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Riesz potential and its commutators on Orlicz spaces

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Abstract

In the present paper, we shall give necessary and sufficient conditions for the strong and weak boundedness of the Riesz potential operator I_α on Orlicz spaces. Cianchi (J. Lond. Math. Soc. 60(1):247-286, 2011) found necessary and sufficient conditions on general Young functions Φ and Ψ ensuring that this operator is of weak or strong type from L^Φ into L^Ψ. Our characterizations for the boundedness of the above-mentioned operator are different from the ones in (Cianchi in J. Lond. Math. Soc. 60(1):247-286, 2011). As an application of these results, we consider the boundedness of the commutators of Riesz potential operator $[b, I_\alpha]$ on Orlicz spaces when b belongs to the BMO and Lipschitz spaces, respectively.

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1 Introduction

Norm inequalities for several classical operators of harmonic analysis have been widely studied in the context of Orlicz spaces. It is well known that many of such operators fail to have continuity properties when they act between certain Lebesgue spaces and, in some situations, the Orlicz spaces appear as adequate substitutes.

The Hardy-Littlewood maximal operator M and the Riesz potential operator I_α ($0 < \alpha < n$) are defined by

$$Mf(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)| \, dy, \qquad I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy.$$

Here and everywhere in the sequel $B(x,r)$ is the ball in $\mathbb{R}^n$ of radius r centered at x and $|B(x,r)| = v_n r^n$ is its Lebesgue measure, where v_n is the volume of the unit ball in $\mathbb{R}^n$.

The commutators generated by a suitable function b and the operator I_α are formally defined by

$$[b, I_\alpha]f = I_\alpha(bf) - bI_\alpha(f),$$

respectively.

Given a measurable function b the operators M_b and $|b, I_\alpha|$ are defined by

$$M_b(f)(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |b(x) - b(y)| |f(y)| \, dy$$

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and

$$|b, I_\alpha| f(x) = \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|}{|x - y|^{n-\alpha}} f(y)\, dy.$$

In [1], Cianchi found necessary and sufficient conditions on general Young functions Φ and Ψ ensuring that the operator I_α is of weak or strong type from L^Φ into L^Ψ. Another boundedness statement with only sufficient conditions for the operator I_α on Orlicz spaces was given by Nakai [2]. Note that in [2] a more general case of generalized fractional integrals was studied. Commutators of classical operators of harmonic analysis play an important role in various topics of analysis and PDE; see for instance [3, 4], and the references therein. In [5], Fu *et al.* gave the sufficient conditions for the boundedness of the commutator $[b, I_\alpha]$ on Orlicz spaces.

The main purpose of this paper is to give characterizations for the strong and weak boundedness of the Riesz potential on Orlicz spaces. Our characterizations for the boundedness of the operator I_α are different from the ones in [1]. As an application of these results, we consider the boundedness of the commutators of Riesz potential operator on Orlicz spaces when b belongs to the BMO and Lipschitz spaces, respectively.

We use the notation $A \lesssim B$, which means that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries; on Young functions and Orlicz spaces

We recall the definition of Young functions.

Definition 2.1 A function $\Phi : [0, \infty) \to [0, \infty]$ is called a Young function if Φ is convex and left-continuous, $\lim_{r \to +0} \Phi(r) = \Phi(0) = 0$ and $\lim_{r \to \infty} \Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing.

The set of Young functions such that

$$0 < \Phi(r) < \infty \quad \text{for } 0 < r < \infty$$

is denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0, \infty)$ and bijective from $[0, \infty)$ to itself.

For a Young function Φ and $0 \leq s \leq \infty$, let

$$\Phi^{-1}(s) = \inf\{r \geq 0 : \Phi(r) > s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ.

It is well known that

$$r \leq \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \leq 2r \quad \text{for } r \geq 0, \tag{2.1}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0, \infty)\}, & r \in [0, \infty), \\ \infty, & r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \leq C\Phi(r), \quad r > 0,$$

for some $C > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2C}\Phi(Cr), \quad r \geq 0,$$

for some $C > 1$.

Definition 2.2 (Orlicz space) For a Young function Φ, the set

$$L^{\Phi}(\mathbb{R}^n) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)\, dx < \infty \text{ for some } k > 0 \right\}$$

is called Orlicz space. If $\Phi(r) = r^p, 1 \leq p < \infty$, then $L^{\Phi}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\Phi(r) = 0, (0 \leq r \leq 1)$ and $\Phi(r) = \infty, (r > 1)$, then $L^{\Phi}(\mathbb{R}^n) = L^{\infty}(\mathbb{R}^n)$. The space $L^{\Phi}_{\mathrm{loc}}(\mathbb{R}^n)$ is defined as the set of all functions f such that $f\chi_B \in L^{\Phi}(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$.

$L^{\Phi}(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}} = \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\}.$$

For a measurable set $\Omega \subset \mathbb{R}^n$, a measurable function f and $t > 0$, let $m(\Omega, f, t) = |\{x \in \Omega : |f(x)| > t\}|$. In the case $\Omega = \mathbb{R}^n$, we for brevity denote it by $m(f, t)$.

Definition 2.3 The weak Orlicz space

$$WL^{\Phi}(\mathbb{R}^n) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{WL^{\Phi}} < \infty \right\}$$

is defined by the norm

$$\|f\|_{WL^{\Phi}} = \inf\left\{ \lambda > 0 : \sup_{t>0} \Phi(t) m\left(\frac{f}{\lambda}, t\right) \leq 1 \right\}.$$

We note that $\|f\|_{WL^{\Phi}} \leq \|f\|_{L^{\Phi}}$,

$$\sup_{t>0} \Phi(t) m(\Omega, f, t) = \sup_{t>0} t m(\Omega, f, \Phi^{-1}(t)) = \sup_{t>0} t m(\Omega, \Phi(|f|), t)$$

and

$$\int_{\Omega} \Phi\left(\frac{|f(x)|}{\|f\|_{L^{\Phi}(\Omega)}}\right) dx \leq 1, \qquad \sup_{t>0} \Phi(t) m\left(\Omega, \frac{f}{\|f\|_{WL^{\Phi}(\Omega)}}, t\right) \leq 1, \tag{2.2}$$

where $\|f\|_{L^{\Phi}(\Omega)} = \|f\chi_{\Omega}\|_{L^{\Phi}}$ and $\|f\|_{WL^{\Phi}(\Omega)} = \|f\chi_{\Omega}\|_{WL^{\Phi}}$.

The following analogue of the Hölder inequality is well known (see, for example, [6]).

Theorem 2.4 *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and functions f and g measurable on Ω. For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid:*

$$\int_\Omega |f(x)g(x)|\, dx \leq 2\|f\|_{L^\Phi(\Omega)}\|g\|_{L^{\widetilde{\Phi}}(\Omega)}.$$

By elementary calculations we have the following property.

Lemma 2.5 *Let Φ be a Young function and B be a set in $\mathbb{R}^n$ with finite Lebesgue measure. Then*

$$\|\chi_B\|_{L^\Phi} = \|\chi_B\|_{WL^\Phi} = \frac{1}{\Phi^{-1}(|B|^{-1})}.$$

By Theorem 2.4, Lemma 2.5 and (2.1) we get the following estimate.

