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A STUDY ON THE k-GENERALIZATIONS OF SOME KNOWN FUNCTIONS AND FRACTIONAL OPERATORS

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Abstract. In this paper, we first draw attention to the relationships between the original definitions and their k-generalizations of some known functions and fractional operators. Using these relationships, we not only easily reacquired the results which can be found in the existing literature for the k-generalizations, but also show how to achieve new results with the help of known properties of the original functions and operators. We conclude our paper by observing that, since the definitions of k-generalizations are closely related to the original definitions (that is, the $k = 1$ case), most of the formulas and results for the $k = 1$ case can be translated rather trivially and simply by appropriate parameter and notational changes to hold true for the corresponding k-case.

1. Introduction

In 2007, Diaz and Pariguan [5] introduced the k-Gamma function as

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \quad k > 0,\ x \in \mathbb{C}\backslash k\mathbb{Z}^-$$

where $(x)_{n,k}$ is the Pochhammer k-symbol which also defined in the same paper as

$$(x)_{n,k} = x(x+k)(x+2k)\cdots\big(x+(n-1)k\big), \quad x \in \mathbb{C},\ k \in \mathbb{R},\ n \in \mathbb{Z}^+.$$

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Then they studied their properties and obtained the following equalities:

$$\begin{aligned}
\Gamma_k(x) &= \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}}\, dt, \quad k>0,\ \Re(x)>0,\\
\Gamma_k(x+k) &= x\Gamma_k(x),\\
\Gamma_k(x) &= k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right), \qquad (1.1)\\
(x)_{n,k} &= \frac{\Gamma_k(x+nk)}{\Gamma_k(x)}, \quad k>0,\ n\in\mathbb{Z},\\
(x)_{n,k} &= k^n\left(\frac{x}{k}\right)_n. \qquad (1.2)
\end{aligned}$$

Remark. *It is clear from the above relationships including* (1.2) *that most of the formulas and results for the $k=1$ case can be translated rather trivially and simply by appropriate parameter and notational changes to hold true for the corresponding k-case.*

Again in the same paper, they defined k-Beta function in terms of k-Gamma function as

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)},$$

and they obtained the following relations:

$$\begin{aligned}
B_k(x,y) &= \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt, \quad k>0,\ \Re(x)>0,\ \Re(y)>0,\\
B_k(x,y) &= \frac{1}{k}B\left(\frac{x}{k},\frac{y}{k}\right). \qquad (1.3)
\end{aligned}$$

Then, they introduced a hypergeometric function of the form

$$F(a,k,b,s)(x) = \sum_{n=0}^{\infty} \frac{(a_1)_{n,k_1}(a_2)_{n,k_2}\cdots(a_p)_{n,k_p}}{(b_1)_{n,s_1}(b_2)_{n,s_2}\cdots(b_q)_{n,s_q}}\frac{x^n}{n!},$$

where $a=(a_1,\ldots,a_p)\in\mathbb{C}^p$, $b=(b_1,\ldots,b_q)\in\mathbb{C}^q$, $k=(k_1,\ldots,k_p)\in(\mathbb{R}^+)^p$ and $s=(s_1,\ldots,s_q)\in(\mathbb{R}^+)^q$ such that $b_i\in\mathbb{C}\backslash s_i\mathbb{Z}^-$. They also mentioned that, this series converges for all x if $p\le q$, diverges if $p>q+1$, and if $p=q+1$ it converges for all $|x|<\frac{s_1\cdots s_q}{k_1\cdots k_p}$.

