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A NOVEL CHELYSHKOV APPROACH TECHNIQUE FOR SOLVING FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH MIXED DELAYS

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Abstract. *This document is to present a improved collocation method based on Chelyshkov polynomials to solve the functional integro-differential equations with mixed delays under the initial-boundary conditions. An efficient error estimation for the Chelyshkov collocation method is also introduced. Some examples from quite different fields of pure and applied mathematics are given to demonstrate the validation and application of the method and a comparison is made between obtained and existing results.*
Keywords: *Delay integro-differential equations; Numerical solutions.*

1. INTRODUCTION

The functional integro-differential equations with proportional delays arise in a wide variety of many phenomena in applied sciences. These equations arise when the rate of change of a time dependent process in its mathematical modelling and are not only determined by its present state but also by a certain past state. Also, because of the inclusion of time delay terms, these equations are becoming more common, appearing in many branches of biological modelling such as infectious disease Dynamics, i.e., primary infection, drug therapy, epidemiology and immune response, and also appeared in the study of chemostat models, circadian rhythms, the respiratory system, tumor growth and neural networks [1-11]. They are also encountered in various fields of science and numerous applications, such as ecology, mechanics and physics [12-16].

In recent years, to solve the mentioned equations, several numerical methods were used such as Adomian decomposition method, the variational iteration method, homotopy perturbation method, the collocation method, the continuous Runge–Kutta methods, Monte-Carlo, Tau and Walsh series methods [17-28]. In addition to these methods mentioned above, such equations in question were solved using the collocation methods based on Sezer's matrix methods, which are derived from special functions as Taylor, Chebyshev, Bessel, Legendre, Laguerre, Hermite and so on [29-35]. Recently, Chelyshkov has introduced sequences of polynomials in [36], which are orthogonal over the interval with respect to the weight function , and are explicitly defined by

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$$C_{Nn}\left(x\right)=\sum_{j=0}^{N-n}\left(-1\right)^{j}\begin{pmatrix}N-n\\ j\end{pmatrix}\begin{pmatrix}N+n+j+1\\ N-n\end{pmatrix}x^{n+j},n=0,1\dots N. \quad (1)$$

This yields the Rodrigues' type representation so that

$$C_{Nn}\left(x\right)=\frac{1}{(N-n)!}\frac{1}{x^{n+1}}\frac{d^{N-n}}{dx^{N-n}}\left(x^{N+n+1}(1-x)^{N-n}\right),n=0,1,\dots,N, \quad (2)$$

By means of the Rodrigues' formula and the Cauchy integral formula for derivatives of an analytic function, one can obtain the integral relation as follows

$$C_{Nn}\left(t\right)=\frac{1}{2\pi i}\frac{1}{t^{n+2}}\int_{\Omega_1}\frac{\overline{z}^{(N+2+n)}(1-z)^{N-n}}{(z-t^{-1})^{N-n+1}}dz,$$

where Ω_1 is a closed curve, which encloses the point $z=t^{-1}$.

The properties of Chelyshkov polynomials are analogous to the properties of the classical orthogonal polynomials. In fact, these polynomials are an example of such alternative orthogonal ones, which are not solutions of the hypergeometric type, but can be expressed in terms of the Jacobi ones. They can also be connected to hypergeometric functions, orthogonal expontional polynomials, and the Jacobi polynomials $P_k^{(\alpha,\beta)}$ by the following relation

$$C_{Nn}x=(-1)^{N-n}x^{N-n}P_{N-n}^{(0,2n+1)}(2x-1),\ n=0,1\dots N.$$

Hence, they keep distinctive attributes of the classical orthogonal polynomials and may be facilitated to different problems on approximation. While investigating more on (1), we deduce that every member of the family of orthogonal polynomials has degree N with $N-k$ simple roots. Hence, for every N the polynomial $C_{N0}(x)$ has exactly N simple roots in (0, 1). Following [6], it can be shown that the sequence of polynomials $\{C_{N0}(x)\}_{N=0}^{\infty}$ generate a family of orthogonal polynomials on [0, 1] which possesses all the properties of other classic orthogonal polynomials e.g. Legendre or Chebyshev polynomials. Hence, for every N if the roots of the polynomial $C_{N0}(x)$ are chosen as node points, then an acurate numerical quadrature can be derived [36, 37].

In this study, we consider the functional integro-differential equations with variable coefficients and variable bounds in the following form

$$\sum_{h=0}^{m_1}\sum_{j=0}^{m_2}P_{hj}(x)y^{(h)}(\alpha_{hj}x+\beta_{hj})+\sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\int_{u_{rs}(x)}^{\vartheta_{rs}(x)}K_{rs}(x,t)y^{(r)}(\mu_{rs}x+\lambda_{rs})dt=f(x),\quad \mathrm{a}\le x,\mathrm{t}\le b \quad (4)$$

subject to the initial-boundary conditions

$$\sum_{h=0}^{m_1-1}\left(a_{ih}y^{(h)}(a)+b_{ih}y^{(h)}(b)\right)=\eta_i,\ i=0,1,...,m_1\text{ -1}. \quad (5)$$

Here, $\alpha_{hj}, \beta_{hj}, \mu_{hj}$ and λ_{hj} are real or complex constants. Meanwhile $P_{hj}(x)$, $K_{rs}(x,t)$, $f(x)$, $u_{rs}(x)$ and $\vartheta_{rs}(x)$ are continuous functions defined within $a \le t, x \le b.$ Our aim in this paper is to find an approximate solution of Eq. (4) under the initial-boundary conditions in Eq. (5) in the truncated Chelyshkov series, based on Eq. (1) or (2). Then, the general form of approximate solution takes the following form

$$y(x) = \sum_{n=0}^{N} a_n C_{n0}(x),$$

or briefly

$$y(x) = \sum_{n=0}^{N} a_n C_n(x) \tag{6}$$

so that $a_n,\ n = 0,1,2,\ldots,N$ are the unknown Chelyshkov coefficients.

