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Characterizations for the Fractional Integral Operators in Generalized Morrey Spaces on Carnot Groups

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Abstract—In this paper, we study the boundedness of the fractional integral operator I_α on Carnot group $\mathbb{G}$ in the generalized Morrey spaces $M_{p,\varphi}(\mathbb{G})$. We shall give a characterization for the strong and weak type boundedness of I_α on the generalized Morrey spaces, respectively. As applications of the properties of the fundamental solution of sub-Laplacian $\mathcal{L}$ on $\mathbb{G}$, we prove two Sobolev–Stein embedding theorems on generalized Morrey spaces in the Carnot group setting.

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Dedicated to Professor Stefan Samko on the occasion of his 75th birthday.

1. INTRODUCTION

Carnot groups appear in quantum physics and many parts of mathematics, including Fourier analysis, several complex variables, geometry and topology. Analysis on these groups is also motivated by their role as the simplest and the most important model in the general theory of vector fields satisfying the Hörmander's condition. The simplest examples of the Carnot groups are Euclidean space $\mathbb{R}^n$, the Heisenberg group $\mathbb{H}_n$ and the Heisenberg-type groups introduced by Kaplan [1].

For $x \in \mathbb{G}$ and $r > 0$, let $D(x,r)$ denote the $\mathbb{G}$-ball centered at x of radius r, and let $^{\complement}D(x,r)$ denote its complement.

Let $f \in L_1^{\mathrm{loc}}(\mathbb{G})$. The maximal operator M and the fractional integral operator I_α are defined by

$$Mf(x) = \sup_{r>0} |D(x,r)|^{-1} \int_{D(x,r)} |f(y)|\, dy,$$

$$I_\alpha f(x) = \int_{\mathbb{G}} \frac{f(y)\, dy}{\rho(x^{-1}y)^{Q-\alpha}}, \qquad 0 < \alpha < Q,$$

where Q is the homogeneous dimension of the homogeneous Carnot group $\mathbb{G}$ and $|D(x,t)|$ is the Haar measure of the $\mathbb{G}$-ball $D(x,t)$.

The operators M and I_α play an important role in real and harmonic analysis and applications (see, for example, [2] and [3]).

The Hardy–Littlewood–Sobolev theorem for the fractional integral operator I_α holds.

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Theorem A ([2], [4]). *Let $0 < \alpha < Q$ and $1 \le p < Q/\alpha$. Then*

(1) *If $1 < p < Q/\alpha$, then the condition $1/p - 1/q = \alpha/Q$ is necessary and sufficient for the boundedness of I_α from $L_p(\mathbb{G})$ to $L_q(\mathbb{G})$.*

(2) *If $p = 1$, then the condition $1 - 1/q = \alpha/Q$ is necessary and sufficient for the boundedness of I_α from $L_1(\mathbb{G})$ to $WL_q(\mathbb{G})$.*

In the present work, we study Spanne-type boundedness of the operator I_α from M_{p,φ_1} to M_{q,φ_2}, $1 < p < q < \infty$, and from the space M_{1,φ_1} to the weak space WM_{q,φ_2}, $1 < q < \infty$. Also we study Adams-type boundedness of the operator I_α from $M_{p,\varphi^{1/p}}(\mathbb{G})$ to $M_{q,\varphi^{1/q}}(\mathbb{G})$, $1 < p < q < \infty$, and from the space $M_{1,\varphi}(\mathbb{G})$ to the weak space $WM_{q,\varphi^{1/q}}(\mathbb{G})$, $1 < q < \infty$. We shall give a characterization for the Spanne and Adams-type boundedness of the operator I_α on generalized Morrey spaces, including weak versions. As applications of the properties of the fundamental solution of sub-Laplacian $\mathcal{L}$ on $\mathbb{G}$, we prove two Sobolev–Stein embedding theorems on generalized Morrey spaces in the Carnot group setting.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. NOTATION AND PRELIMINARY RESULTS

We first recall some preliminaries concerning stratified Lie groups (or so-called Carnot groups). We refer the reader to the books [3], [5] and [6] for analysis on stratified groups.

Let $\mathcal{G}$ be a finite-dimensional, stratified, nilpotent Lie algebra. Assume that there is a direct sum vector space decomposition

$$\mathcal{G} = V_1 \oplus \cdots \oplus V_m, \tag{2.1}$$

so that each element of V_j, $2 \le j \le m$, is a linear combination of $(j-1)$th order commutator of elements of V_1. Equivalently, (2.1) is a stratification provided $[V_i, V_j] = V_{i+j}$ whenever $i + j \le m$ and $[V_i, V_j] = 0$ otherwise. Let $X = X_1, \dots, X_n$ be a basis for V_1 and X_{ij}, $1 \le i \le k_j$, for V_j consisting of commutators of length j. We set $X_{i1} = X_i$, $i = 1, \dots, n$ and $k_1 = n$, and we call X_{i1} a *commutator of length* 1.

If $\mathbb{G}$ is the simply connected Lie group associated with $\mathcal{G}$, then the exponential mapping is a global diffeomorphism from $\mathcal{G}$ to $\mathbb{G}$. Thus, for each $g \in \mathbb{G}$, there is $x = (x_{ij}) \in \mathbb{R}^N$, $1 \le i \le k_j$, $1 \le j \le m$, $N = \sum_{j=1}^m k_j$, such that

$$g = \exp\Big(\sum x_{ij} X_{ij}\Big).$$

A homogeneous norm function $|\cdot|$ on $\mathbb{G}$ is defined by

$$|g| = \bigg(\sum |x_{ij}|^{2m!/j}\bigg)^{1/(2m!)},$$

and $Q = \sum_{j=1}^m j k_j$ is said to be the *homogeneous dimension* of $\mathbb{G}$, since $d(\delta_r x) = r^Q\,dx$ for $r > 0$. The *dilation* δ_r on $\mathbb{G}$ is defined by

$$\delta_r(g) = \exp\Big(\sum r^j x_{ij} X_{ij}\Big), \quad \text{if} \quad g = \exp\Big(\sum x_{ij} X_{ij}\Big).$$

The convolution operation on $\mathbb{G}$ is defined by

$$f * h(x) = \int_{\mathbb{G}} f(xy^{-1}) h(y)\,dy = \int_{\mathbb{G}} f(y) h(y^{-1}x)\,dy,$$

where y^{-1} is the inverse of y and xy^{-1} denotes the group multiplication of x by y^{-1}. It is known that, for any left invariant vector field X on $\mathbb{G}$,

$$X(f * h) = f * (Xh).$$

Since $\mathbb{G}$ is nilpotent, the exponential map is a diffeomorphism from $\mathbb{G}$ onto $\mathbb{G}$ which takes the Lebesgue measure on $\mathbb{G}$ to a bi-invariant Haar measure dx on $\mathbb{G}$. The group identity of $\mathbb{G}$ will be referred to as the *origin* and denoted by e.