Lemma 2.6 *For a Young function Φ and $B = B(x, r)$, the following inequality is valid:*

$$\int_B |f(y)|\, dy \leq 2|B|\Phi^{-1}(|B|^{-1})\|f\|_{L^\Phi(B)}.$$

In the next section, where we prove our main estimates, we use the following theorem.

Theorem 2.7 ([7]) *Let Φ be a Young function.*

(i) *The operator M is bounded from $L^\Phi(\mathbb{R}^n)$ to $WL^\Phi(\mathbb{R}^n)$, and the inequality*

$$\|Mf\|_{WL^\Phi} \leq C_0 \|f\|_{L^\Phi} \tag{2.3}$$

holds with constant C_0 independent of f.

(ii) *The operator M is bounded on $L^\Phi(\mathbb{R}^n)$, and the inequality*

$$\|Mf\|_{L^\Phi} \leq C_0 \|f\|_{L^\Phi} \tag{2.4}$$

holds with constant C_0 independent of f if and only if $\Phi \in \nabla_2$.

3 Riesz potential in Orlicz spaces

In this section we find necessary and sufficient conditions for the strong/weak boundedness of the Riesz potential operator on Orlicz spaces.

We recall that, for functions Φ and Ψ from $[0, \infty)$ into $[0, \infty]$, the function Ψ is said to dominate Φ globally if there exists a positive constant c such that $\Phi(s) \leq \Psi(cs)$ for all $s \geq 0$.

In the theorem below we also use the notation

$$\widetilde{\Psi}_P(s) = \int_0^s r^{P'-1}\big(\mathcal{B}_P^{-1}(r^{P'})\big)^{P'}\, dr, \tag{3.1}$$

where $1 < P \leq \infty$ and $\widetilde{\Psi}_P(s)$ is the Young conjugate function to $\Psi_P(s)$, and

$$\Phi_P(s) = \int_0^s r^{P'-1}\big(\mathcal{A}_P^{-1}(r^{P'})\big)^{P'}\, dr, \tag{3.2}$$

where $\mathcal{B}_P^{-1}(s)$ and $\mathcal{A}_P^{-1}(s)$ are inverses to

$$\mathcal{B}_P(s) = \int_0^s \frac{\Psi(t)}{t^{1+P'}}\, dt \quad \text{and} \quad \mathcal{A}_P(s) = \int_0^s \frac{\widetilde{\Phi}(t)}{t^{1+P'}}\, dt,$$

respectively. These functions $\Psi_P(s)$ and $\Phi_P(s)$ are used below with $P = \frac{n}{\alpha}$.

In [1], Cianchi found the necessary and sufficient conditions for the boundedness of I_α on Orlicz spaces.

Theorem 3.1 ([1]) *Let $0 < \alpha < n$. Let Φ and Ψ Young functions and let $\Phi_{n/\alpha}$ and $\Psi_{n/\alpha}$ be the Young functions defined as in* (3.2) *and* (3.1), *respectively. Then*

(i) *I_α is bounded from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$ if and only if*

$$\int_0^1 \widetilde{\Phi}(t)/t^{1+n/(n-\alpha)}\, dt < \infty \quad \text{and } \Phi_{n/\alpha} \text{ dominates } \Psi \text{ globally}. \tag{3.3}$$

(ii) *I_α is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$ if and only if*

$$\int_0^1 \widetilde{\Phi}(t)/t^{1+n/(n-\alpha)}\, dt < \infty, \qquad \int_0^1 \Psi(t)/t^{1+n/(n-\alpha)}\, dt < \infty, \tag{3.4}$$

Φ dominates $\Psi_{n/\alpha}$ globally and $\Phi_{n/\alpha}$ dominates Ψ globally.

For proving our main results, we need the following estimate.

Lemma 3.2 *If $B_0 := B(x_0, r_0)$, then $r_0^\alpha \leq CI_\alpha \chi_{B_0}(x)$ for every $x \in B_0$.*

Proof If $x, y \in B_0$, then $|x - y| \leq |x - x_0| + |y - x_0| < 2r_0$. Since $0 < \alpha < n$, we get $r_0^{\alpha-n} \leq C|x-y|^{\alpha-n}$. Therefore

$$I_\alpha \chi_{B_0}(x) = \int_{\mathbb{R}^n} \chi_{B_0}(y)|x-y|^{\alpha-n}\, dy = \int_{B_0} |x-y|^{\alpha-n}\, dy \geq Cr_0^{\alpha-n}|B_0| = Cr_0^\alpha. \qquad \Box$$

The following theorem gives necessary and sufficient conditions for the boundedness of the operator I_α from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$ and from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Theorem 3.3 *Let $0 < \alpha < n$ and $\Phi, \Psi \in \mathcal{Y}$.*

(1) *The condition*

$$r^\alpha \Phi^{-1}(r^{-n}) + \int_r^\infty \Phi^{-1}(t^{-n}) t^\alpha \frac{dt}{t} \leq C\Psi^{-1}(r^{-n}) \tag{3.5}$$

for all $r > 0$, where $C > 0$ does not depend on r, is sufficient for the boundedness of I_α from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, the condition (3.5) *is sufficient for the boundedness of I_α from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

(2) *The condition*

$$r^\alpha \Phi^{-1}(r^{-n}) \leq C\Psi^{-1}(r^{-n}) \tag{3.6}$$

for all $r > 0$, where $C > 0$ does not depend on r, is necessary for the boundedness of I_α from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$ and from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

(3) *If the regularity condition*

$$\int_r^\infty \Phi^{-1}(t^{-n}) t^\alpha \frac{dt}{t} \le C r^\alpha \Phi^{-1}(r^{-n}) \tag{3.7}$$

holds for all $r > 0$, where $C > 0$ does not depend on r, then the condition (3.6) *is necessary and sufficient for the boundedness of I_α from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, the condition* (3.6) *is necessary and sufficient for the boundedness of I_α from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

Proof (1) For an arbitrary ball $B = B(x, r)$ we represent f as

$$f = f_1 + f_2, \qquad f_1(y) = f(y)\chi_B(y), \qquad f_2(y) = f(y)\chi_{\complement B}(y),$$

and have

$$I_\alpha f(x) = I_\alpha f_1(x) + I_\alpha f_2(x).$$

For $I_\alpha f_1(x)$, following Hedberg's trick, see [8], we obtain $|I_\alpha f_1(x)| \lesssim r^\alpha M f(x)$. For $I_\alpha f_2(x)$ by Lemma 2.6 we have

$$\begin{aligned}
\int_{\complement B} \frac{|f(y)|}{|x - y|^{n-\alpha}} \, dy &\approx \int_{\complement B} |f(y)| \int_{|x-y|}^\infty \frac{dt}{t^{n+1-\alpha}} \, dy \\
&\approx \int_r^\infty \int_{r \le |x-y| < t} |f(y)| \, dy \frac{dt}{t^{n+1-\alpha}} \\
&\lesssim \int_r^\infty \Phi^{-1}(|B(x,t)|^{-1}) t^{\alpha-1} \|f\|_{L^\Phi(B(x,t))} \, dt.
\end{aligned}$$