2. MATERIALS AND METHODS

2.1. MATERIALS

First, we can write the approximate solution (6) and its derivatives in the matrix forms as follows:

$$y(x) = \boldsymbol{C}(t)\boldsymbol{A} \text{ or } y(x) = \boldsymbol{X}(x)\boldsymbol{B}^0\boldsymbol{\mathcal{C}}\boldsymbol{A},\quad \boldsymbol{B}^0 = \boldsymbol{I}, \tag{7}$$

$$y'(x) = \boldsymbol{C}'(x)\boldsymbol{A} = \boldsymbol{X}(x)\boldsymbol{B}^1\boldsymbol{\mathcal{C}}\boldsymbol{A},$$

$$y''(x) = \boldsymbol{C}''(x)\boldsymbol{A} = \boldsymbol{X}(x)\boldsymbol{B}^2\boldsymbol{\mathcal{C}}\boldsymbol{A},$$

$$\vdots$$

$$y^{(h)}(x) = \boldsymbol{C}^{(h)}(x)\boldsymbol{A} = \boldsymbol{X}(x)\boldsymbol{B}^h\boldsymbol{\mathcal{C}}\boldsymbol{A},\ \mathrm{h} = 0,1,\ldots,m_1, \tag{8}$$

where

$$\boldsymbol{C}(t) = \begin{bmatrix} C_{00}(t) & C_{10}(t) & \cdots & C_{N0}(t) \end{bmatrix},\ \boldsymbol{X}(\mathrm{x}) = \begin{bmatrix} 1 & x & \ldots & x^N \end{bmatrix},$$

$$\boldsymbol{\mathcal{C}} = \begin{bmatrix} \binom{0}{0}\binom{1}{0} & \binom{1}{0}\binom{2}{1} & \binom{2}{0}\binom{3}{2} & \cdots & \binom{N}{0}\binom{N+1}{N} \\ 0 & -\binom{1}{1}\binom{3}{1} & -\binom{2}{1}\binom{4}{2} & \cdots & -\binom{N}{1}\binom{N+2}{N} \\ 0 & 0 & \binom{2}{2}\binom{5}{2} & \cdots & \binom{N}{2}\binom{N+3}{N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (-1)^N\binom{N}{N}\binom{2N+1}{N} \end{bmatrix}_{(N+1)x(N+1)},$$

$$\boldsymbol{B}=\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & N \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{(N+1)x(N+1)} \quad \text{and } \boldsymbol{A}=\begin{bmatrix} a_0 & a_1 & \dots & a_N \end{bmatrix}^T.$$

Replacing $(\alpha_{hj}x+\beta_{hj})$ by x in the relation (8) we have the matrix form

$$y^{(h)}(\alpha_{hj}x+\beta_{hj}) \cong y_N^{(h)}(\alpha_{hj}x+\beta_{hj}) = \boldsymbol{C}^{(h)}(\alpha_{hj}x+\beta_{hj})\boldsymbol{A} = \boldsymbol{X}(\alpha_{hj}x+\beta_{hj})\boldsymbol{B}^h\boldsymbol{\mathcal{C}}\boldsymbol{A}\ ,\ \text{h}=0,1,\dots,m_1. \tag{9}$$

The relation between the matrices $\boldsymbol{X}(\alpha_{hj}x+\beta_{hjs})$ and $\boldsymbol{X}(x)$ is

$$\boldsymbol{X}(\alpha_{hj}x+\beta_{hj}) = \boldsymbol{X}(x)\boldsymbol{M}(\alpha_{hj},\beta_{hj}) \tag{10}$$

such that, for $\alpha_{hj},\beta_{hj} \neq 0$

$$\boldsymbol{M}(\alpha_{hj},\beta_{hj})=\begin{bmatrix} \binom{0}{0}\alpha_{hj}^0\beta_{hj}^0 & \binom{1}{0}\alpha_{hj}^0\beta_{hj}^1 & \binom{2}{0}\alpha_{hj}^0\beta_{hj}^2 & \cdots & \binom{N}{0}\alpha_{hj}^0\beta_{hj}^N \\ 0 & \binom{1}{1}\alpha_{hj}^1\beta_{hj}^0 & \binom{2}{1}\alpha_{hj}^1\beta_{hj}^1 & \cdots & \binom{N}{1}\alpha_{hj}^1\beta_{hj}^{N-1} \\ 0 & 0 & \binom{2}{2}\alpha_{hj}^2\beta_{hj}^0 & \cdots & \binom{N}{2}\alpha_{hj}^2\beta_{hj}^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \binom{N}{N}\alpha_{hj}^N\beta_{hj}^0 \end{bmatrix}$$

and, for $\alpha_{hj} \neq 0$ and $\beta_{hj}=0$

$$\boldsymbol{M}(\alpha_{hj},0)=\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \alpha_{hj} & 0 & \dots & 0 \\ 0 & 0 & \alpha_{hj}^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \alpha_{hj}^N \end{bmatrix}.$$

We have the following matrix relation by substituting Eq.(10) into Eq.(9)

$$[y^{(h)}(\alpha_{hj}x+\beta_{hj})] = \boldsymbol{X}(x)\boldsymbol{M}(\alpha_{hj},\beta_{hj})\boldsymbol{B}^h\boldsymbol{\mathcal{C}}\boldsymbol{A}\ ,\ \text{h}=0,1\dots,m_1. \tag{11}$$

Now, let us construct the matrix form of the integral part of Eq. (4). The kernel functions $K_{rs}(x,t)$ can be approximated by the truncated Taylor series [38] and the truncated Chelyshkov series, respectively,

$$K_{rs}(x,t)=\sum_{m=0}^{N}\sum_{n=0}^{N} {}^{T}k_{mn}^{rs}x^m t^n \text{ and } K_{rs}(x,t)=\sum_{m=0}^{N}\sum_{n=0}^{N} {}^{C}k_{mn}^{rs}C_m(x)C_n(t), \tag{12}$$

where

$${}^{T}k_{mn}^{rs}=\frac{1}{m!n!}\frac{\partial^{m+n}K_{rs}(0,0)}{\partial^m x\partial^n t};\ m,n=0,1,2,\ldots,N,\quad r=0,1,2,...,m_3,\ \mathrm{s}=0,1,2,...,m_4.$$

We write the expressions in Eq. (12) in the forms

$$K_{rs}(x,t)=\mathbf{X}(x)\mathbf{K}_{rs}^{T}\mathbf{X}^{T}(t),\quad \mathbf{K}_{rs}^{T}=[{}^{T}k_{mn}^{rs}] \tag{13}$$

and

$$K_{rs}(x,t)=\boldsymbol{C}(x)\mathbf{K}_{rs}^{T}\boldsymbol{C}^{T}(t),\quad \mathbf{K}_{rs}^{C}=[{}^{C}k_{mn}^{rs}] \tag{14}$$

From Eqs. (7), (13) and (14), we obtain

$$K_{rs}(x,t)=\mathbf{X}(x)\mathbf{K}_{rs}^{T}\mathbf{X}^{T}(t)=\mathbf{C}(x)\mathbf{K}_{rs}^{C}\mathbf{C}^{T}(t)\Rightarrow\ \mathbf{K}_{rs}^{C}=\mathcal{C}^{-1}\mathbf{K}_{rs}^{T}(\mathcal{C}^{T})^{-1}\quad . \tag{15}$$

