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SOME OPERATORS ARISING FROM SCHWARZ BVP IN COMPLEMENTARY LOCAL MORREY-TYPE SPACES ON THE UNIT DISC

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Abstract. In this paper, we prove the boundedness of a class of operators arising from Schwarz BVP in complementary local Morrey-type spaces in the unit disc of the complex plane.

1. Introduction

Let $\mathbb{C}$ be the complex plane and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in $\mathbb{C}$. The Schwarz boundary value problem (Schwarz BVP)

$$g_{\bar{z}} = f \text{ in } \mathbb{D}, \operatorname{Re} g = \gamma \text{ on } \partial\mathbb{D}, \operatorname{Im} g(0) = c, \tag{1.1}$$

is one of the major boundary value problems in complex analysis. It is uniquely solvable for analytic functions [24], and for polyanalytic functions [7]. The solvability of the Schwarz problem for some higher-order linear elliptic complex partial differential equations were investigated in [3] and [2].

Cauchy-Riemann-Poisson-Pompeiu formula given by

$$\begin{aligned} g(z) =& \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} + ic \\ &- \frac{1}{2\pi}\iint_{\mathbb{D}} \left(\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\right) d\xi d\eta, \quad z\in\mathbb{D}, \quad \zeta=\xi+i\eta \end{aligned} \tag{1.2}$$

is the unique solution to the Schwarz BVP, where $f \in L^1(\mathbb{D})$, $\gamma \in C(\partial\mathbb{D}, \mathbb{R})$, $c \in \mathbb{R}$ (see [7]).

The domain integral appearing on the right-hand side of (1.2), denoted by $\widetilde{T}_1$, is a modification of the Pompeiu operator

$$T_1 f(z) := -\frac{1}{\pi}\iint_{\mathbb{D}} f(\zeta)\frac{d\xi d\eta}{\zeta - z}, \quad z \in \mathbb{D},$$

which was studied by Vekua in [26]. The operator $\widetilde{T}_1$ is important for treating complex first-order equations (see, for instance, [26, 11, 4, 5]). Iterating this operator

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with itself by the rule $\widetilde{T}_k f(z) = \widetilde{T}_1(\widetilde{T}_{k-1} f(z))$ generates the operators

$$\widetilde{T}_k f(z) := \frac{(-1)^k}{2\pi(k-1)!} \iint\limits_{\mathbb{D}} \left(\zeta - z + \overline{\zeta - z}\right)^{k-1} \left(\frac{f(\zeta)}{\zeta} \frac{\zeta + z}{\zeta - z} + \frac{\overline{f(\zeta)}}{\bar{\zeta}} \frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}} \right) d\xi d\eta$$

for $k \in \mathbb{N}$ with $\widetilde{T}_0 f(z) = f(z)$.

The two partial derivative operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ are defined by

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and

$$\frac{\partial^l}{\partial z^l} = \frac{\partial}{\partial z}\left(\frac{\partial^{l-1}}{\partial z^{l-1}}\right), \quad \frac{\partial^l}{\partial \bar{z}^l} = \frac{\partial}{\partial \bar{z}}\left(\frac{\partial^{l-1}}{\partial \bar{z}^{l-1}}\right)$$

where $z = x + iy$.

The operators $\widetilde{T}_k f$ satisfy

$$\frac{\partial^l}{\partial \bar{z}^l} \widetilde{T}_k f = \widetilde{T}_{k-l} f, \quad 1 \le l \le k, \tag{1.3}$$

$$\operatorname{Re} \frac{\partial^l}{\partial \bar{z}^l} \widetilde{T}_k f = 0 \quad \text{on} \quad \partial\mathbb{D}, \quad 0 \le l \le k-1, \tag{1.4}$$

$$\operatorname{Im} \frac{\partial^l}{\partial \bar{z}^l} \widetilde{T}_k f(0) = 0, \quad 0 \le l \le k-1, \tag{1.5}$$

see [3]. Note that $\partial_z^l \widetilde{T}_k$ is a weakly singular integral operator for $0 \le l \le k-1$, while