Consequently we have

$$|I_\alpha f(x)| \lesssim r^\alpha M f(x) + \|f\|_{L^\Phi} \int_r^\infty t^\alpha \Phi^{-1}(t^{-n}) \frac{dt}{t}.$$

Thus, by (3.5) we obtain

$$|I_\alpha f(x)| \lesssim M f(x) \frac{\Psi^{-1}(r^{-n})}{\Phi^{-1}(r^{-n})} + \|f\|_{L^\Phi} \Psi^{-1}(r^{-n}).$$

Choose $r > 0$ so that $\Phi^{-1}(r^{-n}) = \frac{M f(x)}{C_0 \|f\|_{L^\Phi}}$. Then

$$\frac{\Psi^{-1}(r^{-n})}{\Phi^{-1}(r^{-n})} = \frac{(\Psi^{-1} \circ \Phi)(\frac{M f(x)}{C_0 \|f\|_{L^\Phi}})}{\frac{M f(x)}{C_0 \|f\|_{L^\Phi}}}.$$

Therefore, we get

$$|I_\alpha f(x)| \le C_1 \|f\|_{L^\Phi} (\Psi^{-1} \circ \Phi)\left(\frac{M f(x)}{C_0 \|f\|_{L^\Phi}} \right). \tag{3.8}$$

Let C_0 be as in (2.3). Then by Theorem 2.7,

$$\sup_{r>0} \Psi(r) m\left(\frac{|I_\alpha f(x)|}{C_1 \|f\|_{L^\Phi}}, r\right) = \sup_{r>0} r m\left(\Psi\left(\frac{|I_\alpha f(x)|}{C_1 \|f\|_{L^\Phi}}\right), r\right)$$
$$\leq \sup_{r>0} r m\left(\Phi\left(\frac{Mf(x)}{C_0 \|f\|_{L^\Phi}}\right), r\right)$$
$$\leq \sup_{r>0} \Phi(r) m\left(\frac{Mf(x)}{\|Mf\|_{WL^\Phi}}, r\right) \leq 1,$$

i.e.

$$\|I_\alpha f\|_{WL^\Psi} \lesssim \|f\|_{L^\Phi}.$$

Let C_0 be as in (2.4). Since $\Phi \in \nabla_2$, by Theorem 2.7, we have

$$\int_{\mathbb{R}^n} \Psi\left(\frac{|I_\alpha f(x)|}{C_1 \|f\|_{L^\Phi}}\right) dx \leq \int_{\mathbb{R}^n} \Phi\left(\frac{Mf(x)}{C_0 \|f\|_{L^\Phi}}\right) dx \leq \int_{\mathbb{R}^n} \Phi\left(\frac{Mf(x)}{\|Mf\|_{L^\Phi}}\right) dx \leq 1,$$

i.e.

$$\|I_\alpha f\|_{L^\Psi} \lesssim \|f\|_{L^\Phi}.$$

(2) We shall now prove the second part. Let $B_0 = B(x_0, r_0)$ and $x \in B_0$. By Lemma 3.2, we have $r_0^\alpha \leq C I_\alpha \chi_{B_0}(x)$. Therefore, by Lemma 2.5, we have

$$r_0^\alpha \lesssim \Psi^{-1}\left(|B_0|^{-1}\right) \|I_\alpha \chi_{B_0}\|_{WL^\Psi(B_0)} \lesssim \Psi^{-1}\left(|B_0|^{-1}\right) \|I_\alpha \chi_{B_0}\|_{WL^\Psi}$$
$$\lesssim \Psi^{-1}\left(|B_0|^{-1}\right) \|\chi_{B_0}\|_{L^\Phi} \lesssim \frac{\Psi^{-1}(r_0^{-n})}{\Phi^{-1}(r_0^{-n})}$$

and

$$r_0^\alpha \lesssim \Psi^{-1}\left(|B_0|^{-1}\right) \|I_\alpha \chi_{B_0}\|_{L^\Psi(B_0)} \lesssim \Psi^{-1}\left(|B_0|^{-1}\right) \|I_\alpha \chi_{B_0}\|_{L^\Psi}$$
$$\lesssim \Psi^{-1}\left(|B_0|^{-1}\right) \|\chi_{B_0}\|_{L^\Phi} \lesssim \frac{\Psi^{-1}(r_0^{-n})}{\Phi^{-1}(r_0^{-n})}.$$

Since this is true for every $r_0 > 0$, we are done.

(3) The third statement of the theorem follows from the first and second parts of the theorem. □

From Theorems 3.1 and 3.3 we have the following corollary.

Corollary 3.4 *Let $0 < \alpha < n$, $\Phi, \Psi \in \mathcal{Y}$ and the regularity condition (3.7) holds, then*:
(1) *Condition (3.3) holds if and only if condition (3.6) holds.*
(2) *Moreover, if $\Phi \in \nabla_2$, then condition (3.4) holds if and only if (3.6) holds.*

The following result is due to Nakai [2].

Theorem 3.5 ([2]) *Let $0 < \alpha < n$ and $\Phi, \Psi \in \mathcal{Y}$. Assume that the conditions* (3.6) *and* (3.7) *hold. Then the operator I_α is bounded from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, then I_α is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

Remark 3.6 Note that in Theorem 3.5 Nakai found the sufficient conditions which ensures the boundedness of the operator I_α from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$, including weak version. Theorem 3.3 improves Theorem 3.5 by adding the necessity. Theorems 3.1 and 3.3 are different characterizations for the boundedness of the operator I_α from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$, including a weak version.

4 Maximal commutator in Orlicz spaces

In this section we investigate the boundedness of the maximal commutator M_b in Orlicz spaces.

We recall the definition of the space of $\mathrm{BMO}(\mathbb{R}^n)$.

Definition 4.1 Suppose that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, let

$$\|f\|_* = \sup_{x \in \mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| \, dy,$$

where

$$f_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) \, dy.$$

Define

$$\mathrm{BMO}(\mathbb{R}^n) = \{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_* < \infty \}.$$

Modulo constants, the space $\mathrm{BMO}(\mathbb{R}^n)$ is a Banach space with respect to the norm $\|\cdot\|_*$. Before proving our theorems, we need the following lemmas and theorem.

Lemma 4.2 ([9]) *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that*

$$|b_{B(x,r)} - b_{B(x,t)}| \le C\|b\|_* \ln \frac{t}{r} \quad \text{for } 0 < 2r < t, \tag{4.1}$$

where C is independent of b, x, r, and t.