These studies were followed by Mansour [16], Kokologiannaki [14], Krasniqi [15] and Merovci [18]. In 2012, Mubeen and Habibullah [21] defined the k-hypergeometric function as

$$_2F_{1,k}(a,b;c;x) = \sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}}\frac{x^n}{n!}, \quad k>0. \qquad (1.4)$$

In the same paper, they introduced integral representations of some k-confluent hypergeometric and k-hypergeometric functions. Mubeen [19] also introduced k-analogue of Kummer's first formula. Same year, Mubeen and Habibullah [22] introduced k-Riemann-Liouville fractional integral using k-gamma function as

$$\left({}_kI_a^\alpha f(t)\right)(x) := \frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1}f(t)dt, \quad k,\alpha\in\mathbb{R}^+. \qquad (1.5)$$

Dorrego and Cerutti introduced the k-Mittag-Leffler function in [8] as

$$E^{\gamma}_{k,\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n+\beta)} \frac{z^n}{n!}. \tag{1.6}$$

In 2013, Mubeen [20] also determined solution of some integral equations involving confluent k-hypergeometric functions, Mubeen et al. [24] introduced k-hypergeometric and confluent k-hypergeometric differential equations.

In 2015, Mubeen et al. [23] defined first k-Appell hypergeometric function as

$$F_{1,k}(a,b,b';c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n,k}(b)_{m,k}(b')_{n,k}}{(c)_{m+n,k}} \frac{x^m}{m!}\frac{y^n}{n!}, \ \max\{|x|,|y|\} < \frac{1}{k}. \tag{1.7}$$

and gave an integral representation of it. Recently, Sarıkaya et al. [26], introduced $(k,s)-$Riemann-Liouville fractional integral as

$$\left({}^{s}_{k}I^{\alpha}_{a} f(t)\right)(x) := \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x (x^{s+1}-t^{s+1})^{\frac{\alpha}{k}-1} t^s f(t)dt, \tag{1.8}$$

where $k,\alpha \in \mathbb{R}^+, s \in \mathbb{R}\setminus\{-1\}$. And finally Tomar et al. [31] defined left-sided and right-sided Hadamard-type k-fractional integral operators as

$$\left(H^{\alpha}_{a^+,k} f(\tau)\right)(t) := \frac{1}{k\Gamma_k(\alpha)} \int_a^t \left(\log\left(\frac{t}{\tau}\right)\right)^{\frac{\alpha}{k}-1} \frac{f(\tau)}{\tau} d\tau, \quad 0<a<t\le b \tag{1.9}$$

$$\left(H^{\alpha}_{b^-,k} f(\tau)\right)(t) := \frac{1}{k\Gamma_k(\alpha)} \int_t^b \left(\log\left(\frac{\tau}{t}\right)\right)^{\frac{\alpha}{k}-1} \frac{f(\tau)}{\tau} d\tau, \quad 0<a\le t<b.$$

It is obvious that most of the given definitions and the obtained results in these papers (and the related ones not mentioned here) coincides with the corresponding original definitions and results when $k=1$.

In this paper, we first emphasize that k-generalizations of some functions and fractional operators are closely related with corresponding original definitions. To draw attention to the importance of these relationships, we give some examples for how to shorten the proofs of some known results about the k-generalizations which can be found in the existing literature. We also define the k-generalizations of Appell hypergeometric functions F_2, F_3, F_4 and give similar relationships with the corresponding original functions to show that, new results can be obtain without long proofs.

2. Observations about k-functions

In this section, we give our attention to the relationships between the definitions of k-functions and their corresponding original definitions.

Conclusion 2.1. *The relationship between the k-hypergeometric function* (1.4) *and its original definition is*

$$\begin{aligned} {}_2F_{1,k}(a,b;c;x) &= \sum_{n=0}^{\infty} \frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}} \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(a/k)_n(b/k)_n}{(c/k)_n} \frac{(kx)^n}{n!} \\ &= {}_2F_1(a/k,b/k;c/k;kx). \end{aligned} \tag{2.1}$$

