrac{1}{j!}$. Then it is easy to see that

$$\lim_{t \to \infty} \frac{1}{e^t} \sum_{j=0}^{\infty} \frac{x_j t^j}{j!} = \lim_{t \to \infty} \frac{1}{e^t} \sum_{j=0}^{\infty} \frac{t^{2j}}{(2j)!} = \lim_{t \to \infty} \frac{1}{e^t} \left\{ \frac{e^t + e^{-t}}{2} \right\} = \frac{1}{2}.$$

So the sequence $x = (x_j)$ is convergent to $\frac{1}{2}$ in the sense of power series method but it is not convergent in the ordinary sense. Note that the power series method is regular if and only if

$$\lim_{t \to R^-} \frac{p_j t^j}{p(t)} = 0$$

holds for each $j \in \mathbb{N}^0$ [12]. Throughout the paper we assume that power series method is regular.

Let $G = [a, b]$ be a bounded interval of the real line $\mathbb{R}$ provided with the Lebesgue measure. We denote by $X(G)$ the space of all real-valued measurable functions on G with equality almost everywhere, by $C(G)$ the space of all continuous real valued functions on G, and by $C^{\infty}(G)$ the space of all infinitely differentiable functions on G. A functional $\varrho : X(G) \to [0, \infty]$ is a modular on $X(G)$ provided that the following conditions hold:

(i) $\varrho[f] = 0$ if and only if $f = 0$ a.e on G,
(ii) $\varrho[-f] = \varrho[f]$ for every $f \in X(G)$,
(iii) $\varrho[\alpha f + \beta g] \leq \varrho[f] + \varrho[g]$ for every $f, g \in X(G)$ and for any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

A modular ϱ is said to be Q-quasi convex if there exists a constant $Q \geq 1$ such that the inequality

$$\varrho[\alpha f + \beta g] \leq Q\alpha\varrho[Qf] + Q\beta\varrho[Qg]$$

holds for every $f, g \in X(G)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. In particular if $Q = 1$, then ϱ is called convex.

A modular ϱ is said to be Q-quasi semiconvex if there exists a constant $Q \geq 1$ such that the inequality

$$\varrho[af] \leq Qa\varrho[Qf]$$

holds for every $f \in X(G)$, $f \geq 0$ and $a \in (0, 1]$. It is clear that every Q-quasi convex modular is Q-quasi semiconvex. A modular ϱ is said to be monotone if $\varrho[f] \leq \varrho[g]$ for all $f, g \in X(G)$ with $|f| \leq |g|$.

We now consider some subspaces of $X(G)$ by means of a modular ϱ as follows

$$L^{\varrho}(G) := \left\{ f \in X(G) : \lim_{\lambda \to 0^+} \varrho[\lambda f] = 0 \right\}$$

and

$$E^{\varrho}(G) := \{ f \in L^{\varrho}(G) : \varrho[\lambda f] < \infty \quad \textit{for all } \lambda > 0 \}$$

are called the modular space generated by ϱ and the space of the finite elements of $L^{\varrho}(G)$, respectively. Observe that if ϱ is Q-quasi semiconvex then the space

$$\{ f \in X(G) : \varrho[\lambda f] < \infty \quad \textit{for some } \lambda > 0 \}$$

coincides with $L^{\varrho}(G)$. The notions about modulars have been introduced and widely discussed in [4–8].

Now recall the convergences in the sense of power series method in modular spaces which have been studied in [25]. Let $\{f_j\}$ be a function sequence whose terms belong to $L^{\varrho}(G)$. Then, $\{f_j\}$ is modularly convergent to a function $f \in L^{\varrho}(G)$ in the sense of power series method if and only if

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\lambda_0 (f_j - f)] = 0 \; for \; some \; \lambda_0 > 0.$$

Also, $\{f_j\}$ is strongly convergent to a function $f \in L^{\varrho}(G)$ in the sense of power series method if and only if

$$\lim_{t \to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \varrho[\lambda(f_j - f)] = 0 \; for \; every \; \lambda > 0.$$

Recall that $\{f_j\}$ is modularly convergent to a function $f \in L^{\varrho}(G)$ if and only if

$$\lim_{j \to \infty} \varrho[\lambda_0(f_j - f)] = 0 \; for \; some \; \lambda_0 > 0,$$

also $\{f_j\}$ is strongly convergent to a function $f \in L^{\varrho}(G)$ if and only if

$$\lim_{j \to \infty} \varrho[\lambda(f_j - f)] = 0 \; for \; every \; \lambda > 0.$$