A homogeneous norm on $\mathbb{G}$ is a continuous function $x \to \rho(x)$ from $\mathbb{G}$ to $[0,\infty)$, which is C^∞ on $\mathbb{G} \setminus \{0\}$ and satisfies $(x^{-1}) = \rho(x)$, $\rho(\delta_t x) = t\rho(x)$ for all $x \in \mathbb{G}$, $t > 0$; $\rho(e) = 0$ (the group identity); moreover, there exists a constant $c_0 \geq 1$ such that

$$\rho(xy) \leq c_0(\rho(x) + \rho(y)) \qquad \text{for all} \quad x, y \in G.$$

We call a curve $\gamma\colon [a,b] \to \mathbb{G}$ a *horizontal curve* connecting two points $x, y \in \mathbb{G}$ if $\gamma(a) = x$, $\gamma(b) = y$ and $\gamma'(t) \in V_1$ for all t. Then the *Carnot–Carathéodory distance* between x, y is defined as

$$d_{cc}(x,y) = \inf_{\gamma} \int_a^b \langle \gamma'(t), \gamma'(t)\rangle^{1/2}\, dt,$$

where the infimum is taken over all horizontal curves γ connecting x and y. It is known that any two points x, y on $\mathbb{G}$ can be joined by a horizontal curve of finite length and then d_{cc} is a left invariant metric on $\mathbb{G}$. We can define the *metric ball* centered at x and with radius r associated with this metric by

$$B_{cc}(x,r) = \{y \in \mathbb{G} : d_{cc}(x,y) < r\}.$$

We note that this metric d_{cc} is equivalent to the pseudo-metric $\rho(x,y) = |x^{-1}y|$ defined by the homogeneous norm $|\cdot|$ in the following sense (see [2]):

$$C^{-1}\rho(x,y) \leq d_{cc}(x,y) \leq C\rho(x,y).$$

We denote the metric ball associated with ρ by $D(x,r) = \{y \in \mathbb{G} : \rho(x,y) < r\}$. An important feature of both of these distance functions is that these distances and thus the associated metric balls are left invariant, namely,

$$d_{cc}(zx, zy) = d_{cc}(x,y), \qquad B_{cc}(x,r) = xB_{cc}(e,r)$$

and

$$\rho(zx, zy) = \rho(x,y), \qquad D(x,r) = xD(e,r).$$

From now on, we will always use the metric d_{cc} and drop the subscript from d_{cc}. Similarly, we will use $B(x,r)$ to denote $B_{cc}(x,r)$.

With this norm, we define the $\mathbb{G}$-ball of radius r centered at x by

$$D(x,r) = \{y \in \mathbb{G} : \rho(y^{-1}x) < r\},$$

and we denote by $D_r = D(e,r) = \{y \in \mathbb{G} : \rho(y) < r\}$ the open ball of radius r centered at e, the identity element of $\mathbb{G}$. By ${}^{\complement}D(x,r) = \mathbb{G} \setminus D(x,r)$ we denote the complement of $D(x,r)$.

One easily recognizes that there exist $c_1 = c_1(\mathbb{G})$, and $c_2 = c_2(\mathbb{G})$ such that

$$|B(x,r)| = c_1 r^Q, \quad |D(x,r)| = c_2 r^Q, \qquad x \in \mathbb{G}, \quad r > 0.$$

The most basic partial differential operator in a Carnot group is the sub-Laplacian associated with X, i.e., the second-order partial differential operator on $\mathbb{G}$ given by $\mathcal{L} = \sum_{i=1}^n X_i^2$.

3. GENERALIZED MORREY SPACES

In the study of local properties of solutions of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $L_{p,\lambda}(\mathbb{G})$ play an important role, see [7]. They were introduced by C. Morrey in 1938 [8]. The Morrey space in a Carnot group is defined as follows: for $1 \leq p \leq \infty$, $0 \leq \lambda \leq Q$, a function $f \in L_{p,\lambda}(\mathbb{G})$ if $f \in L_p^{\text{loc}}(\mathbb{G})$ and

$$\|f\|_{L_{p,\lambda}} := \sup_{x\in\mathbb{G},\, r>0} r^{-\lambda/p} \|f\|_{L_p(D(x,r))} < \infty;$$

$$L_{p,\lambda}(\mathbb{G}) = \begin{cases} L_p(\mathbb{G}) & \text{if } \lambda = 0, \\ L_\infty(\mathbb{G}) & \text{if } \lambda = Q, \\ \Theta & \text{if } \lambda < 0 \text{ or } \lambda > Q, \end{cases}$$

where Θ is the set of all functions equivalent to 0 on $\mathbb{G}$.

We also denote by $WL_{p,\lambda}(\mathbb{G})$ the weak Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{G})$, for which

$$\|f\|_{WL_{p,\lambda}} \equiv \|f\|_{WL_{p,\lambda}(\mathbb{G})} = \sup_{x\in\mathbb{G},\, r>0} r^{-\lambda/p} \|f\|_{WL_p(D(x,r))} < \infty,$$

where $WL_p(D(x,r))$ denotes the weak L_p-space of measurable functions f, for which

$$\|f\|_{WL_p(D(x,r))} = \sup_{t>0} t|\{y \in D(x,r) : |f(y)| > t\}|^{1/p}. \tag{3.1}$$

We find it convenient to define the generalized Morrey spaces as follows.

Definition. Let $1 \le p < \infty$ and $\varphi(x,r)$ be a positive measurable function on $\mathbb{G} \times (0,\infty)$. The *generalized Morrey space* $M_{p,\varphi}(\mathbb{G})$ is defined for all functions $f \in L_p^{\mathrm{loc}}(\mathbb{G})$ by the finite norm

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{G},\, r>0} \frac{r^{-Q/p}}{\varphi(x,r)} \|f\|_{L_p(D(x,r))}.$$

Also the *weak generalized Morrey space* $WM_{p,\varphi}(\mathbb{G})$ is defined for all functions $f \in L_p^{\mathrm{loc}}(\mathbb{G})$ by the finite norm

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{G},\, r>0} \frac{r^{-Q/p}}{\varphi(x,r)} \|f\|_{WL_p(D(x,r))}.$$

Lemma 3.1. *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{G} \times (0,\infty)$.*

(i) *If*

$$\sup_{t<r<\infty} \frac{r^{-Q/p}}{\varphi(x,r)} = \infty \qquad \text{for some} \quad t > 0 \quad \text{and for all} \quad x \in \mathbb{G}, \tag{3.2}$$

then $M_{p,\varphi}(\mathbb{G}) = \Theta$.

(ii) *If*

$$\sup_{0<r<\tau} \varphi(x,r)^{-1} = \infty \qquad \text{for some} \quad t > 0 \quad \text{and for all} \quad x \in \mathbb{G}, \tag{3.3}$$

then $M_{p,\varphi}(\mathbb{G}) = \Theta$.