In general, for the generalized k-hypergeometric function

$$ {}_pF_{q,k}(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_{n,k} \cdots (a_p)_{n,k}}{(b_1)_{n,k} \cdots (b_q)_{n,k}} \frac{x^n}{n!}, $$

one can find a similar relationship with original generalized hypergeometric function as

$$ {}_pF_{q,k}(a_1, \ldots, a_p; b_1, \ldots, b_q; x) = {}_pF_q(a_1/k \cdots a_p/k; b_1/k \cdots b_q/k; k^{p-q}x). \quad (2.2) $$

In fact, relationships (2.1) *and* (2.2) *were given by Diaz and Pariguan with choosing suitable parameters in* [5, Prop. 18] *before. With the help of above relations, most of the known properties of original hypergeometric function can be use for determining the corresponding properties of k-hypergeometric function.*

Example 2.2. *The integral representation of original hypergeometric function is given by* [6]

$$ {}_{m+1}F_m\left(\alpha, \frac{\beta}{m}, \frac{\beta+1}{m}, \ldots, \frac{\beta+m-1}{m}; \frac{\gamma}{m}, \frac{\gamma+1}{m}, \ldots, \frac{\gamma+m-1}{m}; x\right) $$
$$ = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt^m)^{-\alpha}dt, \quad \Re(\gamma) > \Re(\beta) > 0. $$

Using the above representation with (1.1) *and* (2.2) *we can find an integral representation of k-hypergeometric function*

$$ {}_{m+1}F_{m,k}\left(\alpha, \frac{\beta}{m}, \frac{\beta+k}{m}, \ldots, \frac{\beta+(m-1)k}{m}; \frac{\gamma}{m}, \frac{\gamma+k}{m}, \ldots, \frac{\gamma+(m-1)k}{m}; x\right) $$
$$ = {}_{m+1}F_m\left(\frac{\alpha}{k}, \frac{\beta}{km}, \frac{\beta+k}{km}, \ldots, \frac{\beta+(m-1)k}{km}; \frac{\gamma}{km}, \frac{\gamma+k}{km}, \ldots, \frac{\gamma+(m-1)k}{km}; kx\right) $$
$$ = {}_{m+1}F_m\left(\frac{\alpha}{k}, \frac{\frac{\beta}{k}}{m}, \frac{\frac{\beta}{k}+1}{m}, \ldots, \frac{\frac{\beta}{k}+m-1}{m}; \frac{\frac{\gamma}{k}}{m}, \frac{\frac{\gamma}{k}+1}{m}, \ldots, \frac{\frac{\gamma}{k}+m-1}{m}; kx\right) $$
$$ = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)} \int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}(1-kxt^m)^{-\frac{\alpha}{k}}dt, \quad \Re(\gamma) > \Re(\beta) > 0. $$

without an effort. The reader can find the above integral representation in [21, Theorem 3.1] *with a long proof.*

Using relationship (2.1), we give the Mellin transform of ${}_2F_{1,k}$ which probably not exist in the literature.

Theorem 2.3. *The Mellin transform of* ${}_2F_{1,k}$ *is given as*

$$ \mathfrak{M}\left\{{}_2F_{1,k}\left(a, b; c; -x\right)\right\} = \int_0^{\infty} x^{s-1}{}_2F_{1,k}\left(a, b; c; -x\right)dx $$
$$ = k^{1-s}\frac{B_k\left(ks, a-ks\right)B_k\left(ks, b-ks\right)}{B_k(ks, c-ks)} $$

where $0 < \Re(ks) < \min\{\Re(a), \Re(b)\}$.