Lemma 4.3 ([10]) *Let $f \in \mathrm{BMO}(\mathbb{R}^n)$ and Φ be a Young function with $\Phi \in \Delta_2$, then*

$$\|f\|_* \approx \sup_{x \in \mathbb{R}^n, r>0} \Phi^{-1}\left(|B(x,r)|^{-1}\right) \|f(\cdot) - f_{B(x,r)}\|_{L^\Phi(B(x,r))}. \tag{4.2}$$

Theorem 4.4 ([11]) *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $\Phi \in \nabla_2 \cap \mathcal{Y}$.*
  Then the operator M_b is bounded on $L^\Phi(\mathbb{R}^n)$, and the inequality

$$\|M_b f\|_{L^\Phi} \le C_0 \|b\|_* \|f\|_{L^\Phi} \tag{4.3}$$

holds with constant C_0 independent of f.

The following theorem is valid.

Theorem 4.5 *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and Φ be a Young function. Then the condition $\Phi \in \nabla_2$ is necessary for the boundedness of M_b on $L^\Phi(\mathbb{R}^n)$.*

Proof Assume that (4.3) holds. For the particular symbol $b(x) = \log |x| \in \mathrm{BMO}(\mathbb{R}^n)$ and $f(x) = \chi_{B_r}(x)$, (4.3) becomes

$$\|M_b \chi_{B_r}\|_{L^\Phi} \leq C_1 \|\chi_{B_r}\|_{L^\Phi}, \tag{4.4}$$

where $r = (a_1 u v)^{-1/n}$, $B_r = B(0, r)$, $a_r = |B_r|$, $u > 0$ and $v > 1$. By Lemma 2.5 and (2.1), we have

$$\|\chi_{B_r}\|_{L^\Phi} = \frac{1}{\Phi^{-1}(1/|B_r|)} = \frac{1}{\Phi^{-1}(1/(r^n |B_1|))} = \frac{1}{\Phi^{-1}(uv)} \leq \frac{1}{uv} \widetilde{\Phi}^{-1}(uv).$$

On the other hand, if $x \notin B_r$ then $B_r \subset B(x, 2|x|)$ since for $y \in B_r$ we have

$$|x - y| \leq |x| + |y| \leq |x| + r \leq 2|x|.$$

Also for each $y \in B_r$, we have

$$b(x) - b(y) \geq \log\left(\frac{|x|}{r}\right).$$

Therefore

$$M_b \chi_{B_r}(x) \geq \frac{1}{|B(x, 2|x|)|} \int_{B(x, 2|x|) \cap B_r} |b(x) - b(y)| \, dy \geq \left(\frac{r}{2|x|}\right)^n \log\left(\frac{|x|}{r}\right).$$

Following the ideas of [12], for $g = \widetilde{\Phi}^{-1}(u) \chi_{B_s}$ with $s = (a_1 u)^{-1/n}$ we obtain

$$\int_{\mathbb{R}^n} \widetilde{\Phi}(|g(x)|) \, dx \leq u |B_s| = u s^n |B_1| = 1.$$

Since the Luxemburg-Nakano norm is equivalent to the Orlicz norm

$$\|f\|_\Phi^* := \sup\left\{ \int_{\mathbb{R}^n} |f(x) g(x)| \, dx : \|g\|_{L^{\widetilde{\Phi}}} \leq 1 \right\}$$

(more precisely, $\|f\|_{L^\Phi} \leq \|f\|_\Phi^* \leq 2\|f\|_{L^\Phi}$), it follows that

$$\begin{aligned}
\|M_b \chi_{B_r}\|_{L^\Phi}^* &= \sup\left\{ \int_{\mathbb{R}^n} |M_b \chi_{B_r}(x) g(x)| \, dx : \int_{\mathbb{R}^n} \widetilde{\Phi}(|g(x)|) \, dx \leq 1 \right\} \\
&\geq \widetilde{\Phi}^{-1}(u) \int_{B_s} M_b \chi_{B_r}(x) \, dx \geq \widetilde{\Phi}^{-1}(u) \int_{B_s \setminus B_r} \left(\frac{r}{2|x|}\right)^n \log\left(\frac{|x|}{r}\right) dx \\
&= \frac{\widetilde{\Phi}^{-1}(u)}{2^n a_1 u v} \int_{r < |x| < s} \frac{1}{|x|^n} \log\left(\frac{|x|}{r}\right) dx \\
&= \frac{\widetilde{\Phi}^{-1}(u)}{2^{n+1} a_1 u v} n a_1 \left(\log \frac{s}{r}\right)^2 = \frac{\widetilde{\Phi}^{-1}(u)}{2^{n+1} n u v} (\log v)^2.
\end{aligned}$$

Hence (4.4) implies that

$$\frac{\widetilde{\Phi}^{-1}(u)}{2^{n+1}nuv}(\log v)^2 \leq 2C_1 \frac{1}{uv}\widetilde{\Phi}^{-1}(uv)$$

for $u > 0$ and $v > 1$. Thus, taking $v = \exp(\sqrt{nC_1} \cdot 2^{\frac{n+3}{2}})$ we obtain $2\widetilde{\Phi}^{-1}(u) \leq \widetilde{\Phi}^{-1}(u\exp(\sqrt{nC_1} \cdot 2^{\frac{n+3}{2}}))$ for $u > 0$ or $\widetilde{\Phi}(2t) \leq \exp(\sqrt{nC_1} \cdot 2^{\frac{n+3}{2}})\widetilde{\Phi}(t)$ for every $t > 0$, and so $\widetilde{\Phi}$ satisfies the Δ_2 condition. □

By Theorems 4.4 and 4.5 we have the following result.

Corollary 4.6 *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $\Phi \in \mathcal{Y}$. Then the condition $\Phi \in \nabla_2$ is necessary and sufficient for the boundedness of M_b on $L^\Phi(\mathbb{R}^n)$.*

Theorem 4.7 *$b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and Φ be a Young function. The condition $b \in \mathrm{BMO}(\mathbb{R}^n)$ is necessary for the boundedness of M_b on $L^\Phi(\mathbb{R}^n)$.*

Proof Suppose that M_b is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Phi(\mathbb{R}^n)$. Choose any ball $B = B(x, r)$ in $\mathbb{R}^n$, by (2.1)

$$\begin{aligned}
\frac{1}{|B|}\int_B |b(y) - b_B|\, dy &\leq \frac{1}{|B|}\int_B \frac{1}{|B|}\int_B |b(y) - b(z)|\chi_B(z)\, dz\, dy \\
&\leq \frac{1}{|B|}\int_B M_b(\chi_B)(y)\, dy \\
&\leq \frac{2}{|B|}\big\|M_b(\chi_B)\big\|_{L^\Phi(B)}\|1\|_{L^{\widetilde{\Phi}}(B)} \\
&\leq \frac{2}{|B|}\|\chi_B\|_{L^\Phi}\|\chi_B\|_{L^{\widetilde{\Phi}}} \leq C.
\end{aligned}$$

Thus $b \in \mathrm{BMO}(\mathbb{R}^n)$. □

By Theorems 4.4 and 4.7 we have the following result.

Corollary 4.8 *Let Φ be a Young function with $\Phi \in \nabla_2$. Then the condition $b \in \mathrm{BMO}(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of M_b on $L^\Phi(\mathbb{R}^n)$.*

5 Commutators of Riesz potential in Orlicz spaces

In this section we find necessary and sufficient conditions for the boundedness of the commutators of Riesz potential on Orlicz spaces with the help of the previous section.