2.2. METHODS

To construct the fundamental matrix relation, if matrix relations (11) and (15) are substituted into Eq. (4) and arranged, then the following matrix equation is obtained

$$\left\{\sum_{h=0}^{m_1}\sum_{j=0}^{m_2}P_{hj}(x)\boldsymbol{X}(x)\boldsymbol{M}(\alpha_{hj},\beta_{hj})\boldsymbol{B}^h\mathcal{C}+\sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\mathbf{C}(x)\mathbf{K}_{rs}^{C}\boldsymbol{Q}_{rs}(x)\boldsymbol{M}(\mu_{rs},\lambda_{rs})\boldsymbol{B}^r\mathcal{C}\right\}\boldsymbol{A}=f(x), \tag{16}$$

where

$$\mathbf{Q}_{rs}(x)=\int_{u_{rs}(x)}^{\vartheta_{rs}(x)}\mathcal{C}^T\mathbf{X}^T(x)\mathbf{X}(x)\mathcal{C}dt=\mathcal{C}^T\mathbf{H}_{rs}(x)\mathcal{C}$$

and

$$\mathbf{H}_{rs}(x)=\int_{u_{rs}(x)}^{\vartheta_{rs}(x)}\mathbf{X}^T(x)\mathbf{X}(x)dt=[h_{mn}^{rs}(x)];\ h_{mn}^{rs}(x)=\frac{(\vartheta_{rs}(x))^{m+n+1}-(u_{rs}(x))^{m+n+1}}{m+n+1},\quad m,n=0,1,2,\ldots,N.$$

By substituting the collocation points defined by

$$x_i = a + \frac{b-a}{N} i,\ \mathrm{i} = 0,1,...,N \ \text{ or } \ x_i = \frac{a-b}{2}\cos\left(\frac{\pi i}{N}\right) + \frac{a+b}{2}$$

into Eq. (16), we get the following system of the matrix equations

$$\left\{ \sum_{h=0}^{m_1} \sum_{j=0}^{m_2} P_{hj}(x_i) \boldsymbol{X}(x_i) \boldsymbol{M}(\alpha_{hj}, \beta_{hj}) \boldsymbol{B}^h \boldsymbol{\mathcal{C}} + \sum_{r=0}^{m_3} \sum_{s=0}^{m_4} \mathbf{C}(x_i) \mathbf{K}_{rs}^C \boldsymbol{Q}_{rs}(x_i) \boldsymbol{M}(\mu_{rs}, \lambda_{rs}) \boldsymbol{B}^r \boldsymbol{\mathcal{C}} \right\} \boldsymbol{A} = f(x_i)$$

or after arrangend, the fundamental matrix equation is derived as follows:

$$\left\{ \sum_{h=0}^{m_1} \sum_{j=0}^{m_2} \boldsymbol{P}_{hj} \boldsymbol{X} \boldsymbol{M}(\alpha_{hj}, \beta_{hj}) \boldsymbol{B}^h \boldsymbol{\mathcal{C}} + \sum_{r=0}^{m_3} \sum_{s=0}^{m_4} \mathbf{C} \overline{\mathbf{K}}_{rs}^C \overline{\boldsymbol{Q}}_{rs} \overline{\boldsymbol{M}}(\mu_{rs}, \lambda_{rs}) \overline{\boldsymbol{B}}^r \overline{\boldsymbol{\mathcal{C}}} \right\} \boldsymbol{A} = \boldsymbol{F}. \tag{17}$$

Here

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}(x_0) & \boldsymbol{X}(x_1) & \dots & \boldsymbol{X}(x_N) \end{bmatrix}^T = \begin{bmatrix} 1 & x_0 & \dots & x_0^N \\ 1 & x_1 & \dots & x_1^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & \dots & x_N^N \end{bmatrix}, \quad \begin{matrix} \mathbf{C} = \mathrm{Diag}\begin{bmatrix} \boldsymbol{C}(x_0) & \boldsymbol{C}(x_1) & \dots & \boldsymbol{C}(x_N) \end{bmatrix} \\ \overline{\mathbf{C}} = \begin{bmatrix} \boldsymbol{C}(x_0) & \boldsymbol{C}(x_1) & \dots & \boldsymbol{C}(x_N) \end{bmatrix}^T \end{matrix},$$

$$\boldsymbol{P}_{hj}(x) = \mathrm{Diag}\begin{bmatrix} P_{hj}(x_0) & P_{hj}(x_1) & \dots & P_{hj}(x_N) \end{bmatrix}, \ \overline{\mathbf{K}}_{rs}^C = \mathrm{Diag}\begin{bmatrix} \mathbf{K}_{rs}^C & \mathbf{K}_{rs}^C & \dots & \mathbf{K}_{rs}^C \end{bmatrix},$$

$$\overline{\boldsymbol{Q}}_{rs} = \mathrm{Diag}\begin{bmatrix} \boldsymbol{Q}_{rs}(x_0) & \boldsymbol{Q}_{rs}(x_1) & \dots & \boldsymbol{Q}_{rs}(x_N) \end{bmatrix}, \ \overline{\boldsymbol{M}}(\mu_{rs}, \lambda_{rs}) = \mathrm{Diag}\begin{bmatrix} \boldsymbol{M}(\mu_{rs}, \lambda_{rs}) & \boldsymbol{M}(\mu_{rs}, \lambda_{rs}) & \dots & \boldsymbol{M}(\mu_{rs}, \lambda_{rs}) \end{bmatrix},$$

$$\overline{\boldsymbol{B}}^r = \mathrm{Diag}\begin{bmatrix} \boldsymbol{B}^r & \boldsymbol{B}^r & \dots & \boldsymbol{B}^r \end{bmatrix}, \ \overline{\boldsymbol{\mathcal{C}}} = \begin{bmatrix} \boldsymbol{\mathcal{C}} & \boldsymbol{\mathcal{C}} & \dots & \boldsymbol{\mathcal{C}} \end{bmatrix}^T \text{ and } \boldsymbol{F} = \begin{bmatrix} f(x_0) & f(x_1) & \dots & f(x_N) \end{bmatrix}^T.$$

In summary, we can write Eq. (17) in the form

$$\boldsymbol{WA} = \boldsymbol{F} \quad \text{or} \quad [\boldsymbol{W}; \boldsymbol{F}] \tag{18}$$

so that

$$\boldsymbol{W} = \begin{bmatrix} w_{pq} \end{bmatrix} = \sum_{h=0}^{m_1} \sum_{j=0}^{m_2} \boldsymbol{P}_{hj} \boldsymbol{X} \boldsymbol{M}(\alpha_{hj}, \beta_{hj}) \boldsymbol{B}^h \boldsymbol{\mathcal{C}} + \sum_{r=0}^{m_3} \sum_{s=0}^{m_4} \mathbf{C} \overline{\mathbf{K}}_{rs}^C \overline{\boldsymbol{Q}}_{rs} \overline{\boldsymbol{M}}(\mu_{rs}, \lambda_{rs}) \overline{\boldsymbol{B}}^r \overline{\boldsymbol{\mathcal{C}}}.$$