Proof. Using the relationship (1.3) and (2.1), a property of Mellin transform [10, p.307 (2)] and Mellin transform of original hypergeometric function [10, p.336 (3)], we obtain

$$\begin{aligned}\mathfrak{M}\left\{{}_2F_{1,k}\left(a,b;c;-x\right)\right\} &= \mathfrak{M}\left\{{}_2F_1\left(a/k,b/k;c/k;-kx\right)\right\}\\ &= k^{-s}\mathfrak{M}\left\{{}_2F_1\left(a/k,b/k;c/k;-x\right)\right\}\\ &= k^{-s}\frac{B\left(s,\frac{a}{k}-s\right)B\left(s,\frac{b}{k}-s\right)}{B(s,\frac{c}{k}-s)}\\ &= k^{1-s}\frac{B_k\left(ks,a-ks\right)B_k\left(ks,b-ks\right)}{B_k(ks,c-ks)}.\end{aligned}$$

□

Conclusion 2.4. *The relationship between the original Mittag-Leffler function*

$$E^{\gamma}_{\alpha,\beta}(z) := \sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!},$$

and its k-case (1.6) *is*

$$E^{\gamma}_{k,\alpha,\beta}(z) = \left(k^{1-\frac{\beta}{k}}\right)E^{\gamma/k}_{\alpha/k,\beta/k}\left(zk^{1-\frac{\alpha}{k}}\right),$$

which given in [8, Eq. II.11]. *The authors also used this relationship in their proofs, properly.*

Conclusion 2.5. *The relationships between the first Appell hypergeometric function*

$$F_1(a,b,b';c;x,y) = \sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\quad \max\{|x|,|y|\}<1.$$

and its k-case (1.7) *is*

$$F_{1,k}(a,b,b';c;x,y) = F_1(a/k,b/k,b'/k;c/k;kx,ky), \tag{2.3}$$

which is probably not given in existing literature.

Example 2.6. *An integral representation of F_1 is given in* [2, p.77 (4)] *as*

$$F_1(a,b,b';c;x,y) = \frac{1}{B(a,c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b}(1-yt)^{-b'}dt.$$

Using (1.3) *and relationship* (2.3) *in the above representation one can easily get*

$$F_{1,k}(a,b,b';c;x,y) = \frac{1}{kB_k(a,c-a)}\int_0^1 t^{\frac{a}{k}-1}(1-t)^{\frac{c-a}{k}-1}(1-kxt)^{-\frac{b}{k}}(1-kyt)^{-\frac{b'}{k}}dt.$$

Note that, Mubeen et al. also gave this integral representation in [23] *with a long proof.*

Using relationship (2.3), we give the double-Mellin transform of $F_{1,k}$ which probably not exist in the literature.

Theorem 2.7. *The double-Mellin transform of $F_{1,k}$ is given as*

$$\begin{aligned}&\int_0^{\infty}\int_0^{\infty}x^{s_1-1}y^{s_2-1}F_{1,k}(a,b,b';c;-x,-y)dxdy\\ &\quad = k^{2-s_1-s_2}\frac{B_k(ks_1,b-ks_1)B_k(ks_2,b'-ks_2)B_k(a-ks_1-ks_2,c-a)}{B_k(a,c-a)}\end{aligned}$$

$$(0<\Re(ks_1+ks_2)<\Re(a),\ 0<\Re(ks_1)<\Re(b),\ 0<\Re(ks_2)<\Re(b')$$

Proof. Proof is clear from the relationship (1.3) and (2.3), a property of double Mellin transform [13, p.34 (2.7)] and double Mellin transform of original first Appell hypergeometric function [13, p.38 (2.15)]. □

3. Observations about k-fractional operators

Now, we give the relationships between the definitions of k-fractional operators and their corresponding original definitions.

Conclusion 3.1. *The relationship between the k-Riemann-Liouville fractional integral* (1.5) *and the original Riemann-Liouville integral*

$$(I_a^\alpha f(t))(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha \in \mathbb{R}^+ \tag{3.1}$$

is

$$\left({}_kI_a^\alpha f(t)\right)(x) = k^{-\frac{\alpha}{k}} \left(I_a^{\alpha/k} f(t)\right)(x). \tag{3.2}$$

which given in [7, Eq. I.11]. *With the help of this relation, most of the known properties of classic fractional integrals can be use for determining the corresponding properties of k-fractional integral.*