Proof. (i) Let (3.2) be satisfied and f be not equivalent to zero. Then $\sup_{x\in\mathbb{G}} \|f\|_{L_p(D(x,t))} > 0$, and hence

$$\|f\|_{M_{p,\varphi}} \ge \sup_{x\in\mathbb{G}} \sup_{t<r<\infty} \varphi(x,r)^{-1} r^{-Q/p} \|f\|_{L_p(D(x,r))} \ge \sup_{x\in\mathbb{G}} \|f\|_{L_p(D(x,t))} \sup_{t<r<\infty} \varphi(x,r)^{-1} r^{-Q/p}.$$

Therefore, $\|f\|_{M_{p,\varphi}} = \infty$.

(ii) Let $f \in M_{p,\varphi}(\mathbb{G})$ and (3.3) be satisfied. Then there are two possibilities:

Case 1: $\sup_{0<r<t} \varphi(x,r)^{-1} = \infty$ for all $t > 0$.

Case 2: $\sup_{0<r<t} \varphi(x,r)^{-1} < \infty$ for some $s \in (0,\tau)$.

For Case 1, by the Lebesgue differentiation theorem, for almost all $x \in \mathbb{G}$, we have

$$\lim_{r\to 0+} \frac{\|f\chi_{D(x,r)}\|_{L_p}}{\|\chi_{D(x,r)}\|_{L_p}} = |f(x)|. \tag{3.4}$$

We claim that $f(x) = 0$ for all those x. Indeed, fix x and assume $|f(x)| > 0$. Then, by Lemma 3.2 and (3.4), there exists a $t_0 > 0$ such that

$$r^{-Q/p}\|f\|_{L_p(D(x,r))} \geq 2^{-1}c_2^{1/p}|f(x)|$$

for all $0 < r \leq t_0$. Consequently,

$$\|f\|_{M_{p,\varphi}} \geq \sup_{0<r<t_0} \varphi(x,r)^{-1}r^{-Q/p}\|f\|_{L_p(D(x,r))} \geq 2^{-1}c_2^{1/p}|f(x)| \sup_{0<r<t_0} \varphi(x,r)^{-1}.$$

Hence $\|f\|_{M_{p,\varphi}} = \infty$, so $f \notin M_{p,\varphi}(\mathbb{G})$ and we have arrived at a contradiction.

Note that Case 2 implies that $\sup_{s<r<\tau} \varphi(x,r)^{-1} = \infty$, hence

$$\sup_{s<r<\infty} \varphi(x,r)^{-1}r^{-Q/p} \geq \sup_{s<r<\tau} \varphi(x,r)^{-1}r^{-Q/p} \geq \tau^{-Q/p} \sup_{s<r<\tau} \varphi(x,r)^{-1} = \infty,$$

which is the case in (i). $\square$

Remark 3.1. We denote by Ω_p the sets of all positive measurable functions φ on $G \times (0,\infty)$ such that, for all $t > 0$,

$$\sup_{x\in\mathbb{G}}\left\| \frac{r^{-Q/p}}{\varphi(x,r)} \right\|_{L_\infty(t,\infty)} < \infty \qquad \text{and} \qquad \sup_{x\in\mathbb{G}} \|\varphi(x,r)^{-1}\|_{L_\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 3.1, we always assume that $\varphi \in \Omega_p$.

A function $\varphi\colon (0,\infty) \to (0,\infty)$ is said to be *almost increasing* (resp. *almost decreasing*) if there exists a constant $C > 0$ such that

$$\varphi(r) \leq C\varphi(s) \quad (\text{resp.} \quad \varphi(r) \geq C\varphi(s)) \qquad \text{for} \quad r \leq s.$$

Let $1 \leq p < \infty$. Denote by $\mathcal{G}_p$ the set of all almost decreasing functions $\varphi\colon (0,\infty) \to (0,\infty)$, such that $t \in (0,\infty) \mapsto t^{Q/p}\varphi(t) \in (0,\infty)$ is almost increasing.

Apparently, the requirement $\phi \in \mathcal{G}_p$ is superfluous but it turns out that this condition is natural. Indeed, Nakai established that there exists a function ρ such that ρ itself is decreasing, that $\rho(t)t^{n/p} \leq \rho(T)T^{n/p}$ for all $0 < t \leq T < \infty$ and that $M_{p,\phi}(\mathbb{G}) = M_{p,\rho}(\mathbb{G})$.

By elementary calculations, we obtain the following statement, which shows particularly that the spaces $M_{p,\varphi}(\mathbb{G})$ and $WM_{p,\varphi}(\mathbb{G})$ are not trivial; see, for example, [9].

Lemma 3.2. *Let $\varphi \in \mathcal{G}_p$, $1 \leq p < \infty$, $D_0 = D(x_0, r_0)$, and χ_{D_0} is the characteristic function of the ball D_0, then $\chi_{D_0} \in M_{p,\varphi}(\mathbb{G})$. Moreover, there exists $C > 0$ such that*

$$\frac{1}{\varphi(r_0)} \leq \|\chi_{D_0}\|_{WM_{p,\varphi}} \leq \|\chi_{D_0}\|_{M_{p,\varphi}} \leq \frac{C}{\varphi(r_0)}.$$

Proof. Let $\varphi \in \mathcal{G}_p$, $1 \leq p < \infty$, $D_0 = D(x_0, r_0)$ denote an arbitrary ball in $\mathbb{G}$. It is easy to see that

$$\|\chi_{D_0}\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{G},\, r>0} \frac{1}{\varphi(r)}\left(\frac{|D(x,r)\cap D_0|}{|D(x,r)|}\right)^{1/p} \geq \frac{1}{\varphi(r_0)}\left(\frac{|D_0 \cap D_0|}{|D_0|}\right)^{1/p} = \frac{1}{\varphi(r_0)}.$$

Now, if $r \leq r_0$, then $\varphi(r_0) \leq C\varphi(r)$ and

$$\frac{1}{\varphi(r)}\left(\frac{|D(x,r)\cap D_0|}{|D(x,r)|}\right)^{1/p} \leq \frac{1}{\varphi(r)} \leq \frac{C}{\varphi(r_0)}$$

for all $x \in \mathbb{G}$.

On the other hand, if $r_0 \leq r$, we have $\varphi(r_0)r_0^{Q/p} \leq C\varphi(r)r^{Q/p}$ for all $x \in \mathbb{G}$ and

$$\frac{1}{\varphi(r)}\left(\frac{|D(x,r)\cap D_0|}{|D(x,r)|}\right)^{1/p} = \frac{|D(x,r)\cap D_0|^{1/p}}{c_2^{1/p}\varphi(r)r^{Q/p}} \leq \frac{|D_0|^{1/p}}{c_2^{1/p}\varphi(r)r^{Q/p}} = \frac{r_0^{Q/p}}{\varphi(r)r^{Q/p}} \leq \frac{C}{\varphi(r_0)}$$

for all $x \in \mathbb{G}$. This completes the proof. $\square$

The following theorem was proved in [10].