The fundamental matrix equation (18) corresponds to a system of the $(N+1)$ linear algebraic equations with the $(N+1)$ unknown Chelyshkov coefficients $a_i,\ i = 0,1,2,...,N.$

On other hand, using Eq. (8) the matrix forms of conditions are obtained as follows:

$$\boldsymbol{U}_i \boldsymbol{A} = [\eta_i] \ \text{ or } \ [\boldsymbol{U}_i; \eta_i], \ i = 0,1,2,...,m_1 - 1 \tag{19}$$

where

$$\boldsymbol{U}_i = \begin{bmatrix} u_{i0} & u_{i1} & \ldots & u_{iN} \end{bmatrix} = \sum_{h=0}^{m_1} \left\{ \boldsymbol{a}_{ih} \boldsymbol{X}(a) + \boldsymbol{b}_{ih} \boldsymbol{X}(b) \right\} \boldsymbol{B}^h \boldsymbol{\mathcal{C}}.$$

Finally, by replacing the last rows of the augmented matrix (18) by the rows of matrix (19), we have the new augmented matrix as follows

$$\widetilde{\boldsymbol{W}} \widetilde{\boldsymbol{A}} = \widetilde{\boldsymbol{F}} \quad \text{or} \quad \left[\widetilde{\boldsymbol{W}}; \widetilde{\boldsymbol{F}} \right], \tag{20}$$

which corresponds a linear system of algebraic equations. If $\text{rank}\, \tilde{\mathbf{W}} = \text{rank} \left[\tilde{\mathbf{W}}; \widetilde{\boldsymbol{F}} \right] = (N+1)$ in Eq. (20), the unknown Chelyshkov coefficients matrix can be found as

$$\boldsymbol{A} = \widetilde{\boldsymbol{W}}^{-1} \widetilde{\boldsymbol{F}}.$$

Thus, we obtain the approximate solution as

$$y(x) \cong y_N(x) = \boldsymbol{X}(x) \boldsymbol{\mathcal{C}} \boldsymbol{A} .$$

3. RESIDUAL ERROR ANALYSIS

Now, we develop an efficient error estimation for our method and also a technique to obtain the improved solution (high accuracy solution) of the problem (4) and (5) by using the residual correction method [38-40]. According to our purpose, we define the residual function for the present method as

$$R_N(x) = \sum_{h=0}^{m_1} \sum_{j=0}^{m_2} P_{hj}(x) y_N^{(h)}(\alpha_{hj} x + \beta_{hj}) + \sum_{r=0}^{m_3} \sum_{s=0}^{m_4} \int_{u_{rs}(x)}^{\vartheta_{rs}(x)} K_{rs}(x,t) y_N^{(r)}(\mu_{rs} x + \lambda_{rs}) dt - f(x) \tag{21}$$

where $y_N(x)$ is the approximate solution of the problem (4) and (5). Note that, $y_N(x)$ satisfies the following problem

$$\begin{gathered} \sum_{h=0}^{m_1} \sum_{j=0}^{m_2} P_{hj}(x) y_N^{(h)}(\alpha_{hj} x + \beta_{hj}) + \sum_{r=0}^{m_3} \sum_{s=0}^{m_4} \int_{u_{rs}(x)}^{\vartheta_{rs}(x)} K_{rs}(x,t) y_N^{(r)}(\mu_{rs} x + \lambda_{rs}) dt = f(x) + R_N(x) \\ \sum_{h=0}^{m_1 - 1} \left(a_{ih} y_N^{(h)}(a) + b_{ih} y_N^{(h)}(b) \right) = \eta_i . \end{gathered} \tag{22}$$

Also, the error function $e_N(x)$ can be defined as

$$e_N(x) = y(x) - y_N(x), \tag{23}$$

where $y(x)$ is the exact solution of the problem (4) and (5). Substituting Eq. (23) into problem (4) and (5) with the help of Eqs. (21) and (22), we have the error differential equation with homogenous conditions:

$$\sum_{h=0}^{m_1}\sum_{j=0}^{m_2} P_{hj}(x)e_N^{(h)}(\alpha_{hj}x+\beta_{hj})+\sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\int_{u_{rs}(x)}^{\vartheta_{rs}(x)} K_{rs}(x,t)e_N^{(r)}(\mu_{rs}x+\lambda_{rs})dt=-R_N(x)$$
$$\sum_{h=0}^{m_1-1}\left(a_{ih}e_N^{(h)}(a)+b_{ih}e_N^{(h)}(b)\right)=0. \qquad (24)$$

Solving the problem (24) in the same way as in Section 2, we have the approximation of $e_{N,M}(x)$ to $e_N(x), M>N$ which is the error function based on the residual function $R_N(x)$.

Consequently, by means of the Chelyshkov polynomials $y_N(x)$ and $e_{N,M}(x)$, we obtain the improved approximate solution so that

$$y_{N,M}(x)=y_N(x)+e_{N,M}(x)\ .$$

4. NUMERICAL EXAMPLES

In this section, some numerical examples on the problem (4) and (5) are given to illustrate the accuracy and effectiveness of the method.

Example 4.1 Consider the second order integro-differential-functional equation

$$2y(0.5x+1)+(x-1)y'(x-0.75)+\int_{x-1}^{x} e^{x+t}y'(0.5t)dt-\int_{e^{x-0.5}}^{e^x} xty''(t-0.4)dt=5x^2+3x+10+e^{2x}(1+e^{-1}), \qquad (25)$$

with initial conditions $y(0)=0$ and $y'(0)=3$. Here $\alpha_{00}=0.5,\ \beta_{00}=1,\ P_{00}(x)=2, \alpha_{10}=1,$ $\beta_{10}=-0.75,\ P_{10}=x-1,\ \vartheta_{10}=x,\ u_{10}=x-1,\ K_{10}(x,t)=e^{x+t},\ \mu_{10}=0.5,\ \lambda_{10}=0,\ \vartheta_{20}=e^x, u_{20}=e^{x-0.5}$ $K_{20}(x,t)=xt, \mu_{20}=1, \lambda_{20}=-0.4$ and $f(x)=5x^2+3x+10+e^{2x}(1+e^{-1})$.