Example 3.2. *In* [4]*, the Grüss type inequality for Riemann-Liouville fractional integral is given as*

$$\left| \left(I_a^\alpha[p(t)]\right)\left(I_a^\alpha[pfg(t)]\right) - \left(I_a^\alpha[pf(t)]\right)\left(I_a^\alpha[pg(t)]\right) \right| \leq \frac{\left(I_a^\alpha[p(t)]\right)^2}{4}(\Phi-\varphi)(\Psi-\psi),$$

where f and g be two integrable function on $[a,b]$, $\alpha > 0$ with $\varphi < f(t) < \Phi$, $\psi < g < \Psi$ and let p be a positive function on $[a,b]$. In this inequality if we replace α with α/k $(k>0)$, multiply both sides by $k^{-2\alpha/k}$, and use relation (3.2) *we get the Grüss type inequality for k-Riemann-Liouville fractional integral as*

$$\left| \left({}_kI_a^\alpha[p(t)]\right)\left({}_kI_a^\alpha[pfg(t)]\right) - \left({}_kI_a^\alpha[pf(t)]\right)\left({}_kI_a^\alpha[pg(t)]\right) \right| \leq \frac{\left({}_kI_a^\alpha[p(t)]\right)^2}{4}(\Phi-\varphi)(\Psi-\psi).$$

The reader can find the same result in [27, Theorem 3] *with a long proof.*

Conclusion 3.3. *The relationship between the original* (3.1) *and the (k,s)-Riemann-Liouville fractional integral* (1.8) *is*

$$\left({}_k^sI_a^\alpha f(t)\right)(x) = \frac{1}{[k(s+1)]^{\alpha/k}} \left(I_{a^{s+1}}^{\alpha/k} f\left(t^{\frac{1}{s+1}}\right)\right)(x^{s+1}). \tag{3.3}$$

Example 3.4. *Taking $f(t) = (t^{s+1}-a^{s+1})^{\frac{\beta}{k}-1}$ in the above relation, using the equality (see* [12, p. 71]*)*

$$\left(I_a^\alpha (t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1}$$

and relation (1.1)*, we can easily get*

$$\begin{aligned}\left({}_k^sI_a^\alpha (t^{s+1}-a^{s+1})^{\frac{\beta}{k}-1}\right)(x) =& \frac{1}{[k(s+1)]^{\alpha/k}} \left(I_{a^{s+1}}^{\alpha/k} (t-a^{s+1})^{\frac{\beta}{k}-1}\right)(x^{s+1}) \\ =& \frac{1}{(s+1)^{\alpha/k}} \frac{\Gamma_k(\beta)}{\Gamma_k(\beta+\alpha)} (x^{s+1}-a^{s+1})^{\frac{\beta+\alpha}{k}-1}.\end{aligned}$$

without any integration like in [26, Theorem 2.4].

Conclusion 3.5. *The relationship between the original*

$$\left(H_{a^+}^{\alpha} f(\tau)\right)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log\left(\frac{t}{\tau}\right)\right)^{\alpha-1} \frac{f(\tau)}{\tau} d\tau,$$

and the k-Hadamard fractional integral (1.9) *is*

$$\left(H_{a^+,k}^{\alpha} f(\tau)\right)(t) = k^{-\frac{\alpha}{k}} \left(H_{a^+}^{\alpha/k} f(\tau)\right)(t), \tag{3.4}$$

which is similar as (3.2).