One can also study these convergences such a general case:

$$\lim_{t \to R^-} \varrho \left[\lambda_0 \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j (f_j - f) \right] = 0 \; for \; some \; \lambda_0 > 0,$$

$$\lim_{t \to R^-} \varrho \left[\lambda \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j (f_j - f) \right] = 0 \; for \; every \; \lambda > 0.$$

Notice that $\varrho[\lambda_0 \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j (f_j - f)] = \varrho[\lambda_0 (\frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j f_j - f)]$ since

$$p(t) = \sum_{j=0}^{\infty} p_j t^j \; and \; f = \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j f.$$

In the present paper we consider these type of convergences. If there exists a constant $M > 0$ such that

$$\varrho[2u] \leq M \varrho[u]$$

holds for all $u \geq 0$ then it is said to be that ϱ satisfies the Δ_2-condition. A modular ϱ is said to be

- finite if χ_G, the characteristic function associated with G, belongs to $L^{\varrho}(G)$,
- absolutely finite if ϱ is finite and for every $\varepsilon > 0, \; \lambda > 0$ there exists $\delta > 0$ such that $\varrho[\lambda \chi_B] < \varepsilon$ for any measurable subset $B \subset G$ with $|B| < \delta$,
- strongly finite if $\chi_G \in E^{\varrho}(G)$,
- absolutely continuous if there is a positive constant a with the property: for all $f \in X(G)$ with $\varrho[f] < \infty$, the following condition holds: for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\varrho[af \chi_B] < \varepsilon$ whenever B is any measurable subset of G with $|B| < \delta$.

Recall that if a modular ϱ is monotone and finite, then we have $C(G) \subset L^{\varrho}(G)$ [4]. In a similar manner, if ϱ is monotone and strongly finite, then $C(G) \subset E^{\varrho}(G)$.

3 Modular Korovkin theorem by power series method

Let ϱ be monotone and finite modular on $X(G)$. Assume that D is a set satisfying $C^{\infty}(G) \subset D \subset L^{\varrho}(G)$. We can construct such a subset D since ϱ is monotone and finite. Assume further that $T := \{T_j\}$ is a sequence of positive linear operators from D into $X(G)$ for which there exists a subset $X_T \subset D$ containing $C^{\infty}(G)$ such that the inequality

$$\limsup_{t \to R^-} \varrho\big[\lambda \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j (T_j h)\big] \leq P\varrho[\lambda h] \tag{1}$$

holds for every $h \in X_T$, $\lambda > 0$ and for an absolute positive constant P. Throughout the paper we use the test functions defined by $e_i(x) = x^i$, $i = 0, 1, 2, \ldots$.

Theorem 1 *Let ϱ be a strongly finite, monotone, absolutely continuous and Q-quasi semiconvex modular on $X(G)$. Let T_j, $j \in \mathbb{N}^0$, be a sequence of positive linear operators from D into $X(G)$ satisfying (1). If*

$$\lim_{t \to R^-} \varrho\left[\lambda \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j (T_j e_i - e_i)\right] = 0,$$

for every $\lambda > 0$ and $i = 0, 1, 2$, then for every $f \in L^{\varrho}(G)$ such that $f - g \in X_T$ for every $g \in C^{\infty}(G)$

$$\lim_{t \to R^-} \varrho\left[\gamma \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j (T_j f - f)\right] = 0,$$

for some $\gamma > 0$.

Proof Let $g \in C(G) \cap D$ and first we show that

$$\lim_{t \to R^-} \varrho\left[\mu \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j (T_j g - g)\right] = 0, \;\; for\ every\ \mu > 0. \tag{2}$$

Since g is uniformly continuous on G then there exists a constant $M > 0$ such that $|g(x)| \leq M$ for every $x \in G$. Given $\varepsilon > 0$, we can choose $\delta > 0$ such that $|y - x| < \delta$ implies $|g(y) - g(x)| < \varepsilon$ where $x, y \in G$. One can see that for all $x, y \in G$