In [5], Fu *et al.* found the sufficient conditions for the boundedness of the commutator $[b, I_\alpha]$ on Orlicz spaces as follows.

Theorem 5.1 ([5]) *Let $0 < \alpha < n$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Let Φ be a Young function and Ψ defined, via its inverse, by setting, for all $t \in (0, \infty)$, $\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n}$. If $\Phi, \Psi \in \Delta_2 \cap \nabla_2$, then $[b, I_\alpha]$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

The following lemma is the analogue of the Hedberg trick for $[b, I_\alpha]$.

Lemma 5.2 *If $0 < \alpha < n$ and $f, b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then for all $x \in \mathbb{R}^n$ and $r > 0$ we get*

$$\int_{B(x,r)} \frac{|f(y)|}{|x-y|^{n-\alpha}} \big| b(x) - b(y) \big| \, dy \lesssim r^\alpha M_b f(x).$$

Proof We have

$$\int_{B(x,r)} \frac{|f(y)|}{|x-y|^{n-\alpha}} \big| b(x) - b(y) \big| \, dy$$

$$= \sum_{j=0}^\infty \int_{2^{-j-1}r \le |x-y| < 2^{-j}r} \frac{|f(y)|}{|x-y|^{n-\alpha}} \big| b(x) - b(y) \big| \, dy$$

$$\lesssim \sum_{j=0}^\infty \left(2^{-j} r \right)^\alpha \left(2^{-j} r \right)^{-n} \int_{|x-y| < 2^{-j}r} |f(y)| \big| b(x) - b(y) \big| \, dy \lesssim r^\alpha M_b f(x). \qquad \square$$

Lemma 5.3 *If $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $B_0 := B(x_0, r_0)$, then*

$$r_0^\alpha \big| b(x) - b_{B_0} \big| \le C |b, I_\alpha| \chi_{B_0}(x)$$

for every $x \in B_0$, where $b_{B_0} = \frac{1}{|B_0|} \int_{B_0} b(y) \, dy$.

Proof If $x, y \in B_0$, then $|x - y| \le |x - x_0| + |y - x_0| < 2r_0$. Since $0 < \alpha < n$, we get $r_0^{\alpha-n} \le C|x-y|^{\alpha-n}$. Therefore

$$|b, I_\alpha| \chi_{B_0}(x) = \int_{B_0} \big| b(x) - b(y) \big| |x-y|^{\alpha-n} \, dy \ge C r_0^{\alpha-n} \int_{B_0} \big| b(x) - b(y) \big| \, dy$$

$$\ge C r_0^{\alpha-n} \left| \int_{B_0} \big(b(x) - b(y) \big) \, dy \right| = C r_0^\alpha \big| b(x) - b_{B_0} \big|. \qquad \square$$

The following theorem gives necessary and sufficient conditions for the boundedness of the operator $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Theorem 5.4 *Let $0 < \alpha < n$, $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $\Phi, \Psi \in \mathcal{Y}$.*
(1) *If $\Phi \in \nabla_2$ and $\Psi \in \Delta_2$, then the condition*

$$r^\alpha \Phi^{-1}\big(r^{-n}\big) + \int_r^\infty \left(1 + \ln \frac{t}{r} \right) \Phi^{-1}\big(t^{-n}\big) t^\alpha \frac{dt}{t} \le C \Psi^{-1}\big(r^{-n}\big) \tag{5.1}$$

for all $r > 0$, where $C > 0$ does not depend on r, is sufficient for the boundedness of $[b, I_\alpha]$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.
(2) *If $\Psi \in \Delta_2$, then the condition (3.6) is necessary for the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*
(3) *Let $\Phi \in \nabla_2$ and $\Psi \in \Delta_2$. If the condition*

$$\int_r^\infty \left(1 + \ln \frac{t}{r} \right) \Phi^{-1}\big(t^{-n}\big) t^\alpha \frac{dt}{t} \le C r^\alpha \Phi^{-1}\big(r^{-n}\big) \tag{5.2}$$

holds for all $r > 0$, where $C > 0$ does not depend on r, then the condition (3.6) is necessary and sufficient for the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Proof (1) For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f \chi_{2B}$ and $f_2 = f \chi_{\complement(2B)}$.

For $x \in B$ we have

$$
\big|[b, I_\alpha] f_2(x)\big| \lesssim \int_{\mathbb{R}^n} \frac{|b(y) - b(x)|}{|x - y|^{n-\alpha}} \big|f_2(y)\big| \, dy \approx \int_{\complement(2B)} \frac{|b(y) - b(x)|}{|x_0 - y|^{n-\alpha}} \big|f(y)\big| \, dy
$$

$$
\lesssim \int_{\complement(2B)} \frac{|b(y) - b_B|}{|x_0 - y|^{n-\alpha}} \big|f(y)\big| \, dy + \int_{\complement(2B)} \frac{|b(x) - b_B|}{|x_0 - y|^{n-\alpha}} \big|f(y)\big| \, dy = J_1 + J_2(x),
$$

since $x \in B$ and $y \in \complement(2B)$ implies $|x - y| \approx |x_0 - y|$.

Let us estimate J_1.

$$
J_1 = \int_{\complement(2B)} \frac{|b(y) - b_B|}{|x_0 - y|^{n-\alpha}} \big|f(y)\big| \, dy \approx \int_{\complement(2B)} \big|b(y) - b_B\big| \big|f(y)\big| \int_{|x_0-y|}^\infty \frac{dt}{t^{n+1-\alpha}} \, dy
$$

$$
\approx \int_{2r}^\infty \int_{2r \leq |x_0-y| \leq t} \big|b(y) - b_B\big| \big|f(y)\big| \, dy \frac{dt}{t^{n+1-\alpha}} \lesssim \int_{2r}^\infty \int_{B(x_0,t)} \big|b(y) - b_B\big| \big|f(y)\big| \, dy \frac{dt}{t^{n+1-\alpha}}.
$$

Applying Hölder's inequality, by (2.1), (4.1), (4.2) and Lemma 2.6 we get

$$
J_1 \lesssim \int_{2r}^\infty \int_{B(x_0,t)} \big|b(y) - b_{B(x_0,t)}\big| \big|f(y)\big| \, dy \frac{dt}{t^{n+1-\alpha}}
$$

$$
+ \int_{2r}^\infty \big|b_{B(x_0,r)} - b_{B(x_0,t)}\big| \int_{B(x_0,t)} \big|f(y)\big| \, dy \frac{dt}{t^{n+1-\alpha}}
$$

$$
\lesssim \int_{2r}^\infty \big\|b(\cdot) - b_{B(x_0,t)}\big\|_{L^{\widetilde\Phi}(B(x_0,t))} \|f\|_{L_\Phi(B(x_0,t))} \frac{dt}{t^{n+1-\alpha}}
$$

$$
+ \int_{2r}^\infty \big|b_{B(x_0,r)} - b_{B(x_0,t)}\big| \|f\|_{L_\Phi(B(x_0,t))} \Phi^{-1}\big(\big|B(x_0,t)\big|^{-1}\big) \frac{dt}{t^{1-\alpha}}
$$

$$
\lesssim \|b\|_* \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_\Phi(B(x_0,t))} \Phi^{-1}\big(\big|B(x_0,t)\big|^{-1}\big) \frac{dt}{t^{1-\alpha}}
$$

$$
\lesssim \|b\|_* \|f\|_{L_\Phi} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right) \Phi^{-1}(t^{-n}) t^\alpha \frac{dt}{t}.
$$