The exact solution of the problem is $y(x)=3x+2x^2$. Now, let us seek the Chelyshkov polynomial solution in the form

$$y_2(x)=\sum_{n=0}^{2} a_n C_n(x)$$

for $N=2$ in [0,1]. Then, the collocation points are $x_0=0,\ x_1=\frac{1}{2}$ and $x_3=1$ and the fundamental matrix equation is defined by

$$\left\{\boldsymbol{P}_{00}\boldsymbol{X}\boldsymbol{M}(\alpha_{00},\beta_{00})\boldsymbol{\mathcal{C}}+\boldsymbol{P}_{10}\boldsymbol{X}\boldsymbol{M}(\alpha_{10},\beta_{10})\boldsymbol{B}\boldsymbol{\mathcal{C}}+\mathbf{C}\overline{\mathbf{K}}^{C}_{10}\overline{\boldsymbol{Q}}_{10}\overline{\boldsymbol{M}}(\mu_{10},\lambda_{10})\overline{\boldsymbol{B}}\overline{\boldsymbol{\mathcal{C}}}-\mathbf{C}\overline{\mathbf{K}}^{C}_{20}\overline{\boldsymbol{Q}}_{20}\overline{\boldsymbol{M}}(\mu_{20},\lambda_{20})\overline{\boldsymbol{B}}^{2}\overline{\boldsymbol{\mathcal{C}}}\right\}\boldsymbol{A}=\boldsymbol{F},$$

where

$$\boldsymbol{P}_{00}=\mathrm{Diag}[1\quad 1\quad 1],\ \boldsymbol{P}_{10}=\mathrm{Diag}[-1\quad -1/2\quad 0],\ \boldsymbol{X}=[\boldsymbol{X}(0)\quad \boldsymbol{X}(1/2)\quad \boldsymbol{X}(1)]^T,$$

$$\boldsymbol{X}(0)=[1\quad 0\quad 0],\quad \boldsymbol{X}(1/2)=[1\quad 1/2\quad 1/4],\ \boldsymbol{X}(1)=[1\quad 1\quad 1],$$

$$\boldsymbol{M}(\alpha_{00},\beta_{00})=\boldsymbol{M}(0.5,1)=\begin{bmatrix}1 & 1 & 1\\ 0 & 0.5 & 1\\ 0 & 0 & 0.25\end{bmatrix},\ \boldsymbol{M}(\alpha_{10},\beta_{10})=\boldsymbol{M}(1,-0.75)=\begin{bmatrix}1 & -0.75 & 0.5625\\ 0 & 1 & -1.5\\ 0 & 0 & 1\end{bmatrix},$$

$$\overline{\boldsymbol{M}}(\mu_{10},\lambda_{10})=\overline{\boldsymbol{M}}(0.5,0)=\text{Diag}[\boldsymbol{M}(0.5,0)\quad \boldsymbol{M}(0.5,0)\quad \boldsymbol{M}(0.5,0)],\ \boldsymbol{M}(0.5,0)=\text{Diag}[1\quad 0.5\quad 0.25],$$

$$\overline{\boldsymbol{M}}(\mu_{20},\lambda_{20})=\overline{\boldsymbol{M}}(1,-0.4)=\text{Diag}[\boldsymbol{M}(1,-0.4)\quad \boldsymbol{M}(1,-0.4)\quad \boldsymbol{M}(1,-0.4)],\ \boldsymbol{M}(1,-0.4)=\begin{bmatrix}1 & -0.4 & 0.16\\ 0 & 1 & -0.8\\ 0 & 0 & 1\end{bmatrix},$$

$$\overline{\boldsymbol{Q}}_{10}=\text{Diag}\left[\boldsymbol{Q}_{10}\quad \boldsymbol{Q}_{10}\quad \boldsymbol{Q}_{10}\right],\boldsymbol{Q}_{10}=\text{Diag}\left[\boldsymbol{Q}_{10}(0)\quad \boldsymbol{Q}_{10}(0.5)\quad \boldsymbol{Q}_{10}(1)\right],\ \overline{\mathbf{K}}_{10}^{C}=\text{Diag}\left[\mathbf{K}_{10}^{C}\quad \mathbf{K}_{10}^{C}\quad \mathbf{K}_{10}^{C}\right],$$

$$\mathcal{C}=\begin{bmatrix}1 & 2 & 3\\ 0 & -3 & -12\\ 0 & 0 & 10\end{bmatrix},\ \mathbf{C}=\text{Diag}\left[\mathbf{C}(0)\quad \mathbf{C}(1/2)\quad \mathbf{C}(1)\right],\ \overline{\boldsymbol{B}}=\text{Diag}\left[\boldsymbol{B}\quad \boldsymbol{B}\quad \boldsymbol{B}\right],\ \boldsymbol{B}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 2\\ 0 & 0 & 0\end{bmatrix},$$

$$\overline{\mathcal{C}}=\left[\mathcal{C}\quad \mathcal{C}\quad \mathcal{C}\right],\ \boldsymbol{A}=\left[a_0\quad a_1\quad a_2\right]^T \text{ and } \boldsymbol{F}=\left[f(0)\quad f(0.5)\quad f(1)\right]^T.$$

The augmented matrix for this fundamental matrix equation is calculated as

$$\left[\boldsymbol{W};\boldsymbol{F}\right]=\begin{bmatrix}2 & -1 & 18.0833333332 & ; & 11+\mathrm{e}^{-1}\\ 2 & -7.078125000 & -11.799742476 & ; & 55/4+\mathrm{e}\\ 2 & -17.50000000 & 57.7494093713 & ; & 18+\mathrm{e}+\mathrm{e}^2\end{bmatrix}.$$

From Eq.(19), the matrix form for initial conditions is computated as

$$\left[\boldsymbol{U};\eta_i\right]=\begin{bmatrix}1 & 2 & 3 & ; & 0\\ 0 & -3 & -12 & ; & 3\end{bmatrix}.$$

Hence, the new augmented matrix based on conditions from Eq. (20) can be obtained as follows:

$$\left[\widetilde{\boldsymbol{W}};\widetilde{\boldsymbol{F}}\right]=\begin{bmatrix}2 & -1 & 18.0833333332 & ; & 11+\mathrm{e}^{-1}\\ 1 & 2 & 3 & ; & 0\\ 0 & -3 & -12 & ; & 3\end{bmatrix}.$$

By solving this system, substituting the resulting unknown Chelyshkov coefficients matrix into Eq.(6), we obtain the following approximation solution

$$y_2(x)=3x+1.9847935920529x^2.$$

The error problem for this solution becomes

$$\begin{cases}2e_2(0.5x+1)+(x-1)e_2'(x-0.75)+\int_{x-1}^{x}e^{x+t}e_2'(0.5t)dt-\int_{e^{x-0.5}}^{e^x}xte_2''(t-0.4)dt=-R_2(x),\ 0\le \text{x,t}\le 1\\ e_2(0)=e_2'(0)=0.\end{cases}$$

By means of similar process for $M=3,$ we get the approximate solution of the error problem as follows:

$$e_{2,3}(x)=0.015173438522945x^2+0.000039704166532347x^3 .$$

Thus, we have the improved approximate solution such as

$$y_{2,3}(x)=3x+1.9999670305758x^2+0.39704166532347.10^{-4}x^3 .$$

Table 1 demonstrates the comparison of the exact and approximate solutions for $N=2,4$ values. The results of residual error function support the evaluation of the improved new Chelyshkov polynomial solutions for various values of *N*.