$$|g(y) - g(x)| < \varepsilon + \frac{2M}{\delta^2}(y - x)^2.$$

Since $\{T_j\}$ is a sequence of positive linear operators, we get

$$\begin{aligned}
&\left|\frac{1}{p(t)}\sum_{j=0}^{\infty} p_j t^j T_j(g(.); x) - g(x)\right| \\
&= \left|\frac{1}{p(t)}\sum_{j=0}^{\infty} p_j t^j T_j(g(.) - g(x); x) + \frac{1}{p(t)}\sum_{j=0}^{\infty} p_j t^j T_j(g(x); x) - g(x)\right| \\
&\leq \frac{1}{p(t)}\sum_{j=0}^{\infty} p_j t^j T_j(|g(.) - g(x)|; x) + |g(x)| \left|\frac{1}{p(t)}\sum_{j=0}^{\infty} p_j t^j T_j(e_0(.); x) - e_0(x)\right| \\
&\leq \frac{1}{p(t)}\sum_{j=0}^{\infty} p_j t^j T_j(\varepsilon + \frac{2M}{\delta^2}(. - x)^2; x) + M \left|\frac{1}{p(t)}\sum_{j=0}^{\infty} p_j t^j T_j(e_0(.); x) - e_0(x)\right| \\
&\leq \varepsilon + \varepsilon\left|\frac{1}{p(t)}\sum_{j=0}^{\infty} p_j t^j T_j(e_0(.); x) - 1\right| + \frac{2M}{\delta^2}\frac{1}{p(t)}\sum_{j=0}^{\infty} p_j t^j T_j((. - x)^2; x) \\
&\quad + M\left|\frac{1}{p(t)}\sum_{j=0}^{\infty} p_j t^j T_j(e_0(.); x) - e_0(x)\right| \\
&\leq \varepsilon + (\varepsilon + M)\left|\frac{1}{p(t)}\sum_{j=0}^{\infty} p_j t^j T_j(e_0(.); x) - e_0(x)\right| \\
&\quad + \frac{2M}{\delta^2}\Bigg[(\frac{1}{p(t)}\sum_{j=0}^{\infty} p_j t^j T_j(e_2(.); x) - e_2(x)) \\
&\quad - 2e_1(x)(\frac{1}{p(t)}\sum_{j=0}^{\infty} p_j t^j T_j(e_1(.); x) - e_1(x)) \\
&\quad + e_2(x)(\frac{1}{p(t)}\sum_{j=0}^{\infty} p_j t^j T_j(e_0(.); x) - e_0(x))\Bigg] \\
&\leq \varepsilon + \left(\varepsilon + M + \frac{2Mr^2}{\delta^2}\right)\left|\frac{1}{p(t)}\sum_{j=0}^{\infty} p_j t^j T_j(e_0(.); x) - e_0(x)\right| \\
&\quad + \frac{4Mr}{\delta^2}|T_j(e_1(.); x) - e_1(x)| \\
&\quad + \frac{2M}{\delta^2}|T_j(e_2(.); x) - e_2(x)|
\end{aligned}$$

where $r := \max\{|a|, |b|\}$. So the last inequality gives for any $\mu > 0$ that

$$\begin{aligned}\mu\left|\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jT_j(g;x)-g(x)\right| \le \mu\varepsilon+\mu K\left|\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jT_j(e_0;x)-e_0(x)\right|\\ +\mu K\left|\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jT_j(e_1;x)-e_1(x)\right|\\ +\mu K\left|\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jT_j(e_2;x)-e_2(x)\right|\end{aligned}$$

where $K:=\max\{\varepsilon+M+\frac{2Mr^2}{\delta^2},\frac{4Mr}{\delta^2},\frac{2M}{\delta^2}\}$. By applying the modular ϱ in the both sides of the above inequality, since ϱ is monotone, we have

$$\begin{aligned}\varrho\left[\mu(\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jT_j(g)-g)\right] \le \varrho\Bigg[\mu\varepsilon+\mu K\left|\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jT_je_0-e_0\right|\\ +\mu K\left|\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jT_je_1-e_1\right|\\ +\mu K\left|\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jT_je_2-e_2\right|\Bigg].\end{aligned}$$

So we may write that

$$\begin{aligned}&\varrho\left[\mu\left(\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jT_j(g)-g\right)\right] \le \varrho[4\mu\varepsilon]\\ &+\varrho\left[4\mu K\left(\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jT_je_0-e_0\right)\right]+\varrho\left[4\mu K\left(\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jT_je_1-e_1\right)\right]\\ &+\varrho\left[4\mu K\left(\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jT_je_2-e_2\right)\right].\end{aligned}$$