A geometric observation shows $2B \subset B(x, 3r)$ for all $x \in B$. Using Lemma 5.2, we get

$$
J_0(x) := \big|[b, I_\alpha] f_1(x)\big| \lesssim \int_{2B} \frac{|b(y) - b(x)|}{|x - y|^{n-\alpha}} \big|f(y)\big| \, dy
$$

$$
\lesssim \int_{B(x,3r)} \frac{|b(y) - b(x)|}{|x - y|^{n-\alpha}} \big|f(y)\big| \, dy \lesssim r^\alpha M_b f(x).
$$

Consequently, we have

$$
J_0(x) + J_1 \lesssim \|b\|_* r^\alpha M_b f(x) + \|b\|_* \|f\|_{L_\Phi} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right) \Phi^{-1}(t^{-n}) t^\alpha \frac{dt}{t}.
$$

Thus, by (5.1) we obtain

$$
J_0(x) + J_1 \lesssim \|b\|_* \left(M_b f(x) \frac{\Psi^{-1}(r^{-n})}{\Phi^{-1}(r^{-n})} + \Psi^{-1}(r^{-n}) \|f\|_{L_\Phi} \right).
$$

Choose $r > 0$ so that $\Phi^{-1}(r^{-n}) = \frac{M_b f(x)}{C_0 \|b\|_* \|f\|_{L^\Phi}}$. Then

$$\frac{\Psi^{-1}(r^{-n})}{\Phi^{-1}(r^{-n})} = \frac{(\Psi^{-1} \circ \Phi)\left(\frac{M_b f(x)}{C_0 \|b\|_* \|f\|_{L^\Phi}}\right)}{\frac{M_b f(x)}{C_0 \|b\|_* \|f\|_{L^\Phi}}}.$$

Therefore, we get

$$J_0(x) + J_1 \le C_1 \|b\|_* \|f\|_{L^\Phi} \left(\Psi^{-1} \circ \Phi\right)\left(\frac{M_b f(x)}{C_0 \|b\|_* \|f\|_{L^\Phi}}\right).$$

Let C_0 be as in (4.3). Consequently by Theorem 4.4 we have

$$\int_B \Psi\left(\frac{J_0(x) + J_1}{C_1 \|b\|_* \|f\|_{L^\Phi}}\right) dx \le \int_B \Phi\left(\frac{M_b f(x)}{C_0 \|b\|_* \|f\|_{L^\Phi}}\right) dx \le \int_{\mathbb{R}^n} \Phi\left(\frac{M_b f(x)}{\|M_b f\|_{L^\Phi}}\right) dx \le 1,$$

i.e.

$$\left\|J_0(\cdot) + J_1\right\|_{L^\Psi(B)} \lesssim \|b\|_* \|f\|_{L^\Phi}. \tag{5.3}$$

In order to estimate J_2, by (4.2), Lemma 2.6 and condition (5.1), we also get

$$\begin{aligned}
\|J_2\|_{L^\Psi(B)} &= \left\|\int_{\complement(2B)} \frac{|b(\cdot) - b_B|}{|x_0 - y|^{n-\alpha}} |f(y)| \, dy\right\|_{L^\Psi(B)} \\
&\approx \left\|b(\cdot) - b_B\right\|_{L^\Psi(B)} \int_{\complement(2B)} \frac{|f(y)|}{|x_0 - y|^{n-\alpha}} \, dy \\
&\lesssim \|b\|_* \frac{1}{\Psi^{-1}(|B|^{-1})} \int_{\complement(2B)} \frac{|f(y)|}{|x_0 - y|^{n-\alpha}} \, dy \\
&\approx \|b\|_* \frac{1}{\Psi^{-1}(|B|^{-1})} \int_{\complement(2B)} |f(y)| \int_{|x_0 - y|}^\infty \frac{dt}{t^{n+1-\alpha}} \, dy \\
&\approx \|b\|_* \frac{1}{\Psi^{-1}(|B|^{-1})} \int_{2r}^\infty \int_{2r \le |x_0 - y| < t} |f(y)| \, dy \frac{dt}{t^{n+1-\alpha}} \\
&\lesssim \|b\|_* \frac{1}{\Psi^{-1}(|B|^{-1})} \int_{2r}^\infty \int_{B(x_0, t)} |f(y)| \, dy \frac{dt}{t^{n+1-\alpha}} \\
&\lesssim \|b\|_* \frac{1}{\Psi^{-1}(|B|^{-1})} \int_{2r}^\infty \|f\|_{L^\Phi(B(x_0, t))} \Phi^{-1}\left(|B(x_0, t)|^{-1}\right) t^{\alpha - 1} \, dt \\
&\lesssim \|b\|_* \frac{1}{\Psi^{-1}(|B|^{-1})} \|f\|_{L^\Phi} \int_{2r}^\infty t^\alpha \Phi^{-1}\left(t^{-n}\right) \frac{dt}{t} \\
&\lesssim \|b\|_* \|f\|_{L^\Phi}.
\end{aligned}$$

Consequently, we have

$$\|J_2\|_{L^\Psi(B)} \lesssim \|b\|_* \|f\|_{L^\Phi}. \tag{5.4}$$

Combining (5.3) and (5.4), we get

$$\left\|[b, I_\alpha] f\right\|_{L^\Psi(B)} \lesssim \|b\|_* \|f\|_{L^\Phi}. \tag{5.5}$$

By taking the supremum over B in (5.5), we get

$$\big\|[b,I_\alpha]f\big\|_{L^\Psi} \lesssim \|b\|_* \|f\|_{L^\Phi},$$

since the constants in (5.5) do not depend on x_0 and r.

(2) We shall now prove the second part. Let $B_0 = B(x_0, r_0)$ and $x \in B_0$. By Lemma 5.3, we have $r_0^\alpha |b(x) - b_{B_0}| \le C|b, I_\alpha|\chi_{B_0}(x)$. Therefore, by Lemmas 4.3 and 2.5

$$r_0^\alpha \lesssim \frac{\big\||b,I_\alpha|\chi_{B_0}\big\|_{L^\Psi(B_0)}}{\|b(\cdot) - b_{B_0}\|_{L^\Psi(B_0)}} \lesssim \Psi^{-1}\big(|B_0|^{-1}\big)\big\||b,I_\alpha|\chi_{B_0}\big\|_{L^\Psi(B_0)}$$

$$\lesssim \Psi^{-1}\big(|B_0|^{-1}\big)\big\||b,I_\alpha|\chi_{B_0}\big\|_{L^\Psi} \lesssim \Psi^{-1}\big(|B_0|^{-1}\big)\|\chi_{B_0}\|_{L^\Phi} \lesssim \frac{\Psi^{-1}(r_0^{-n})}{\Phi^{-1}(r_0^{-n})}.$$

Since this is true for every $r_0 > 0$, we are done.