$$\Pi_k f(z) := \frac{\partial^k}{\partial z^k} \widetilde{T}_k f(z) = \frac{(-1)^k k}{\pi} \iint_{\mathbb{D}} \left[\left(\frac{\overline{\zeta - z}}{\zeta - z} \right)^{k-1} \frac{f(\zeta)}{(\zeta - z)^2} + \left(\frac{\zeta - z + \overline{\zeta - z}}{1 - z\bar{\zeta}} \bar{\zeta} - 1 \right)^{k-1} \frac{\overline{f(\zeta)}}{(1 - z\bar{\zeta})^2} \right] d\xi d\eta \tag{1.6}$$

is a strongly singular integral operator. It is known that $\|\Pi_1\|_{L^2(\mathbb{D})} = 1$ (see [26, 11]). Π_k are shown to be bounded in the space L^p for $1 < p < \infty$ and in particular their L^2 norms are estimated in [1]. These operators are investigated by decomposing them into two parts as $\Pi_k = T_{-k,k} + P_k$, where

$$T_{-k,k} f(z) = \frac{(-1)^k k}{\pi} \iint_{\mathbb{D}} \left(\frac{\overline{\zeta - z}}{\zeta - z} \right)^{k-1} \frac{f(\zeta)}{(\zeta - z)^2} \, d\xi d\eta, \tag{1.7}$$

and

$$P_k f(z) = \frac{(-1)^k k}{\pi} \iint_{\mathbb{D}} \left(\frac{\zeta - z + \overline{\zeta - z}}{1 - z\bar{\zeta}} \bar{\zeta} - 1 \right)^{k-1} \frac{\overline{f(\zeta)}}{(1 - z\bar{\zeta})^2} \, d\xi d\eta, \tag{1.8}$$

which are investigated extensively in [6] and [1], respectively, and the boundedness of $T_{-k,k}$ and P_k in $L_p(\mathbb{D})$ are proved.

It is mentioned in [6] that the integral in (1.7) must be viewed as a Cauchy principal value integral,

$$T_{-k,k} f(z) = \lim_{\varepsilon \to 0} \iint_{\mathbb{D}_\varepsilon} K_{-k,k}(z - \zeta) w(\zeta) \, d\xi d\eta, \tag{1.9}$$

where $\mathbb{D}_\varepsilon$ is the domain $\mathbb{D}\backslash\{\zeta : |\zeta - z| \leq \varepsilon\}$, and the limit is taken in the norm of $L^p(\mathbb{D})$. Here

$$K_{-k,k}(z) := \frac{(-1)^k k}{\pi} z^{-k-1}\bar{z}^{k-1}.$$

These integrals can be analyzed with the well-known theory of Calderón and Zygmund [9, 10, 22] concerning singular integrals. The boundedness of P_k in $L_p(\mathbb{D})$ was proved in [1] using Schur's test (see, for instance, [27]) and Forelli-Rudin lemma in [12].

The well-known Morrey spaces $\mathcal{M}_{p,\lambda}$ introduced by C.B. Morrey in 1938 [19] in relation to the study of partial differential equations, were widely investigated during the last decades, including the study of classical operators of harmonic and real analysis - maximal, singular and potential operators - in generalizations of these spaces (so-called local Morrey-type spaces).The local Morrey-type spaces and the complemenary local Morrey-type spaces introduced by Guliyev in the doctoral thesis [16] (see also [17]). The main purpose of [16] (also of [17, 18]) is to give some sufficient conditions for the boundedness of fractional integral operators and singular integral operators in complementary local Morrey-type spaces ${}^{^c}LM_{p\theta,\omega}(G)$ defined on homogeneous Lie groups G.

The research on complementary local Morrey-type spaces mainly includes the study of classical operators in these spaces (see, for instance, [8]). However, recently in a series of papers, authors started to study the structure of complementary local Morrey-type spaces and relation of these spaces with other known function spaces (see, for instance, [13, 14, 20]).

The aim of this paper is to study the boundedness of integral operators (1.6) in ${}^{^c}LM_{p\theta,\omega}(\mathbb{D})$. Our main result is Theorem 5.1. This statement allows us to obtain apriori estimate for the solution of Schwarz BVP (1.1) with $\gamma = c = 0$ in ${}^{^c}LM_{p\theta,\omega}(\mathbb{D})$ (see Theorem 5.3).