Example 3.6. *In* [3, Theorem 3.1]*, the Pólya-Szegő inequality for the Hadamard fractional integral is given as*

$$\frac{H_{1^+}^{\alpha}\left(\psi_1\psi_2 f^2\right)(t) H_{1^+}^{\alpha}\left(\varphi_1\varphi_2 g^2\right)(t)}{\left(H_{1^+}^{\alpha}\left((\psi_1\varphi_1 + \psi_2\varphi_2) f g\right)(t)\right)^2} \leq \frac{1}{4},$$

such that for all $\tau \in [1, t], (t > 1)$

$$0 < \psi_1(\tau) \leq g(\tau) \leq \psi_2(\tau), \quad 0 < \varphi_1(\tau) \leq f(\tau) \leq \varphi_2(\tau),$$

where f and g are real integrable functions, $\psi_1, \psi_2, \varphi_1, \varphi_2$ are integrable functions defined on $[1, \infty)$*. If we take α with $\alpha/k, (k > 0)$, multiply numerator and the denominator by $k^{-2\alpha/k}$, and use relation* (3.4) *for $a = 1$, we get the Pólya-Szegő inequality for the Hadamard fractional integral as*

$$\frac{H_{1^+,k}^{\alpha}\left(\psi_1\psi_2 f^2\right)(t) H_{1^+,k}^{\alpha}\left(\varphi_1\varphi_2 g^2\right)(t)}{\left(H_{1^+,k}^{\alpha}((\psi_1\varphi_1 + \psi_2\varphi_2) f g)(t)\right)^2} \leq \frac{1}{4}.$$

Note that, the reader can find the Pólya-Szegő inequality for $H_{a^+}^{\alpha}$ in [31, Lemma 2.1] *with a long proof.*

4. k-Appell Functions

The aim of this section is to show how easily access the results (probably not in the existing literature) by using the relationships between the original functions and their k-generalizations. To do this, we first define the k-Appell functions $F_{2,k}, F_{3,k}, F_{4,k}$ in a similar way with $F_{1,k}$, and we give their relationships with corresponding original definitions. Throughout this section we assume that $k > 0$.

Definition 4.1. *The k-Appell hypergeometric functions can be defined as*

$$F_{2,k}(a, b, b'; c, c'; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n,k}(b)_{m,k}(b')_{n,k}}{(c)_{m,k}(c')_{n,k}} \frac{x^m}{m!} \frac{y^n}{n!}, \quad |x| + |y| < \frac{1}{k} \tag{4.1}$$

$$F_{3,k}(a, a', b, b'; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m,k}(a')_{n,k}(b)_{m,k}(b')_{n,k}}{(c)_{m+n,k}} \frac{x^m}{m!} \frac{y^n}{n!}, \ \max\{|x|, |y|\} < \frac{1}{k} \tag{4.2}$$

$$F_{4,k}(a, b; c, c'; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n,k}(b)_{m+n,k}}{(c)_{m,k}(c')_{n,k}} \frac{x^m}{m!} \frac{y^n}{n!}, \quad \sqrt{|x|} + \sqrt{|y|} < \frac{1}{\sqrt{k}} \tag{4.3}$$

Theorem 4.2. *The relationships between k-Appell hypergeometric functions and original Appell hypergeometric functions are given as*

$$F_{2,k}(a,b,b';c,c';x,y) = F_2(a/k,b/k,b'/k;c/k,c'/k;kx,ky), \tag{4.4}$$

$$F_{3,k}(a,a',b,b';c;x,y) = F_3(a/k,a'/k,b/k,b'/k;c/k;kx,ky), \tag{4.5}$$

$$F_{4,k}(a,b;c,c';x,y) = F_4(a/k,b/k;c/k,c'/k;kx,ky). \tag{4.6}$$

Proof. These relationships are direct consequences of the definitions of k-Appell hypergeometric functions and property (1.2). □

Conclusion 4.3. *Using some properties of original Appell hypergeometric functions and relationships* (2.3), (4.4)-(4.6) *together, corresponding properties of k-Appell hypergeometric functions can be found.*

Now we give integral representations of k-Appell hypergeometric functions using the known integral representations of original Appell hypergeometric functions and the above relationships.