Theorem 3.1. *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfies the condition*

$$\sup_{r<t<\infty} t^{-Q/p} \operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{Q/p} \le C\varphi_2(x,r), \tag{3.5}$$

where C does not depend on x and r. Then, for $p > 1$, the operator M is bounded from $M_{p,\varphi_1}(\mathbb{G})$ to $M_{p,\varphi_2}(\mathbb{G})$, and for $p = 1$, the operator M is bounded from $M_{1,\varphi_1}(\mathbb{G})$ to $WM_{1,\varphi_2}(\mathbb{G})$.

4. FRACTIONAL INTEGRAL OPERATOR IN THE SPACES $M_{p,\varphi}(\mathbb{G})$

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_t^\infty g(s) w(s)\, ds, \qquad 0 < t < \infty,$$

where w is a weight.

The following theorem was proved in [11] (see also [12]).

Theorem 4.1. *Let v_1, v_2, let w be weights on $(0, \infty)$, and let $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t) H_w g(t) \le C \operatorname*{ess\,sup}_{t>0} v_1(t) g(t) \tag{4.1}$$

holds for some $C > 0$ for all nonnegative and nondecreasing g on $(0, \infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)\, ds}{\operatorname{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Moreover, the value $C = B$ is the best constant for (4.1).

4.1. Spanne-Type Result

The following local estimates are valid.

Theorem 4.2. *Let $1 \le p < \infty$, $0 < \alpha < Q/p$, $1/q = 1/p - \alpha/Q$, and $f \in L_p^{\mathrm{loc}}(\mathbb{G})$. Then, for $p > 1$*

$$\|I_\alpha f\|_{L_q(D(x,t))} \le C t^{Q/q} \int_{2c_0 t}^\infty r^{-Q/q-1} \|f\|_{L_p(D(x,r))}\, dr \tag{4.2}$$

and for $p = 1$

$$\|I_\alpha f\|_{WL_q(D(x,t))} \le C t^{Q/q} \int_{2c_0 t}^\infty r^{-Q/q-1} \|f\|_{L_1(D(x,r))}\, dr, \tag{4.3}$$

where C does not depend on f, $x \in \mathbb{G}$, and $t > 0$.

Proof. For a given ball $D = D(x,t)$, we split the function f as $f = f_1 + f_2$, where $f_1 = f\chi_{2c_0 D}$, $f_2 = f\chi_{\complement(2c_0 D)}$, and $2c_0 D = D(x, 2c_0 t)$; and then

$$I_\alpha f(x) = I_\alpha f_1(x) + I_\alpha f_2(x).$$

Let $1 < p < \infty$, $0 < \alpha < Q/p$, $1/q = 1/p - \alpha/Q$. Since $f_1 \in L_p(\mathbb{G})$, by the boundedness of the operator I_α from $L_p(\mathbb{G})$ to $L_q(\mathbb{G})$ (see Theorem A) it follows that

$$\|I_\alpha f_1\|_{L_q(D)} \le C\|f_1\|_{L_p(\mathbb{G})} = C\|f\|_{L_p(2c_0 D)} \le C t^{Q/q} \int_{2c_0 t}^\infty r^{-Q/q-1} \|f\|_{L_p(D(x,r))}\, dr, \tag{4.4}$$

where the constant C is independent of f.

Observe that the conditions $z \in D$ and $y \in \complement(2c_0D)$ imply

$$\frac{1}{2c_0}\,\rho(y^{-1}z) \le \rho(x^{-1}y) \le \frac{3c_0}{2}\,\rho(y^{-1}z).$$

Then, for all $z \in D$, we obtain

$$|I_\alpha f_2(z)| \le \left(\frac{3c_0}{2}\right)^{Q-\alpha} \int_{\complement(2c_0D)} \rho(x^{-1}y)^{\alpha-Q}|f(y)|\,dy.$$

By Fubini's theorem, we have

$$\int_{\complement(2c_0D)} \rho(x^{-1}y)^{\alpha-Q}|f(y)|\,dy \approx \int_{\complement(2c_0D)} |f(y)|\,dy \int_{\rho(x^{-1}y)}^{\infty} \tau^{\alpha-Q-1}\,d\tau$$
$$\approx \int_{2c_0t}^{\infty} \int_{2c_0t\le\rho(x^{-1}y)<\tau} |f(y)|\,dy\tau^{\alpha-Q-1}\,d\tau \lesssim \int_{2c_0t}^{\infty} \int_{D(x,\tau)} |f(y)|\,dy\tau^{\alpha-Q-1}\,d\tau.$$

Applying Hölder's inequality, we obtain

$$\int_{\complement(2c_0D)} \rho(x^{-1}y)^{\alpha-Q}|f(y)|\,dy \lesssim \int_{2c_0t}^{\infty} \|f\|_{L_p(D(x,\tau))}\tau^{-Q/q-1}\,d\tau$$

and for all $z \in D$

$$|I_\alpha f_2(z)| \lesssim \int_{2c_0t}^{\infty} \|f\|_{L_p(D(x,\tau))}\tau^{-Q/q-1}\,d\tau. \tag{4.5}$$

Moreover, for all $p \in [1,\infty)$ the inequality

$$\|I_\alpha f_2\|_{L_q(D)} \lesssim t^{Q/q} \int_{2c_0t}^{\infty} r^{-Q/q-1}\|f\|_{L_p(D(x,r))}\,dr \tag{4.6}$$

is valid. Thus, from (4.4) and (4.6), we deduce the inequality (4.2).

Finally, in the case $p = 1$ by the weak $(1,q)$-boundedness of I_α (see Theorem A) it follows that

$$\|I_\alpha f_1\|_{WL_q(D)} \le C\|f_1\|_{L_1(\mathbb{G})} \le Ct^{Q/q} \int_{2c_0t}^{\infty} r^{-Q/q-1}\|f\|_{L_1(D(x,r))}\,dr, \tag{4.7}$$

where C does not depend on x, t. Then, from (4.6) and (4.7) we get the inequality (4.3). $\square$

Theorem 4.3. *Let $1 \le p < \infty$, $0 < \alpha < Q/p$, $1/q = 1/p - \alpha/Q$, $\varphi_1 \in \Omega_p$, $\varphi_2 \in \Omega_q$, and let the pair (φ_1,φ_2) satisfy the condition*

$$\int_t^{\infty} \frac{\operatorname{ess\,inf}_{r<s<\infty} \varphi_1(x,s)s^{Q/p}}{r^{Q/q}}\frac{dr}{r} \le C\varphi_2(x,t), \tag{4.8}$$

where C does not depend on x and r. Then, for $p > 1$ the operator I_α is bounded from $M_{p,\varphi_1}(\mathbb{G})$ to $M_{q,\varphi_2}(\mathbb{G})$ and for $p = 1$ the operator I_α is bounded from $M_{1,\varphi_1}(\mathbb{G})$ to $WM_{q,\varphi_2}(\mathbb{G})$.