Since ϱ is Q-quasi semiconvex and strongly finite, we have

$$\begin{aligned}\varrho\left[\mu(\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jT_j(g)-g)\right] \le Q\varepsilon\varrho[4\mu Q]+\varrho\left[4\mu K(\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jT_je_0-e_0)\right]\\ +\varrho\left[4\mu K(\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jT_je_1-e_1)\right]\\ +\varrho\left[4\mu K(\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jT_je_2-e_2)\right]\end{aligned}$$

without loss of generality where $0 < \varepsilon \leq 1$. By taking limit superior as $t \to R^-$ in the both sides and by using hypothesis, we get

$$\lim_{t \to R^-} \varrho[\mu \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j (T_j g - g)] = 0$$

which proves our claim. Now let $f \in L^{\varrho}(G)$ satisfying $f - g \in X_T$ for every $g \in C^{\infty}(G)$. Since $|G| < \infty$ and ϱ is strongly finite and absolutely continuous, it is known that ϱ is also absolutely finite on $X(G)$ (see [3]). Using the properties of ϱ and it is also known from [8] that the space $C^{\infty}(G)$ is modularly dense in $L^{\varrho}(G)$, i.e., there exists a sequence $\{g_k\} \subset C^{\infty}(G)$ such that

$$\lim_{k} \varrho[3\lambda_0(g_k - f)] = 0 \quad \textit{for some } \lambda_0 > 0.$$

This means that, for every $\varepsilon > 0$, there is a positive number $k_0 = k_0(\varepsilon)$ so that

$$\varrho[3\lambda_0(g_k - f)] < \varepsilon \quad \textit{for every } k \geq k_0.$$

On the other hand, by linearity and positivity of the operators T_j we may write that

$$\begin{aligned}\lambda_0 \left| \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j T_j(f; x) - f(x) \right| \leq{}& \lambda_0 \left| \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j T_j(f - g_{k_0}; x) \right| \\ &+ \lambda_0 \left| \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j T_j(g_{k_0}; x) - g_{k_0} \right| \\ &+ \lambda_0 |g_{k_0}(x) - f(x)|.\end{aligned}$$

Applying the modular ϱ in the both sides of the above inequality, since ϱ is monotone

$$\begin{aligned}\varrho \left[\lambda_0 \left(\frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j T_j f - f \right) \right] \leq{}& \varrho \left[3\lambda_0 \left(\frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j T_j (f - g_{k_0}) \right) \right] \\ &+ \varrho \left[3\lambda_0 \left(\frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j T_j g_{k_0} - g_{k_0} \right) \right] \\ &+ \varrho \left[3\lambda_0 \left(g_{k_0} - f \right) \right].\end{aligned}$$

Then it follows from the above inequalities that

$$\varrho\left[\lambda_0\left(\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jT_jf-f\right)\right]\leq\varrho\left[3\lambda_0\left(\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jT_j(f-g_{k_0})\right)\right]$$
$$+\varrho\left[3\lambda_0\left(\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jT_jg_{k_0}-g_{k_0}\right)\right]+\varepsilon.$$

Hence, using the facts that $g_{k_0}\in C^{\infty}(G)$ and $f-g_{k_0}\in X_T$, and taking limit superior as $t\to R^-$ in both sides, we obtain that

$$\limsup_{t\to R^-}\varrho\left[\lambda_0\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^j(T_jf-f)\right]$$
$$\leq\varepsilon+P\varrho\left[3\lambda_0\left(f-g_{k_0}\right)\right]+\limsup_{t\to R^-}\varrho\left[3\lambda_0\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^j\left(T_jg_{k_0}-g_{k_0}\right)\right] \tag{3}$$

which gives

$$\limsup_{t\to R^-}\varrho\left[\lambda_0\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^j(T_jf-f)\right]$$
$$\leq\varepsilon+\varepsilon P+\limsup_{t\to R^-}\varrho\left[3\lambda_0\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^j\left(T_jg_{k_0}-g_{k_0}\right)\right].$$

By (2), we get

$$\limsup_{t\to R^-}\varrho\left[3\lambda_0\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^j\left(T_jg_{k_0}-g_{k_0}\right)\right]=0$$

and this implies

$$\limsup_{t\to R^-}\varrho\left[\lambda_0\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^j(T_jf-f)\right]\leq\varepsilon+\varepsilon P.$$

Since ε is arbitrary positive real number, we have

$$\limsup_{t\to R^-}\varrho\left[\lambda_0\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^j(T_jf-f)\right]=0$$

and also $\varrho[\lambda_0 \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j (T_j f - f)]$ is nonnegative then

$$\lim_{t \to R^-} \varrho \left[\lambda_0 \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j (T_j f - f) \right] = 0.$$

This completes the proof. □

If the modular ϱ satisfies the Δ_2-condition, then one can get the following result from the above theorem.