(3) The third statement of the theorem follows from the first and second parts of the theorem. $\qquad\square$

Remark 5.5 Theorems 5.1 and 5.4 give different sufficient conditions for the boundedness of the operator $[b, I_\alpha]$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$. But in Theorem 5.4 we also have necessary conditions for the boundedness of the operator $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

The following theorem is valid.

Theorem 5.6 *Let $0 < \alpha < n$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\Phi, \Psi \in \mathcal{Y}$.*

(1) *If $\Phi \in \nabla_2$, $\Psi \in \Delta_2$ and the condition (5.1) holds, then the condition $b \in \mathrm{BMO}(\mathbb{R}^n)$ is sufficient for the boundedness of $[b, I_\alpha]$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

(2) *If $\Psi^{-1}(t) \lesssim \Phi^{-1}(t)t^{-\alpha/n}$, then the condition $b \in \mathrm{BMO}(\mathbb{R}^n)$ is necessary for the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

(3) *If $\Phi \in \nabla_2$, $\Psi \in \Delta_2$, $\Psi^{-1}(t) \approx \Phi^{-1}(t)t^{-\alpha/n}$ and the condition (5.2) holds, then the condition $b \in \mathrm{BMO}(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

Proof (1) The first statement of the theorem follows from the first part of Theorem 5.4.

(2) We shall now prove the second part. Choose any ball $B = B(x, r)$ in $\mathbb{R}^n$, by Lemmas 2.5 and 2.6

$$\frac{1}{|B|}\int_B \big|b(y) - b_B\big|\,dy = \frac{1}{|B|}\int_B \bigg|\frac{1}{|B|}\int_B \big(b(y) - b(z)\big)\,dz\bigg|\,dy$$

$$\le \frac{1}{|B|^{1+\frac{\alpha}{n}}}\int_B \frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B \big|b(y) - b(z)\big|\chi_B(z)\,dz\,dy$$

$$\le \frac{C}{|B|^{1+\frac{\alpha}{n}}}\int_B \int_B \frac{|b(y) - b(z)|}{|y - z|^{n-\alpha}}\chi_B(z)\,dz\,dy$$

$$\le \frac{C}{|B|^{1+\frac{\alpha}{n}}}\int_B |b, I_\alpha|(\chi_B)(y)\,dy$$

$$\le \frac{C}{|B|^{\frac{\alpha}{n}}}\frac{\Psi^{-1}(|B|^{-1})}{\Phi^{-1}(|B|^{-1})} \le C.$$

Thus $b \in \mathrm{BMO}(\mathbb{R}^n)$.

(3) The third statement of the theorem follows from the first and second parts of the theorem. □

6 Characterization of Lipschitz spaces via commutators

In this section, as an application of Theorem 3.3 we consider the boundedness of $[b, I_\alpha]$ on Orlicz spaces when b belongs to the Lipschitz space, by which some new characterizations of the Lipschitz spaces are given. Such a characterization was given in [13] as an application of the boundedness of M_b on Lebesgue spaces.

Definition 6.1 Let $0 < \beta < 1$, we say a function b belongs to the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ if there exists a constant C such that, for all $x, y \in \mathbb{R}^n$,

$$|b(x) - b(y)| \leq C|x - y|^\beta.$$

The smallest such constant C is called the $\dot{\Lambda}_\beta(\mathbb{R}^n)$ norm of b and is denoted by $\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}$.

To prove the theorems, we need auxiliary results. The first one is the following characterization of Lipschitz space, which is due to DeVore and Sharply [14].

Lemma 6.2 *Let* $0 < \beta < 1$, *we have*

$$\|f\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \approx \sup_B \frac{1}{|B|^{1+\beta/n}} \int_B |f(x) - f_B|\, dx,$$

where $f_B = \frac{1}{|B|} \int_B f(y)\, dy$.

Lemma 6.3 *Let* $0 < \beta < 1$, $0 < \alpha < n$, $0 < \alpha + \beta < n$ *and* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, *then the following pointwise estimate holds*:

$$|b, I_\alpha|(|f|)(x) \lesssim \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} I_{\alpha+\beta}(|f|)(x).$$

Proof If $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, then

$$|b, I_\alpha|(|f|)(x) = \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|}{|x - y|^{n-\alpha}} |f(y)|\, dy$$

$$\lesssim \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} I_{\alpha+\beta}(|f|)(x). \qquad \square$$

The following theorem is valid.

Theorem 6.4 *Let* $0 < \beta < 1$, $0 < \alpha < n$, $0 < \alpha + \beta < n$, $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, $\Phi, \Psi \in \mathcal{Y}$.
(1) *If* $\Phi \in \nabla_2$ *and the conditions*

$$\int_t^\infty \Phi^{-1}(r^{-n}) r^{\alpha+\beta} \frac{dr}{r} \leq Ct^{\alpha+\beta} \Phi^{-1}(t^{-n}), \tag{6.1}$$

$$t^{-\frac{\alpha+\beta}{n}} \Phi^{-1}(t) \leq C\Psi^{-1}(t), \tag{6.2}$$

hold for all $t > 0$, *where* $C > 0$ *does not depend on* t, *then the condition* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *is sufficient for the boundedness of* $[b, I_\alpha]$ *from* $L^\Phi(\mathbb{R}^n)$ *to* $L^\Psi(\mathbb{R}^n)$.

(2) *If the condition*

$$\Psi^{-1}(t) \leq C\Phi^{-1}(t)t^{-\frac{\alpha+\beta}{n}}, \tag{6.3}$$

holds for all $t > 0$, where $C > 0$ does not depend on t, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary for the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

3. *If $\Phi \in \nabla_2$, condition (6.1) holds and $\Psi^{-1}(t) \approx \Phi^{-1}(t)t^{-\frac{\alpha+\beta}{n}}$, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.*

Proof (1) The first statement of the theorem follows from Theorem 3.3 and Lemma 6.3.