Theorem 4.4 (Integral representations of k-Appell hypergeometric functions)**.**

$$F_{1,k}(a,b,b';c;x,y) = \frac{1}{k^2 B_k(b,c-b)B_k(b',c-b-b')} \cdot \iint_D u^{\frac{b}{k}-1} v^{\frac{b'}{k}-1} (1-u-v)^{\frac{c-b-b'}{k}-1} (1-kxu-kyv)^{-\frac{a}{k}} dudv$$
$$(\Re(b)>0,\ \Re(b')>0,\ \Re(c-b-b')>0),$$

$$F_{1,k}(a,b,b';c;x,y) = \frac{k^{\frac{b+b'}{k}}}{k^2\Gamma_k(b)\Gamma_k(b')} \int_0^\infty\int_0^\infty u^{\frac{b}{k}-1} v^{\frac{b'}{k}-1} e^{-u-v} {}_1F_{1,k}\,(a;c;kxu+kyv)\,dudv$$
$$(\Re(b)>0,\ \Re(b')>0),$$

$$F_{2,k}(a,b,b';c,c';x,y) = \frac{1}{k^2 B_k(b,c-b)B_k(b',c'-b')} \cdot \int_0^1\int_0^1 u^{\frac{b}{k}-1} v^{\frac{b'}{k}-1} (1-u)^{\frac{c-b}{k}-1} (1-v)^{\frac{c'-b'}{k}-1} (1-kxu-kyv)^{-\frac{a}{k}} dudv$$
$$(\Re(c)>\Re(b)>0,\ \Re(c')>\Re(b')>0),$$

$$F_{2,k}(a,b,b';c,c';x,y) = \frac{1}{kB_k(b,c-b)} \cdot \int_0^1 u^{\frac{b}{k}-1} (1-u)^{\frac{c-b}{k}-1} (1-kxu)^{-\frac{a}{k}} {}_2F_{1,k}\left(a,b';c';\frac{y}{1-kxu}\right) du$$
$$(\Re(c)>\Re(b)>0),$$

$$F_{2,k}(a,b,b';c,c';x,y) = \frac{k^{\frac{a}{k}}}{k\Gamma_k(a)} \int_0^\infty u^{\frac{a}{k}-1} e^{-u} {}_1F_{1,k}\,(b;c;kxu)\, {}_1F_{1,k}\,(b';c';kyu)\, du$$
$$(\Re(a)>0),$$

$$F_{3,k}(a,a',b,b';c;x,y) = \frac{1}{k^2 B_k(b,c-b)B_k(b',c-b-b')} \cdot \iint_D u^{\frac{b}{k}-1} v^{\frac{b'}{k}-1} (1-kxu)^{-\frac{a}{k}} (1-kyv)^{-\frac{a'}{k}} (1-u-v)^{\frac{c-b-b'}{k}-1} dudv$$

$$(\Re(b) > 0,\ \Re(b') > 0,\ \Re(c-b-b') > 0),$$

$$F_{3,k}(a,a',b,b';c;x,y) = \frac{k^{\frac{a+a'+b+b'}{k}}}{k^4\Gamma_k(a)\Gamma_k(a')\Gamma_k(b)\Gamma_k(b')}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-u_1-u_2-v_1-v_2}$$
$$\cdot u_1^{\frac{a}{k}-1}u_2^{\frac{a'}{k}-1}v_1^{\frac{b}{k}-1}v_2^{\frac{b'}{k}-1}{}_0F_{1,k}(-;c;k^2xu_1v_1+k^2yu_2v_2)du_1du_2dv_1dv_2$$
$$(\Re(a) > 0,\ \Re(a') > 0,\ \Re(b) > 0,\ \Re(b') > 0),$$

$$F_{4,k}(a,b;c,c';x,y) = \frac{k^{\frac{a+b}{k}}}{k^2\Gamma_k(a)\Gamma_k(b)}$$
$$\cdot\int_0^\infty\int_0^\infty e^{-u-v}u^{\frac{a}{k}-1}v^{\frac{b}{k}-1}{}_0F_{1,k}(-;c;k^2xuv){}_0F_{1,k}(-;c';k^2yuv)dudv$$
$$(\Re(a) > 0,\ \Re(b) > 0),$$

where $D = \{(u,v) : u \geq 0,\ v \geq 0,\ u+v \leq 1\}$.