Theorem 2 *Let ϱ and $T = \{T_j\}$ be as in the above theorem. If ϱ satisfies the Δ_2-condition, then the followings are equivalent:*

- $\lim_{t \to R^-} \varrho[\lambda \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j (T_j e_i - e_i)] = 0,$ *for every* $\lambda > 0$ *and* $i = 0, 1, 2.$
- $\lim_{t \to R^-} \varrho[\gamma \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j (T_j f - f)] = 0,$ *for every* $\gamma > 0$ *then every* $f \in L^{\varrho}(G)$ *such that* $f - g \in X_T,$ *for every* $g \in C^{\infty}(G).$

4 Concluding remarks

In this section we provide an example which satisfies our main theorem.
In order to provide the example, take $G = [0, 1]$ and let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous function for which the following conditions hold:

- φ is convex,
- $\varphi(0) = 0, \varphi(u) > 0$ *for* $u > 0$ and $\lim_{u \to +\infty} \varphi(u) = \infty.$

Here, consider the functional ρ^{φ} on $X(G)$ defined by

$$\rho^{\varphi}[f] := \int_0^1 \varphi(|f(x)|)dx, \ \ for \ f \in X(G).$$

In this case, ρ^{φ} is a convex modular on $X(G)$ (see [4]). Consider the Orlicz space generated by φ as follows

$$L_{\varphi}^{\rho}(G) := \{f \in X(G) : \rho^{\varphi}[\lambda f] < \infty \quad for\ some\ \lambda > 0\}.$$

Let us consider a finite sequence $\Gamma_j = (v_{j,k})_{k=0,1,2,\ldots,r(j),r(j)+1} \subset G$ satisfying the following assumption:

$$0 < a_j \leq \gamma_{j,k} := v_{j,k+1} - v_{j,k} \leq b_j, \ k = 0, 1, 2, \ldots, r(j)$$

where a_j, b_j are positive real numbers and $\lim_{j\to\infty} b_j = 0$. Then, consider a sequence $\mathbb{U} := \{U_j\}$ on the space $L_\varphi^\rho(G)$ (see [4]) which is defined by

$$U_j(f;x) := \sum_{k=0}^{r(j)} K_j(x, v_{j,k}) \frac{1}{\gamma_{j,k}} \int_{v_{j,k}}^{v_{j,k+1}} f(t)dt \ ; \ x \in G.$$

Here the kernel $(K_j)_{j\in\mathbb{N}^0}$, $K_j : G \times \Gamma_j \to \mathbb{R}$ is a sequence of nonnegative functions such that

$$\sum_{k=0}^{r(j)} K_j(x, v_{j,k}) = 1, \ \ for\ every\ j \in \mathbb{N}^0\ and\ x \in [0, 1];$$
$$\lim_{j\to\infty} m_n(K_j, .) = 0, n = 1, 2$$

and

$$\int_G K_j(x, v_{j,k})dx \le \xi_j$$

where ξ_j is a bounded sequence of positive numbers independent of k and

$$m_n(K_j, x) := \sum_{k=0}^{r(j)} K_j(x, v_{j,k})(v_{j,k} - x)^n.$$

Also assume that $\frac{\xi_j}{a_j} \le M$ for every $j \in \mathbb{N}^0$ and an absolute constant $M > 0$. Then it is known that U_j map the Orlicz space $L_\varphi^\rho(G)$ into itself [4]. Moreover, the property $\limsup_{j\to\infty} \varrho^\varphi[\lambda(U_j h)] \le M\varrho^\varphi[\lambda h]$ is satisfied with the choice of $X_U := L_\varphi^\rho(G)$ and for every function $f \in L_\varphi^\rho(G)$, $\{U_j f\}$ is modularly convergent to f. Using the operators $\{U_j\}$ define the sequence of positive linear operators $V := \{V_j\}$ on $L_\varphi^\rho(G)$ as follows:

$$V_j(f;x) = (1 + s_j)U_j(f;x), \ \ for\ f \in L_\varphi^\rho(G), \ x \in [0, 1]\ and\ j \in \mathbb{N}^0, \tag{4}$$