Table 1. Numerical solutions of Eq. (25) for (N,M)=(2,3), (4,5).

	Exact Solution	Approximate solution		Corrected approximate solution	
x_i	$y(x_i)$	$y_2(x_i)$	$y_4(x_i)$	$y_{2,3}(x_i)$	$y_{4,5}(x_i)$
0.0	0	0	0	0	0
0.2	0.680000	0.67939174	0.68000838	0.67999900	0.67999993
0.4	1.520000	1.51756697	1.52003688	1.51999727	1.51999998
0.6	2.520000	2.51452569	2.52007289	2.51999671	2.51999998
0.8	3.680000	3.67026790	3.68008030	3.67999923	3.68000001
1.0	5.000000	4.98479359	4.99999950	5.00000673	5.00000007

Table 2 displays the corrected absolute errors by our method for $N=2,4$ and $M=3,5$. The results support the idea that when *N* and *M* values are chosen large enough, the absolute error and residual error are diminished.

Table 2. Comparisons of the absolute errors for (N,M)=(2,3), (4,5) of Eq. (25)

	Actual absolute error		Estimated absolute errors	
x_i	$\lvert e_2(x_i)\rvert$	$\lvert e_4(x_i)\rvert$	$\lvert e_{2,3}(x_i)\rvert$	$\lvert e_{4,5}(x_i)\rvert$
0.0	0	0	0	0
0.2	$6.08256318.10^{-4}$	$8.38458115.10^{-6}$	$1.00114364.10^{-6}$	$7.16550000.10^{-9}$
0.4	$2.43302527.10^{-3}$	$3.68837753.10^{-5}$	$2.73404120.10^{-6}$	$1.87290000.10^{-8}$
0.6	$5.47430686.10^{-3}$	$7.28904567.10^{-5}$	$3.29289270.10^{-6}$	$1.84180000.10^{-8}$
0.8	$9.73210109.10^{-3}$	$8.02970968.10^{-5}$	$7.71898200.10^{-7}$	$8.63170000.10^{-9}$
1.0	$1.52064079.10^{-2}$	$5.04235200.10^{-7}$	$6.73474230.10^{-6}$	$7.18256000.10^{-8}$

Example 4.2 [39]. Let us consider the linear delay integro-differential equations with variable coefficients given by

$$y^{(4)}(x)-y^{(2)}(x-1)+xy(x+0.5)=f(x)+\int_0^1 x\cos(t)y'(t+1)dt+\int_0^1 x\sin(t)y^{(2)}(t-0.5)dt,\ \ 0\le x,t\le 1 \qquad (26)$$

with the initial conditions $y(0)=0,\ y^{(1)}(0)=1,\ y^{(2)}(0)=0,\ y^{(3)}(0)=-1$ and the exact solution $y(x)=\sin x.$ Here,

$$P_{40}(x)=P_{20}(x)=1,\ P_{00}(x)=x,\ \alpha_{40}=\alpha_{20}=\alpha_{00}=1,\ \beta_{40}=0,\ \beta_{20}=-1,\ \beta_{00}=0.5,\ \lambda_{20}=-0.5,$$
$$\vartheta_{10}=\vartheta_{20}=1,\ u_{10}=u_{20}=0,\ K_{10}(x,t)=x\cos(t),\ K_{20}(x,t)=x\sin(t),\ \ \mu_{10}=\ \mu_{20}=\lambda_{10}=1,$$
$$f(x)=\sin(x)+\sin(x-1)+x\sin(x+0.5)+\frac{x}{2}\big(\cos(0.5)-\cos(1)\big)+\frac{x}{4}\big(\sin(1)-\sin(3)-\sin(0.5)-\sin(1.5)\big).$$

From Eq.(17), the fundamental matrix equation of the problem is

$$\left\{ \mathbf{P}_{40}\mathbf{X}\mathbf{M}(\alpha_{40},\beta_{40})\mathbf{B}^4\mathcal{C} - \mathbf{P}_{20}\mathbf{X}\mathbf{M}(\alpha_{20},\beta_{20})\mathbf{B}^2\mathcal{C} + \mathbf{P}_{00}\mathbf{X}\mathbf{M}(\alpha_{00},\beta_{00})\mathcal{C} - \overline{\mathbf{C}}\mathbf{K}_{10}^{C}\mathbf{Q}_{10}\mathbf{M}(\mu_{10},\lambda_{10})\mathbf{B}\mathcal{C} - \overline{\mathbf{C}}\mathbf{K}_{20}^{C}\mathbf{Q}_{20}\mathbf{M}(\mu_{20},\lambda_{20})\mathbf{B}^2\mathcal{C} \right\} \mathbf{A} = \mathbf{F}$$

and by following procedure in Example 4.1, we get the approximate solutions.

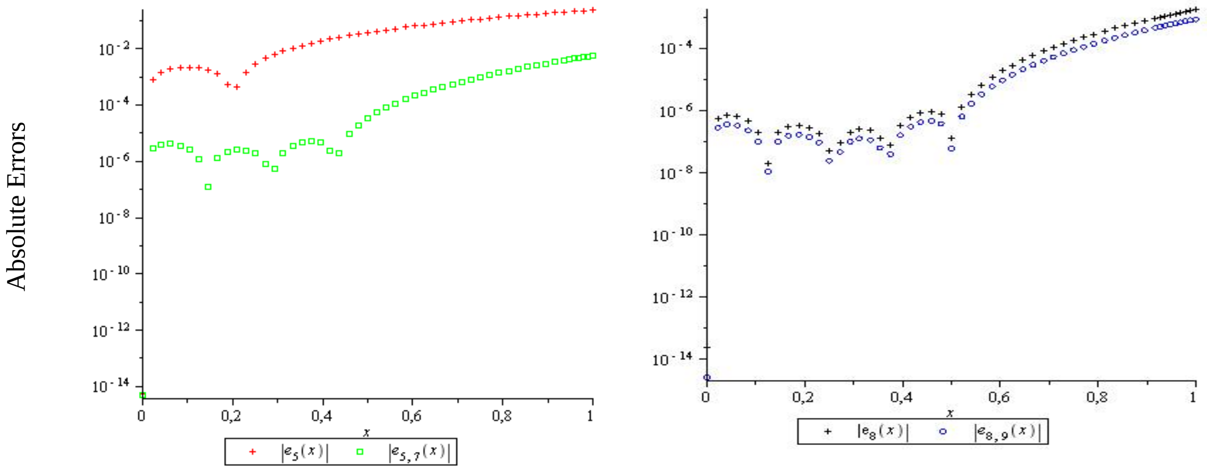
In Table 3, we give numerical results obtained by using the presented method for different values of N and M for Example 4.2.