Proof. By Theorems 4.1 and 4.2 with $v_2(r) = \varphi_2(x,r)^{-1}$, $v_1(r) = \varphi_1(x,r)^{-1}r^{-Q/p}$, and $w(r) = r^{-Q/q}$ we have for $p > 1$

$$\|I_\alpha f\|_{M_{q,\varphi_2}} \lesssim \sup_{x\in\mathbb{G},\,t>0} \varphi_2(x,t)^{-1} \int_t^{\infty} r^{-Q/q-1}\|f\|_{L_p(D(x,r))}\,dr$$
$$\lesssim \sup_{x\in\mathbb{G},\,t>0} \varphi_1(x,t)^{-1}t^{-Q/p}\|f\|_{L_p(D(x,t))} = \|f\|_{M_{p,\varphi_1}},$$

and for $p = 1$

$$\|I_\alpha f\|_{WM_{q,\varphi_2}} \lesssim \sup_{x\in\mathbb{G},\,t>0} \varphi_2(x,t)^{-1} \int_t^{\infty} r^{-Q/q-1}\|f\|_{L_1(D(x,r))}\,dr$$
$$\lesssim \sup_{x\in\mathbb{G},\,t>0} \varphi_1(x,t)^{-1}t^{-Q}\|f\|_{L_1(D(x,t))} = \|f\|_{M_{1,\varphi_1}}. \quad \square$$

Remark 4.1. Note that, in the case $\mathbb{G} = \mathbb{H}_n$ Theorems 4.2 and 4.3 were proved in [13, Lemma 5.1 and Theorem 5.2]; see also [14], [15].

For proving our main results, we need the following estimate.

Lemma 4.1. *If $D_0 := D(x_0, r_0)$, then*

$$r_0^\alpha \le c_2(2c_0)^{Q-\alpha} I_\alpha \chi_{D_0}(x) \qquad \text{for every} \quad x \in D_0.$$

Proof. If $x, y \in D_0$, then

$$\rho(x^{-1}y) \le c_0(\rho(x^{-1}x_0) + \rho(x_0^{-1}y)) < 2c_0 r_0.$$

Since $0 < \alpha < Q$, we get $r_0^{\alpha-Q} \le (2c_0)^{Q-\alpha}\rho(x^{-1}y)^{\alpha-Q}$. Therefore,

$$I_\alpha \chi_{D_0}(x) = \int_{\mathbb{G}} \chi_{D_0}(y)\rho(x^{-1}y)^{\alpha-Q}\,dy = \int_{D_0} \rho(x^{-1}y)^{\alpha-Q}\,dy \ge c_2(2c_0)^{Q-\alpha} r_0^\alpha. \quad \square$$

The following theorem is one of our main results.

Theorem 4.4. *Let $0 < \alpha < Q$, $p, q \in [1, \infty)$, $\varphi_1 \in \Omega_p$, and $\varphi_2 \in \Omega_q$.*

(1) *If $1 \le p < Q/\alpha$ and $1/q = 1/p - \alpha/Q$, then condition* (4.8) *is sufficient for the boundedness of I_α from $M_{p,\varphi_1}(\mathbb{G})$ to $WM_{q,\varphi_2}(\mathbb{G})$. Moreover, if $1 < p < Q/\alpha$, condition* (4.8) *is sufficient for the boundedness of I_α from $M_{p,\varphi_1}(\mathbb{G})$ to $M_{q,\varphi_2}(\mathbb{G})$.*

(2) *If the function $\varphi_1 \in \mathcal{G}_p$, then the condition*

$$t^\alpha \varphi_1(t) \le C\varphi_2(t) \qquad \text{for all} \quad t > 0, \tag{4.9}$$

where $C > 0$, does not depend on t, is necessary for the boundedness of I_α from $M_{p,\varphi_1}(\mathbb{G})$ to $WM_{q,\varphi_2}(\mathbb{G})$ and $M_{p,\varphi_1}(\mathbb{G})$ to $M_{q,\varphi_2}(\mathbb{G})$.

(3) *Let $1 \le p < Q/\alpha$ and $1/q = 1/p - \alpha/Q$. If $\varphi_1 \in \mathcal{G}_p$ satisfies the regularity condition*

$$\int_t^\infty r^{\alpha-1}\varphi_1(r)\,dr \le Ct^\alpha\varphi_1(t), \tag{4.10}$$

for all $t > 0$, where $C > 0$ does not depend on t, then condition (4.9) *is necessary and sufficient for the boundedness of I_α from $M_{p,\varphi_1}(\mathbb{G})$ to $WM_{q,\varphi_2}(\mathbb{G})$. Moreover, if $1 < p < Q/\alpha$, then condition* (4.9) *is necessary and sufficient for the boundedness of I_α from $M_{p,\varphi_1}(\mathbb{G})$ to $M_{q,\varphi_2}(\mathbb{G})$.*

Proof. The first part of the theorem was proved in Theorem 4.3.

We shall now prove the second part. Let $D_0 = D(x_0, t_0)$ and $x \in D_0$. By Lemma 4.1 we have $t_0^\alpha \le CI_\alpha\chi_{D_0}(x)$. Therefore, by Lemma 3.2 and Lemma 4.1

$$t_0^\alpha \lesssim |D_0|^{-1/p}\|I_\alpha\chi_{D_0}\|_{L_q(D_0)} \lesssim \varphi_2(t_0)\|I_\alpha\chi_{D_0}\|_{M_{q,\varphi_2}} \lesssim \varphi_2(t_0)\|\chi_{D_0}\|_{M_{p,\varphi_1}} \lesssim \frac{\varphi_2(t_0)}{\varphi_1(t_0)}$$

or

$$t_0^\alpha \lesssim \frac{\varphi_2(t_0)}{\varphi_1(t_0)} \quad \text{for all} \quad t_0 > 0 \qquad \Longleftrightarrow \qquad t_0^\alpha\varphi_1(t_0) \lesssim \varphi_2(t_0) \quad \text{for all} \quad t_0 > 0.$$

Since this is true for every $t_0 > 0$, we are done.

The third statement of the theorem follows from first and second parts of the theorem. $\square$

Remark 4.2. If we take $\varphi_1(t) = t^{(\lambda-Q)/p}$ and $\varphi_2(t) = t^{(\mu-Q)/q}$ in Theorem 4.4, then conditions (4.10) and (4.9) are equivalent to $0 < \lambda < Q - \alpha p$ and $\lambda/p = \mu/q$, respectively. Therefore, we obtain the following Spanne result for Morrey spaces on Carnot groups.