where $s_j = 1$, if j is square and 0 otherwise. Let $R = 1$, $p(t) = \frac{1}{1-t}$ and for $j \in \mathbb{N}^0$ $p_j = 1$. Observe that $\{s_j\}$ is a sequence of zeros and ones which is nonconvergent but convergent to 0 in the sense of power series method which coincides with the Abel method under this choice. Let also $\varphi(x) = x^p$ for $1 \le p < \infty$. In this case we have $L_\varphi^\rho(G) = L_p(G)$ and $\rho^\varphi[f] = \|f\|_{L_p}^p$ [21]. So one can get that

$$\rho^{\varphi}\left[\lambda\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jV_jh\right]=\left\|\lambda\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^j(1+s_j)U_jh\right\|_{L_p}^p$$
$$\leq 2^p\left\|\lambda\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jU_jh\right\|_{L_p}^p\leq 2^p|\lambda|^p\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^j\|U_jh\|_{L_p}^p$$

and

$$\limsup_{t\to R^-}\rho^{\varphi}\left[\lambda\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jV_jh\right]\leq 2^p|\lambda|^pM\|\lambda h\|_{L_p}^p=N\rho^{\varphi}[\lambda h]$$

where $N=2^p|\lambda|^pM$. Now, we show that conditions in Theorem 1 holds. First note that

$$V_j(e_0;x)=1+s_j$$

since $U_j(e_0)=1$. Therefore

$$\rho^{\varphi}\left[\lambda(\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jV_j(e_0)-e_0)\right]=\left\|\lambda(\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jV_j(e_0)-e_0)\right\|_{L_p}^p$$
$$=\left\|\lambda\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^js_j\right\|_{L_p}^p,$$

holds and one can get $\lim_{t\to R^-}\|\lambda\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^js_j\|_{L_p}^p=0\ for\ every\ \lambda>0$ by taking limit as $t\to R^-$. Similarly

$$V_j(e_1;x)=(1+s_j)U_j(e_1;x)$$

and it is known that $\lim_{j\to\infty}\rho^{\varphi}[\lambda(U_j(e_1)-e_1)]=0,\ \ for\ every\ \lambda>0$ [4]. Since the Abel method is regular, one can again get that

$$\lim_{t\to R^-}\rho^{\varphi}\left[\lambda(\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jV_j(e_1)-e_1)\right]$$
$$=\lim_{t\to R^-}\left\|\lambda\left(\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jV_j(e_1)-e_1\right)\right\|_{L_p}^p$$
$$\leq 2^p\lim_{t\to R^-}\left\|\lambda\left(\frac{1}{p(t)}\sum_{j=0}^{\infty}p_jt^jU_j(e_1)-e_1\right)\right\|_{L_p}^p$$

$$\leq 2^p \lim_{t\to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \left\| \lambda(U_j(e_1) - e_1) \right\|_{L_p}^p$$

$$\leq 2^p \lim_{t\to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \rho^{\varphi}[\lambda(U_j(e_1) - e_1)] = 0, \ \textit{for every}\ \lambda > 0.$$

Now

$$V_j(e_2; x) = (1 + s_j)U_j(e_2; x),$$

and it is also known that $\lim_{j\to\infty} \rho^{\varphi}[\lambda(U_j(e_2) - e_2)] = 0,\ \textit{for every}\ \lambda > 0$. Since the Abel method is regular, we have that

$$\lim_{t\to R^-} \rho^{\varphi}\left[\lambda(\frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j V_j(e_2) - e_2)\right]$$

$$= \lim_{t\to R^-} \left\| \lambda\left(\frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j V_j(e_2) - e_2 \right) \right\|_{L_p}^p$$

$$\leq 2^p \lim_{t\to R^-} \left\| \lambda\left(\frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j U_j(e_2) - e_2 \right) \right\|_{L_p}^p$$

$$\leq 2^p \lim_{t\to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \left\| \lambda(U_j(e_2) - e_2) \right\|_{L_p}^p$$

$$\leq 2^p \lim_{t\to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j \rho^{\varphi}[\lambda(U_j(e_2) - e_2)] = 0, \ \textit{for every}\ \lambda > 0.$$

So we can say that the sequence $V := \{V_j\}$ satisfies all assumptions of Theorem 1.

Remark 1 It is known that

- in the case of $R = 1$, $p\,(t) = \dfrac{1}{1-t}$ and for $j \in \mathbb{N}^0$, $p_j = 1$ the power series method coincides with Abel method which is a sequence-to-function transformation,
- in the case of $R = \infty$, $p\,(t) = e^t$ and for $j \in \mathbb{N}^0$, $p_j = \dfrac{1}{j!}$ the power series method coincides with Borel method.

We can therefore give all of the theorems of this paper for Abel and Borel convergences.

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