Table 3. Numerical results of the exact and the approximate solutions of Eq. (26) for N=5,8 and M=7,9.

x_i	Exact Value	Approximate solutions		Corrected approximate solutions	
	$\sin x_i$	$y_5(x_i)$	$y_8(x_i)$	$y_{5,7}(x_i)$	$y_{8,9}(x_i)$
0.0	0	0	0	0	0
0.2	0.19866933079506	0.19866495974607	0.19866822899025	0.19866925803630	0.1986692949292
0.4	0.38941834230865	0.38935586432582	0.38940218096657	0.38941723620881	0.3894178171866
0.6	0.56464247339504	0.56436638686758	0.56456717548223	0.56463711589275	0.5646400322275
0.8	0.71735609089952	0.71662209765348	0.71713607608036	0.71733971568804	0.7173489769144
1.0	0.84147098480790	0.84005427703524	0.84097189357243	0.84143143981590	0.8414548995521

Table 4. Comparison of the absolute errors for N=5,8 and M=7,9.

x_i	Müntz-Legendre Method				Present method			
	$\|e_5(x_i)\|$	$\|e_8(x_i)\|$	$\|e_{5,7}(x_i)\|$	$\|e_{8,9}(x_i)\|$	$\|e_5(x_i)\|$	$\|e_8(x_i)\|$	$\|e_{5,7}(x_i)\|$	$\|e_{8,9}(x_i)\|$
0.0	1.2406e-015	2.2329e-015	1.7745e-019	6.2660e-019	0	0	0	0
0.2	1.2020e-005	9.2512e-007	1.2815e-005	1.3853e-006	4.3710e-006	1.1018e-006	7.2759e-008	3.5866e-008
0.4	1.6817e-004	1.1910e-005	1.8080e-004	2.4918e-005	6.2478e-005	1.6161e-005	1.1061e-006	5.2512e-007
0.6	7.2667e-004	1.1940e-004	7.9102e-004	1.4216e-004	2.7609e-004	7.5298e-005	5.3575e-006	2.4412e-006
0.8	1.8910e-003	4.5459e-004	2.0979e-003	5.0473e-004	7.3399e-004	2.2001e-004	1.6375e-005	7.1141e-006
1.0	3.5897e-003	1.3046e-003	4.1067e-003	1.3761e-003	1.4167e-003	4.9909e-004	3.9545e-005	1.6085e-005



a) Comparison of the error functions $|e_5(x)|$ and $|e_{5,7}(x)|$

b) Comparison of the error functions $|e_8(x)|$ and $|e_{8,9}(x)|$

Figure 1. Comparison of the error functions $|e_N(x)|$ and the estimated error functions $|e_{N,M}(x)|$ for N=5,8 and M=7,9

In Table 4, the absolute errors are compared with results obtained by the Müntz-Legendre polynomial solution [39] and our method, whereas a comparison of the estimated absolute error functions and the actual absolute error functions for the present method is given in Fig. 1. From Table 4, we observe that the present method gives better results without using the residual correction technique according to other method.

Example 4.3 [40]. Finally, we solve the problem

$$y'(x) = 2e^{1-x} - 3y(x) - \int_{x-1}^{x} y(t)dt - \int_{x-1}^{x} y'(t)dt, \qquad (27)$$

with the initial condition, $y(0)=1$ on the interval $[0,2]$. The exact solution of this problem is $y(x)=e^{-x}$. Here, $P_{10}(x)=1,\ P_{00}(x)=3,\ \alpha_{10}=\alpha_{00}=1, \beta_{10}=\beta_{00}=0,\ \vartheta_{10}=\vartheta_{00}=x,$ $u_{10}=u_{00}=x-1, K_{10}(x,t)=1, K_{00}(x,t)=3,\ \mathrm{f}(x)=2e^{1-x}.$

The main matrix equation of the problem is witten as

$$\left\{\boldsymbol{P}_{00}\boldsymbol{XM}(\alpha_{00},\beta_{00})\boldsymbol{\mathcal{C}} + \boldsymbol{P}_{10}\boldsymbol{XM}(\alpha_{10},\beta_{10})\boldsymbol{B\mathcal{C}} + \mathbf{C}\overline{\mathbf{K}}_{10}^{C}\overline{\boldsymbol{Q}}_{10}\overline{\boldsymbol{M}}(\mu_{10},\lambda_{10})\overline{\boldsymbol{\mathcal{C}}} + \mathbf{C}\overline{\mathbf{K}}_{10}^{C}\overline{\boldsymbol{Q}}_{10}\overline{\boldsymbol{M}}(\mu_{10},\lambda_{10})\overline{\boldsymbol{B\mathcal{C}}}\right\}\mathbf{A} = \boldsymbol{F}.$$

After applying our method for N=6,8,10 and M=8,9,10,13, we get the numerical results of the approximate solutions in Table 5.

Table 5. Numerical results of the exact and the approximate solutions for *N*=6,8,10 and *M*=8,9,10,13.

x_i	Exact solution	Approximate solutions			Corrected approximate solutions			
	e^{-x_i}	$y_6(x_i)$	$y_8(x_i)$	$y_{10}(x_i)$	$y_{6,8}(x_i)$	$y_{8,9}(x_i)$	$y_{10,10}(x_i)$	$y_{10,13}(x_i)$
0.2	0.8187317531	0.8187631952	0.8187303359	0.818729259	0.8187304835	0.8187305136	0.8187307258	0.8187307532
0.4	0.6703200460	0.6703598623	0.6703189733	0.670318210	0.6703190776	0.6703197779	0.6703200426	0.6703200459
0.6	0.5488116361	0.5488352148	0.5488105197	0.548810534	0.5488105138	0.5488114972	0.5488116614	0.5488116357
0.8	0.4493289641	0.4493303625	0.4493283456	0.449328826	0.4493282708	0.4493289700	0.4493289972	0.4493289638
1.0	0.3678794412	0.3678694310	0.3678793800	0.367879830	0.3678793106	0.3678795133	0.3678794620	0.3678794409
1.2	0.3011942119	0.3011869615	0.3011944259	0.301194586	0.3011943999	0.3011942693	0.3011942180	0.3011942116
1.4	0.2465969639	0.2465955930	0.2465971660	0.246597060	0.2465971722	0.2465969749	0.2465969620	0.2465969644
1.6	0.2018965180	0.2018940249	0.2018965522	0.201896378	0.2018965716	0.2018964967	0.2018965180	0.2018965176
1.8	0.1652988882	0.1653152609	0.1652988003	0.165298714	0.1652988096	0.1652988666	0.1652988930	0.1652988886
2.0	0.1353352832	0.1355151345	0.1353367370	0.135334706	0.1353364548	0.1353351807	0.1353352990	0.1353352827