Corollary 4.1. *Let $0 < \alpha < Q$, $1 \le p < Q/\alpha$, $0 < \lambda < Q - \alpha p$, and $1/q = 1/p - \alpha/Q$. Then the operator I_α is bounded from $L_{p,\lambda}(\mathbb{G})$ to $WL_{q,\mu}(\mathbb{G})$ if and only if $\lambda/p = \mu/q$. Moreover, if $1 < p < Q/\alpha$, then the operator I_α is bounded from $L_{p,\lambda}(\mathbb{G})$ to $L_{q,\mu}(\mathbb{G})$ if and only if $\lambda/p = \mu/q$.*

4.2. Adams-Type Results

The following pointwise estimate plays a key role in the proof of our main results.

Theorem 4.5. *Let $1 \le p < \infty$, $0 < \alpha < Q$, and $f \in L_p^{\mathrm{loc}}(\mathbb{G})$. Then*

$$|I_\alpha f(x)| \le C t^\alpha M f(x) + C \int_t^\infty r^{\alpha - Q/p - 1} \|f\|_{L_p(D(x,r))}\, dr, \tag{4.11}$$

where C does not depend on f, x, and t.

Proof. Write $f = f_1 + f_2$, where $f_1 = f\chi_{2c_0D}$, $f_2 = f\chi_{\complement(2c_0D)}$, and $D = D(x,t)$. Then

$$I_\alpha f(x) = I_\alpha f_1(x) + I_\alpha f_2(x).$$

For $I_\alpha f_1(x)$, following Hedberg's trick (see for instance [3, p. 354]), for all $z \in \mathbb{G}$ we obtain $|I_\alpha f_1(z)| \le C_1 t^\alpha M f(z)$. For $I_\alpha f_2(z)$ with $z \in D$, from (4.5) we have

$$|I_\alpha f_2(z)| \le \int_{\complement(2c_0D)} \rho(x^{-1}y)^{\alpha - Q} |f(y)|\, dy \le C \int_{2c_0 t}^\infty r^{\alpha - Q/p - 1} \|f\|_{L_p(D(x,r))}\, dr, \tag{4.12}$$

which proves (4.11). □

The following is a result of Adams type for the fractional integral on Carnot groups (see [16]).

Theorem 4.6 (Adams-type result)**.** *Let $1 \le p < q < \infty$, $0 < \alpha < Q/p$, and let $\varphi \in \Omega_p$ satisfy the following conditions*:

$$\sup_{r<t<\infty} t^{-Q} \operatorname*{ess\,inf}_{t<s<\infty} \varphi(x,s) s^Q \le C\varphi(x,r), \tag{4.13}$$

$$\int_r^\infty t^{\alpha-1} \varphi(x,t)^{1/p}\, dt \le C r^{-\alpha p/(q-p)}, \tag{4.14}$$

where C does not depend on $x \in \mathbb{G}$ and $r > 0$. Then, for $p > 1$, the operator I_α is bounded from $M_{p,\varphi^{1/p}}(\mathbb{G})$ to $M_{q,\varphi^{1/q}}(\mathbb{G})$ and the operator I_α is bounded from $M_{1,\varphi}(\mathbb{G})$ to $WM_{q,\varphi^{1/q}}(\mathbb{G})$ for $p = 1$.

Proof. Let $1 \le p < \infty$ and $f \in M_{p,\varphi}(\mathbb{G})$. By Theorem 4.5, the inequality (4.11) is valid. Then, from condition (4.14) and inequality (4.11), we obtain

$$\begin{aligned} |I_\alpha f(x)| &\lesssim t^\alpha M f(x) + \int_t^\infty r^{\alpha - Q/p - 1} \|f\|_{L_p(D(x,r))}\, dr \\ &\le t^\alpha M f(x) + \|f\|_{M_{p,\varphi^{1/p}}} \int_t^\infty r^{\alpha-1} \varphi(x,r)^{1/p}\, dr \\ &\le t^\alpha M f(x) + t^{-\alpha p/(q-p)} \|f\|_{M_{p,\varphi^{1/p}}}. \end{aligned} \tag{4.15}$$

Hence, choosing

$$t = \left(\frac{M_{p,\varphi^{1/p}}}{M f(x)} \right)$$

for every $x \in \mathbb{G}$, we can write

$$|I_\alpha f(x)| \lesssim (M f(x))^{p/q} \|f\|_{M_{p,\varphi^{1/p}}}^{1-p/q}.$$

Hence the statement of the theorem follows in view of the boundedness of the maximal operator M in $M_{p,\varphi}(\mathbb{G})$ provided by Theorem 3.1, by virtue of condition (4.13):

$$\begin{aligned} \|I_\alpha f\|_{\mathcal{M}_{q,\varphi^{1/q}}} &\lesssim \|f\|_{M_{p,\varphi^{1/p}}}^{1-p/q} \sup_{x\in\mathbb{G},\, t>0} \varphi(x,t)^{-p/q} t^{-Q/q} \|Mf\|_{L_p(D(x,t))}^{p/q} \\ &\lesssim \|f\|_{M_{p,\varphi^{1/p}}}^{1-p/q} \|Mf\|_{M_{p,\varphi^{1/p}}}^{p/q} \lesssim \|f\|_{M_{p,\varphi^{1/p}}}, \end{aligned}$$

if $1 < p < q < \infty$ and

$$\begin{aligned}\|I_\alpha f\|_{WM_{q,\varphi^{1/q}}} &\lesssim \|f\|_{\mathcal{M}_{1,\varphi}}^{1-1/q} \sup_{x\in\mathbb{G},\, t>0} \varphi(x,t)^{-1/q} t^{-Q/q} \|Mf\|_{WL_1(D(x,t))}^{1/q} \\ &\lesssim \|f\|_{M_{1,\varphi}}^{1-1/q} \|Mf\|_{M_{1,\varphi}}^{1/q} \lesssim \|f\|_{\mathcal{M}_{1,\varphi}},\end{aligned}$$

if $p = 1 < q < \infty$. □

Remark 4.3. Note that, in the case $\mathbb{G} = \mathbb{H}_n$ Theorem 4.6 was proved in [13, Theorem 5.3].

The following theorem is one of our main results.

Theorem 4.7. *Let $0 < \alpha < Q$, $1 \le p < q < \infty$, and $\varphi \in \Omega_p$.*

(1) *If $\varphi(x,t)$ satisfies condition* (4.13)*, then condition* (4.14) *is sufficient for the boundedness of I_α from $M_{p,\varphi^{1/p}}(\mathbb{G})$ to $WM_{q,\varphi^{1/q}}(\mathbb{G})$. Moreover, if $1 < p < q < \infty$, then condition* (4.14) *is sufficient for the boundedness of I_α from $M_{p,\varphi^{1/p}}(\mathbb{G})$ to $M_{q,\varphi^{1/q}}(\mathbb{G})$.*

(2) *If $\varphi \in \mathcal{G}_p$, then the condition*

$$r^\alpha \varphi(r)^{1/p} \le C r^{-\alpha p/(q-p)} \tag{4.16}$$

for all $r > 0$, where $C > 0$ does not depend on r, is necessary for the boundedness of I_α from $M_{p,\varphi^{1/p}}(\mathbb{G})$ to $WM_{q,\varphi^{1/q}}(\mathbb{G})$ and from $M_{p,\varphi^{1/p}}(\mathbb{G})$ to $M_{q,\varphi^{1/q}}(\mathbb{G})$.