Proof. The above integral representations can be obtain using equalities (1.1)-(1.3) and relationships (2.1)-(2.3), (4.4)-(4.6) in the known integral representations of original Appell hypergeometric functions (see [2], [9], [11], [17], [25], [28]-[30]). □

Finally, we easily give double Mellin transforms of k-Appell hypergeometric functions $F_{2,k}, F_{3,k}, F_{4,k}$ which probably not exist in the literature, using the known double Mellin transforms of original Appell hypergeometric functions (see [13]) and the above relationships. Note that, we also give the double Mellin transform of $F_{1,k}$ in Theorem 2.7, before.

Theorem 4.5 (Double Mellin Transforms of k-Appell hypergeometric functions)**.**

$$\int_0^\infty\int_0^\infty x^{s_1-1}y^{s_2-1}F_{2,k}(a,b,b';c,c';-x,-y)dxdy = k^{2-s_1-s_2}$$
$$\cdot\frac{B_k(ks_1,b-ks_1)B_k(ks_2,b'-ks_2)B_k(a-ks_1-ks_2,ks_1+ks_2)B_k(ks_1,ks_2)}{B_k(ks_1,c-ks_1)B_k(ks_2,c'-ks_2)}$$
$$(0 < \Re(ks_1+ks_2) < \Re(a),\ 0 < \Re(ks_1) < \Re(b),\ 0 < \Re(ks_2) < \Re(b')$$

$$\int_0^\infty\int_0^\infty x^{s_1-1}y^{s_2-1}F_{3,k}(a,a',b,b';c;-x,-y)dxdy = k^{2-s_1-s_2}$$
$$\cdot\frac{B_k(ks_1,a-ks_1)B_k(ks_2,a'-ks_2)B_k(ks_1,b-ks_1)B_k(ks_2,b'-ks_2)}{B_k(ks_1+ks_2,c-ks_1-ks_2)B_k(ks_1,ks_2)}$$
$$(0 < \Re(ks_1) < \min\{\Re(a),\Re(b)\},\ 0 < \Re(ks_2) < \min\{\Re(a'),\Re(b')\}$$

$$\int_0^\infty\int_0^\infty x^{s_1-1}y^{s_2-1}F_{4,k}(a,b;c,c';-x,-y)dxdy = k^{2-s_1-s_2}$$
$$\cdot\frac{B_k(a-ks_1-ks_2,ks_1+ks_2)B_k(b-ks_1-ks_2,ks_1+ks_2)B_k^2(ks_1,ks_2)}{B_k(ks_1,c-ks_1)B_k(ks_2,c'-ks_2)}$$
$$(0 < \Re(ks_1+ks_2) < \min\{\Re(a),\Re(b)\}$$

Proof. Proof is clear from the relationships (1.3) and (4.4)-(4.6), a property of double Mellin transform [13, p.34 (2.7)] and double Mellin transform of original hypergeometric functions [13, p.38 (2.16-2.18)]). □

5. Concluding Remarks

It should be observed in conclusion that, since the definitions of k-generalizations are closely related to the original definitions (that is, the $k = 1$ case), most of the formulas and results for the $k = 1$ case can be translated rather trivially and simply by appropriate parameter and notational changes to hold true for the corresponding k-case. In this connection, use can and should be made of the relationships given in the equations (1.1) and (1.2). For example, the reader can define k-generalizations of multivariable hypergeometric functions such as Lauricella, Horn, Srivastava, etc. and can be obtain these kinds of relationships with their corresponding original definitions. As a result, lots of properties for these new functions can be easily discovered with the help of know results, without an effort.

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