(2) We shall now prove the second part. Suppose that $\Psi^{-1}(t) \lesssim \Phi^{-1}(t)t^{-(\alpha+\beta)/n}$ and $|b, I_\alpha|$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$. Choose any ball B in $\mathbb{R}^n$, by Lemmas 2.5 and 2.6

$$
\begin{aligned}
\frac{1}{|B|^{1+\frac{\beta}{n}}} \int_B |b(y) - b_B|\, dy &= \frac{1}{|B|^{1+\frac{\alpha+\beta}{n}}} \int_B \left| \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B (b(y) - b(z))\, dz \right| dy \\
&\leq \frac{C}{|B|^{1+\frac{\alpha+\beta}{n}}} \int_B \int_B \frac{|b(y) - b(z)|}{|y-z|^{n-\alpha}} \chi_B(z)\, dz\, dy \\
&\leq \frac{C}{|B|^{1+\frac{\alpha+\beta}{n}}} \int_B |b, I_\alpha|(\chi_B)(y)\, dy \\
&\leq \frac{C\Psi^{-1}(|B|^{-1})}{|B|^{\frac{\alpha+\beta}{n}}} \big\| |b, I_\alpha|(\chi_B) \big\|_{L^\Psi(B)} \\
&\leq \frac{C}{|B|^{\frac{\alpha+\beta}{n}}} \frac{\Psi^{-1}(|B|^{-1})}{\Phi^{-1}(|B|^{-1})} \leq C.
\end{aligned}
$$

Thus by Lemma 6.2 we get $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

(3) The third statement of the theorem follows from the first and second parts of the theorem. □

The following theorem is valid.

Theorem 6.5 *Let $0 < \beta < 1$, $0 < \alpha < n$, $0 < \alpha + \beta < n$, $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, $\Phi, \Psi \in \mathcal{Y}$.*

(1) *If the conditions (6.1) and (6.2) are satisfied, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is sufficient for the boundedness of $[b, I_\alpha]$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$.*

(2) *If the condition (6.3) holds and $\frac{t^{1+\varepsilon}}{\Psi(t)}$ is almost decreasing for some $\varepsilon > 0$, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary for the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$.*

(3) *If $\Psi^{-1}(t) \approx \Phi^{-1}(t)t^{-\frac{\alpha+\beta}{n}}$, condition (6.1) holds and $\frac{t^{1+\varepsilon}}{\Psi(t)}$ is almost decreasing for some $\varepsilon > 0$, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$.*

Proof (1) The first statement of the theorem follows from Theorem 3.3 and Lemma 6.3.

(2) For any fixed ball B_0 such that $x \in B_0$ by Lemma 5.3 we have $|B_0|^{\alpha/n}|b(x) - b_{B_0}| \lesssim |b, I_\alpha|\chi_{B_0}(x)$. Thus, together with the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$ and

Lemma 2.5,

$$\left|\left\{x \in B_0 : |B_0|^{\alpha/n}\left|b(x) - b_{B_0}\right| > \lambda\right\}\right| \le \left|\left\{x \in B_0 : |b, I_\alpha|\chi_{B_0}(x) > \lambda\right\}\right|$$

$$\le \frac{1}{\Psi\left(\frac{\lambda}{C\|\chi_{B_0}\|_{L^\Phi}}\right)} = \frac{1}{\Psi\left(\frac{\lambda \Phi^{-1}(|B_0|^{-1})}{C}\right)}.$$

Let $t > 0$ be a constant to be determined later, then

$$\int_{B_0}\left|b(x) - b_{B_0}\right| dx = |B_0|^{-\alpha/n}\int_0^\infty\left|\left\{x \in B_0 : \left|b(x) - b_{B_0}\right| > |B_0|^{-\alpha/n}\lambda\right\}\right| d\lambda$$

$$= |B_0|^{-\alpha/n}\int_0^t\left|\left\{x \in B_0 : \left|b(x) - b_{B_0}\right| > |B_0|^{-\alpha/n}\lambda\right\}\right| d\lambda$$

$$+ |B_0|^{-\alpha/n}\int_t^\infty\left|\left\{x \in B_0 : \left|b(x) - b_{B_0}\right| > |B_0|^{-\alpha/n}\lambda\right\}\right| d\lambda$$

$$\le t|B_0|^{1-\alpha/n} + |B_0|^{-\alpha/n}\int_t^\infty \frac{1}{\Psi\left(\frac{\lambda \Phi^{-1}(|B_0|^{-1})}{C}\right)} d\lambda$$

$$\lesssim t|B_0|^{1-\alpha/n} + \frac{|B_0|^{-\alpha/n}t}{\Psi\left(\frac{t\Phi^{-1}(|B_0|^{-1})}{C}\right)},$$

where we use $\frac{t^{1+\varepsilon}}{\Psi(t)}$ being almost decreasing in the last step.

Set $t = C|B_0|^{\frac{\alpha+\beta}{n}}$ in the above estimate, we have

$$\int_{B_0}\left|b(x) - b_{B_0}\right| dx \lesssim |B_0|^{1+\beta/n}.$$

Thus by Lemma 6.2 we get $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ since B_0 is an arbitrary ball in $\mathbb{R}^n$.

(3) The third statement of the theorem follows from the first and second parts of the theorem.   □

7 Conclusions

We have obtained necessary and sufficient conditions for the boundedness of the Riesz potential and its commutators on Orlicz spaces. We have also compared our results with the existing results. Lastly, we conclude this paper by remarking that some new characterizations of the Lipschitz spaces have been given as an application of the above-mentioned results.

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References
1. Cianchi, A: Strong and weak type inequalities for some classical operators in Orlicz spaces. J. Lond. Math. Soc. **60**(1), 247-286 (1999)
2. Nakai, E: On generalized fractional integrals. Taiwan. J. Math. **5**, 587-602 (2001)
3. Polidoro, S, Ragusa, MA: Hölder regularity for solutions of ultraparabolic equations in divergence form. Potential Anal. **14**(4), 341-350 (2001)
4. Ragusa, MA: Cauchy-Dirichlet problem associated to divergence form parabolic equations. Commun. Contemp. Math. **6**(3), 377-393 (2004)
5. Fu, X, Yang, D, Yuan, W: Generalized fractional integrals and their commutators over non-homogeneous metric measure spaces. Taiwan. J. Math. **18**(2), 509-557 (2014)
6. Rao, MM, Ren, ZD: Theory of Orlicz Spaces. Dekker, New York (1991)
7. Kokilashvili, V, Krbec, MM: Weighted Inequalities in Lorentz and Orlicz Spaces. World Scientific, Singapore (1991)
8. Hedberg, LI: On certain convolution inequalities. Proc. Am. Math. Soc. **36**, 505-510 (1972)
9. Janson, S: Mean oscillation and commutators of singular integral operators. Ark. Mat. **16**, 263-270 (1978)
10. Ho, K-P: Characterization of BMO in terms of rearrangement-invariant Banach function spaces. Expo. Math. **27**, 363-372 (2009)
11. Agcayazi, M, Gogatishvili, A, Koca, K, Mustafayev, R: A note on maximal commutators and commutators of maximal functions. J. Math. Soc. Jpn. **67**(2), 581-593 (2015)
12. Maligranda, L, Matsuoka, K: Maximal function in Beurling-Orlicz and central Morrey-Orlicz spaces. Colloq. Math. **138**(2), 165-181 (2015)
13. Zhang, P: Characterization of Lipschitz spaces via commutators of the Hardy-Littlewood maximal function. C. R. Acad. Sci. Paris, Ser. I **355**, 336-344 (2017)
14. DeVore, RA, Sharpley, RC: Maximal functions measuring smoothness. Mem. Am. Math. Soc. **47**(293), 1-115 (1984)