(3) *If $\varphi \in \mathcal{G}_p$ satisfies the regularity condition*

$$\int_r^\infty t^{\alpha-1} \varphi(t)^{1/p}\, dt \le C r^\alpha \varphi(r)^{1/p} \tag{4.17}$$

for all $r > 0$, where $C > 0$ does not depend r, then condition (4.16) *is necessary and sufficient for the boundedness of I_α from $M_{p,\varphi^{1/p}}(\mathbb{G})$ to $WM_{q,\varphi^{1/q}}(\mathbb{G})$. Moreover, if $1 < p < q < \infty$, then condition* (4.16) *is necessary and sufficient for the boundedness of I_α from $M_{p,\varphi^{1/p}}(\mathbb{G})$ to $M_{q,\varphi^{1/q}}(\mathbb{G})$.*

Proof. The first part of the theorem is a corollary of Theorem 4.6.

We shall now prove the second part. Let $D_0 = D(x_0, t_0)$ and $x \in D_0$. By Lemma 4.1, we have $t_0^\alpha \le C I_\alpha \chi_{D_0}(x)$. Therefore, by Lemma 3.2 and Lemma 4.1, we have

$$\begin{aligned} t_0^\alpha &\lesssim |D_0|^{-1/q} \|I_\alpha \chi_{D_0}\|_{L_q(D_0)} \\ &\lesssim \varphi(t_0)^{1/q} \|I_\alpha \chi_{D_0}\|_{M_{q,\varphi^{1/q}}} \lesssim \varphi(t_0)^{1/q} \|\chi_{D_0}\|_{M_{p,\varphi^{1/p}}} \lesssim \varphi(t_0)^{1/q-1/p}, \end{aligned}$$

or

$$t_0^\alpha \varphi(t_0)^{1/p-1/q} \lesssim 1 \quad \text{for all} \quad t_0 > 0 \qquad \Longleftrightarrow \qquad t_0^\alpha \varphi(t_0)^{1/p} \lesssim t_0^{-\alpha p/(q-p)}.$$

Since this is true for every $x \in \mathbb{G}$ and $t_0 > 0$, we are done.

The third statement of the theorem follows from first and second parts of the theorem. □

The following is a result of Adams type for the fractional integral on Carnot groups.

Theorem 4.8 (Adams-type result)**.** *Let $0 < \alpha < Q$, $1 \le p < q < \infty$, and $\varphi \in \Omega_p$ satisfy condition* (4.13) *and*

$$t^\alpha \varphi(x,t) + \int_t^\infty r^{\alpha-1} \varphi(x,r)\, dr \le C \varphi(x,t)^{p/q}, \tag{4.18}$$

where C does not depend on $x \in \mathbb{G}$ and $r > 0$. Then, for $p > 1$, the operator I_α is bounded from $M_{p,\varphi^{1/p}}(\mathbb{G})$ to $M_{q,\varphi^{1/q}}(\mathbb{G})$ and the operator I_α is bounded from $M_{1,\varphi}(\mathbb{G})$ to $WM_{q,\varphi^{1/q}}(\mathbb{G})$ for $p = 1$.

Proof. Let $1 \le p < \infty$ and $f \in M_{p,\varphi}(\mathbb{G})$. By Theorem 4.5, the inequality (4.11) is valid. Then, from condition (4.14) and inequality (4.11), we obtain

$$\begin{aligned} |I_\alpha f(x)| &\lesssim t^\alpha Mf(x) + \int_t^\infty r^{\alpha-Q/p-1}\|f\|_{L_p(D(x,r))}\,dr \\ &\le t^\alpha Mf(x) + \|f\|_{M_{p,\varphi}} \int_t^\infty r^{\alpha-1}\varphi(x,r)\,dr. \end{aligned} \tag{4.19}$$

Thus, by (4.18) and (4.19), we have

$$\begin{aligned} |I_\alpha f(x)| &\lesssim \min\{\varphi(x,t)^{p/q-1}Mf(x), \varphi(x,t)^\beta \|f\|_{M_{p,\varphi}}\} \\ &\lesssim \sup_{s>0}\min\{s^{p/q-1}Mf(x), s^{p/q}\|f\|_{M_{p,\varphi}}\} = (Mf(x))^{p/q}\|f\|_{M_{p,\varphi}}^{1-p/q}; \end{aligned} \tag{4.20}$$

where we have used the fact that the supremum is achieved when the minimum parts are balanced. From Theorem 3.1 and (4.20), we obtain

$$\|I_\alpha f\|_{\mathcal{M}_{q,\varphi^{1/q}}} \lesssim \|f\|_{M_{p,\varphi^{1/p}}}^{1-p/q}\|Mf\|_{M_{p,\varphi^{1/p}}}^{p/q} \lesssim \|f\|_{M_{p,\varphi^{1/p}}},$$

if $1 < p < q < \infty$ and

$$\|I_\alpha f\|_{WM_{q,\varphi^{1/q}}} \lesssim \|f\|_{M_{1,\varphi}}^{1-1/q}\|Mf\|_{M_{1,\varphi}}^{1/q} \lesssim \|f\|_{\mathcal{M}_{1,\varphi}},$$

if $p = 1 < q < \infty$. ☐

The following theorem is one of our main results.

Theorem 4.9. *Let $0 < \alpha < Q$, $1 \le p < q < \infty$, and $\varphi \in \Omega_p$.*

(1) If $\varphi(x,t)$ satisfy condition (4.13), then condition (4.18) is sufficient for the boundedness of I_α from $M_{p,\varphi^{1/p}}(\mathbb{G})$ to $M_{q,\varphi^{1/q}}(\mathbb{G})$. Moreover, if $1 < p < q < \infty$, then condition (4.18) is sufficient for the boundedness of I_α from $M_{p,\varphi^{1/p}}(\mathbb{G})$ to $M_{q,\varphi^{1/q}}(\mathbb{G})$.

(2) If $\varphi \in \mathcal{G}_p$, then the condition

$$r^\alpha \varphi(r)^{1/p} \le C\varphi(r)^{1/q} \tag{4.21}$$

for all $r > 0$, where $C > 0$ does not depend on r, is necessary for the boundedness of the operator I_α from $M_{p,\varphi^{1/p}}(\mathbb{G})$ to $WM_{q,\varphi^{1/q}}(\mathbb{G})$ and from $M_{p,\varphi^{1/p}}(\mathbb{G})$ to $M_{q,\varphi^{1/q}}(\mathbb{G})$.