Also we give the exact and approximate solutions in Fig. 2. In Table 6, our absolute errors are compared with the results of the other methods [40,41] and maximum absolute errors from our methods in [0,2] is given in Table 7.

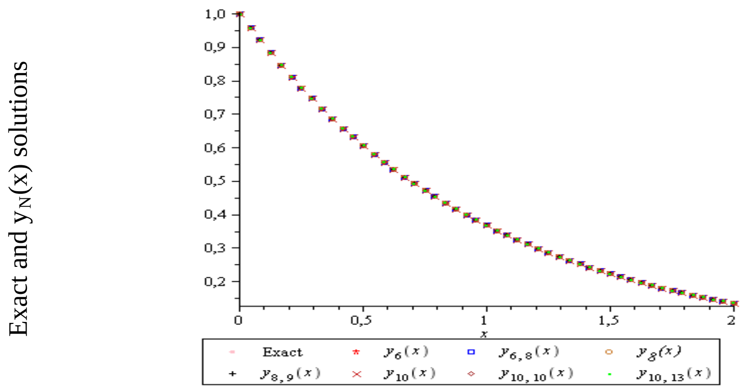


Figure 2. Comparison of the exact solution $y(x)$ and the approximate solutions $y_{N,M}(x)$ of Eq. (27)

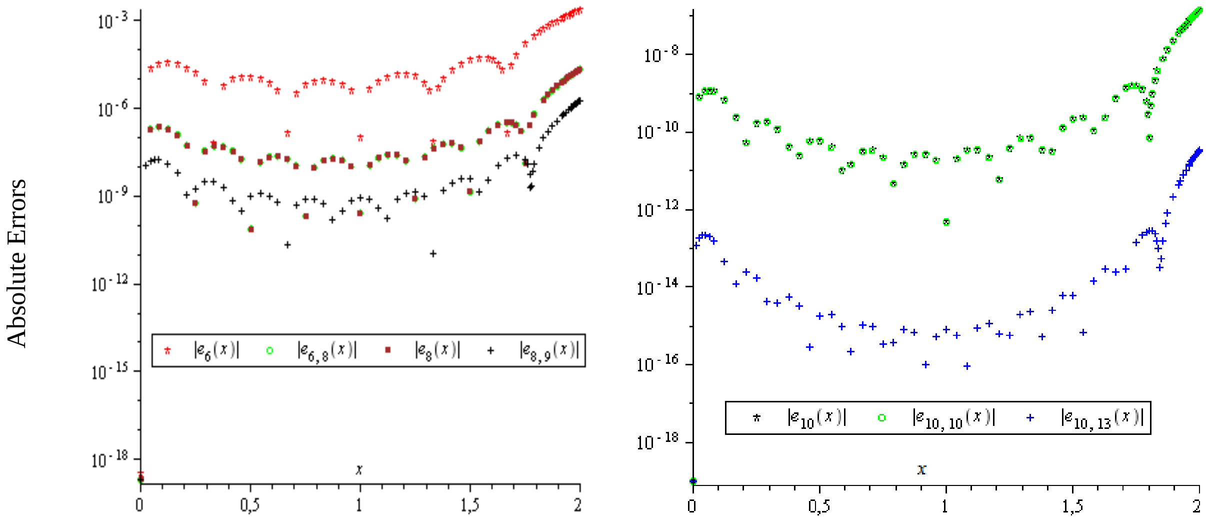
Table 6. Comparison of the maximum absolute errors obtained by the different methods in [0,2]

The Maximum absolute errors			
Gauss (by using the collocation points 30)	8.33e-09	Gauss (by using the collocation points 60)	1.30e-10
Radau II (by using the collocation points 30)	1.43e-07	Radau II (by using the collocation points 60)	4.67e-09
Obatto (by using the collocation points 30)	1.81e-06	Obatto (by using the collocation points 60)	1.13e-07
Gauss I (by using the collocation points 20)	9.10e-07	Gauss I (by using the collocation points 40)	6.57e-08
Other (by using the collocation points 30)	4.25e-06	Other (by using the collocation points 60)	5.85e-07
Laguerre (by using the collocation points 10, *N*=*M*=10)	4.807e-08	Laguerre (*N*=*M*=15)	2.930e-10
Our Method (by using the collocation points 10, *N*=*M*=10)	2.882e-08	Our Method (*N*=10, *M*=13)	2.979e-10

Table 7. The maximum errors for the estimated absolute error functions $|e_{N,M}(x)|$ of Eq. (27)

(N,M)	$(6,8)$	$(8,9)$	$(10,10)$	$(10,13)$
Error	1,1743e-006	2.6853e-007	2.8821e-008	2.9789e-010

It follows from Tables 6 and 7 that our method is more effective than most other methods for the lower values of N. In Fig. 3 is given a comparison of the estimated absolute error functions and the estimated absolute error functions for the present method.



a) Comparison of the error functions $|e_6(x)|$, $|e_{6,8}(x)|$,$|e_8(x)|$ and $|e_{8,9}(x)|$

b) Comparison of the error functions $|e_{10}(x)|$, $|e_{10,10}(x)|$ and $|e_{10,13}(x)|$

Figure 3. Comparison of the absolute error functions $|e_N(x)|$ and the estimated absolute error functions $|e_{N,M}(x)|$ for *N*=6,8,10 and *M*=8,9,10,13

5. CONCLUSION

In this paper, we have presented a modified collocation method and used it for the mentioned linear functional integro-differential equations with variable coefficients and bounds. The comparison of the obtained results shows that the present method is a powerful mathematical tool for finding the numerical solutions of these type equations. One of the considerable advantages of the method is that the approximate solutions are found very easily by using avaible software since the method is based on matrix operations. Moreover, the proposed method in this work can be adapted to solve the systems of these functional differential equations which play an important role in physics, biology and engineering.

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