(3) If $\varphi \in \mathcal{G}_p$ satisfies the regularity condition (4.17), then condition (4.21) is necessary and sufficient for the boundedness of I_α from $M_{p,\varphi^{1/p}}(\mathbb{G})$ to $WM_{q,\varphi^{1/q}}(\mathbb{G})$. Moreover, if $1 < p < q < \infty$, then condition (4.21) is necessary and sufficient for the boundedness of I_α from $M_{p,\varphi^{1/p}}(\mathbb{G})$ to $M_{q,\varphi^{1/q}}(\mathbb{G})$.

Proof. The first part of the theorem is a corollary of Theorem 4.8.

We shall now prove the second part. Let $D_0 = D(x_0, t_0)$ and $x \in D_0$. By Lemma 4.1 we have $t_0^\alpha \le CI_\alpha\chi_{D_0}(x)$. Therefore, by Lemma 3.2 and Lemma 4.1 we have

$$\begin{aligned} t_0^\alpha &\lesssim |D_0|^{-1/q}\|I_\alpha\chi_{D_0}\|_{L_q(D_0)} \\ &\lesssim \varphi(t_0)^{1/q}\|I_\alpha\chi_{D_0}\|_{M_{q,\varphi^{1/q}}} \lesssim \varphi(t_0)^{1/q}\|\chi_{D_0}\|_{M_{p,\varphi^{1/p}}} \lesssim \varphi(t_0)^{1/q-1/p}, \end{aligned}$$

or

$$t_0^\alpha\varphi(t_0)^{1/p-1/q} \lesssim 1 \quad \text{for all} \quad t_0 > 0 \qquad \Longleftrightarrow \qquad t_0^\alpha\varphi(t_0)^{1/p} \lesssim \varphi(t_0)^{1/q}.$$

Since this is true for every $x \in \mathbb{G}$ and $t_0 > 0$, we are done.

The third statement of the theorem follows from first and second parts of the theorem. ☐

Remark 4.4. If we take $\varphi(t) = t^{\lambda-Q}$ in Theorem 4.9, then condition (4.17) will be equivalent to $0 < \lambda < Q - \alpha p$, and condition (4.16) will be equivalent to $1/p - 1/q = \alpha/(Q - \lambda)$. Therefore, we obtain the following Adams result for Morrey spaces in Carnot groups.

Corollary 4.2. *Let $0 < \alpha < Q$, $1 \le p < q < \infty$, and $0 < \lambda < Q - \alpha p$. Then the operator I_α is bounded from $L_{p,\lambda}(\mathbb{G})$ to $WL_{q,\lambda}(\mathbb{G})$ if and only if $1/p - 1/q = \alpha/(Q - \lambda)$. Moreover, if $1 < p < q < \infty$, then the operator I_α is bounded from $L_{p,\lambda}(\mathbb{G})$ to $L_{q,\lambda}(\mathbb{G})$ if and only if $1/p - 1/q = \alpha/(Q - \lambda)$.*

Remark 4.5. Note that, in the case $\mathbb{G} = \mathbb{R}^n$, the sufficient part of Corollary 4.6 was proved in [17].

5. SOME APPLICATIONS

It is known that (see [18, p. 247]) if ρ is a homogeneous norm on $\mathbb{G}$, then there exists a positive constant β_ρ such that $\Gamma(x) = \beta_\rho \rho(x)^{2-Q}$ is the fundamental solution of $\mathcal{L}$.

From Theorems 4.4 and 4.7 one easily obtains an inequality extending the classical Sobolev embedding theorem to the homogeneous Carnot groups.

Theorem 5.1 (Sobolev–Stein embedding on generalized Morrey space). *Let $\mathcal{L}$ be the sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$ of homogeneous dimension Q. Let also $1 < p < Q$ and $1/q = 1/p - 1/Q$, $\varphi_1 \in \Omega_p$ and $\varphi_2 \in \Omega_q$ satisfy condition (4.8). Then there exists a positive constant C such that*

$$\|u\|_{M_{q,\varphi_2}} \le C\|\nabla_{\mathcal{L}} u\|_{M_{p,\varphi_1}} \qquad \textit{for every} \quad u \in L_p(\mathbb{G}) \cap M_{p,\varphi_1}(\mathbb{G}).$$

Proof. Let $u \in C_0^\infty(\mathbb{G})$. By using the integral representation formula for the fundamental solution (see [18, p. 237]), we have

$$u(x) = \int_{\mathbb{G}} \Gamma(x^{-1}y)\mathcal{L}u(y)\,dy. \tag{5.1}$$

Keeping in mind that $\mathcal{L} = \Sigma_{i=1}^n X_i^2$ and $X_i^* = -X_i$, by integrating the right-hand side of (5.1) by parts, we obtain

$$u(x) = \int_{\mathbb{G}} (\nabla_{\mathcal{L}}\Gamma)(x^{-1}y)\nabla_{\mathcal{L}} u(y)\,dy. \tag{5.2}$$

On the other hand, out of the origin, we have

$$\nabla_{\mathcal{L}}\Gamma(x) = \beta_\rho \nabla_{\mathcal{L}}(\rho(x)^{2-Q}) = (2-Q)\beta_\rho \rho(x)^{1-Q}\nabla_{\mathcal{L}}\rho(x),$$

so that, since $\nabla_{\mathcal{L}}\rho$ is smooth in $\mathbb{G} \setminus \{0\}$ and δ_λ-homogeneous of degree zero,

$$\nabla_{\mathcal{L}}\Gamma(x) \le C\rho(x)^{1-Q}$$

for a suitable constant $C > 0$ depending only on $\mathcal{L}$. Using this inequality in (5.2), we can write

$$|u(x)| \le C\int_{\mathbb{G}} |\nabla_{\mathcal{L}} u(y)|\rho(x)^{1-Q}\,dy = CI_1(|\nabla_{\mathcal{L}} u|)(x). \tag{5.3}$$

Then, by Theorem 4.4,

$$\|u\|_{M_{q,\varphi_2}} \le C\|I_1(|\nabla_{\mathcal{L}} u|)\|_{M_{q,\varphi_2}} \le C\|\nabla_{\mathcal{L}} u\|_{M_{p,\varphi_1}}, \qquad \text{where} \quad 1 < p < Q, \qquad \frac{1}{q} = \frac{1}{p} - \frac{1}{Q}. \quad \square$$

The following theorem can be proved in a similar way.

Theorem 5.2 (Sobolev-Stein embedding on generalized Morrey space). *Let $\mathcal{L}$ be the sub-Laplacian on the homogeneous Carnot group $\mathbb{G}$ of homogeneous dimension Q. Let also $1 < p < q < \infty$, $\varphi \in \Omega_p$ satisfy conditions (4.13) and (4.14). Then there exists a positive constant C such that*

$$\|u\|_{M_{q,\varphi^{1/q}}} \le C\|\nabla_{\mathcal{L}} u\|_{M_{p,\varphi^{1/p}}} \qquad \textit{for every} \quad u \in L_p(\mathbb{G}) \cap M_{p,\varphi^{1/p}}(\mathbb{G}).$$

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