The paper is organized as follows. Some notations and definitions are given in Section 2. Some local estimates of sublinear operators satisfying Soria-Weiss condition (see (3.1) below) are obtained in Section 3 (see Theorem 3.1). The boundedness of such operators in complementary local Morrey-type spaces are proved in Section 4 (see Theorem 4.1). Finally our main results are presented in Section 5.

2. Notations and Preliminaries

Now we make some conventions. Throughout the paper, we always denote by c or C a positive constant which is independent of the main parameters, but it may vary from line to line. However a constant with subscript such as c_1 does not change in different occurrences. By $a \lesssim b$, ($b \gtrsim a$) we mean that $a \leq \lambda b$, where $\lambda > 0$ depends on inessential parameters. If $a \lesssim b$ and $b \lesssim a$, we write $a \approx b$ and say that a and b are equivalent. For a measurable set E, χ_E denotes the characteristic function of E. We define the Lebesgue measure of E by $|E|$. For $0 < \rho < 1$, let $B(z,\rho) := \{\zeta \in \mathbb{D} : |z - \zeta| < \rho\}$ be the open ball centered at $z \in \mathbb{D}$ of radius ρ and ${}^{^c}B(z,\rho) := {}^{^c}B(z,\rho)$.

The symbol $\mathfrak{M}^+$ stands for the collection of all measurable functions on $(0,\infty)$ which are non-negative, while $\mathfrak{M}^\downarrow$ is used to denote the subset of those functions which are non-increasing on $(0,\infty)$.

For $0 < p \leq \infty$ and w a weight function on a measurable subset E of $\mathbb{C}$, that is, locally integrable real-valued non-negative function on E, let us denote by

$L_{p,w}(E)$ the weighted Lebesgue space defined as the set of all measurable functions $f : E \to \mathbb{C}$ for which the quantity

$$\|f\|_{L_{p,w}(E)} = \begin{cases} \left(\iint_E |f(\zeta)|^p w(\zeta)\, d\xi d\eta\right)^{\frac{1}{p}} & \text{for } p < \infty, \\ \operatorname*{ess\,sup}_{\zeta \in E} |f(\zeta)| w(\zeta) & \text{for } p = \infty \end{cases} \tag{2.1}$$

is finite. When $w \equiv 1$, we write simply $L_p(E)$ and $\|\cdot\|_{L_p(E)}$ instead of $L_{p,w}(E)$ and $\|\cdot\|_{L_{p,w}(E)}$.

Recall the definition of weak Lebesgue space:

$$WL_p(E) := \left\{ f : E \to \mathbb{C} \text{ meas.} : \|f\|_{WL_p(E)} := \sup_{t>0} t \, |\{z \in E : |f(z)| > t\}|^{\frac{1}{p}} < \infty \right\}.$$

Convention 2.1. We adopt the following conventions:

- Throughout the paper we put $0 \cdot \infty = 0$, $\infty/\infty = 0$ and $0/0 = 0$.
- For a fixed p with $p \in [1, \infty]$, p' denoted the dual exponent of p, namely,

$$p' := \begin{cases} \infty & \text{if } \; p = 1, \\ \frac{p}{p-1} & \text{if } \; 1 < p < \infty, \\ 1 & \text{if } \; p = \infty. \end{cases} \tag{2.2}$$

Recall the following complete characterization of the weighted Hardy inequality on the cone of non-increasing functions. We will use the notations:

$$U(t) := \int_0^t u(x)\, dx, \qquad V(t) := \int_0^t v(x)\, dx, \qquad W_*(t) := \int_t^\infty w(x)\, dx, \quad t > 0.$$

Theorem 2.2 ([15], Theorems 2.5, 3.15, 3.16)**.** *Let $0 < q, p \le \infty$ and u, v, w be weight functions on $(0, \infty)$. Then inequality*

$$\|H_u(f)\|_{L_{q.w}(0,\infty)} \le c \|f\|_{L_{p,v}(0.\infty)}, \quad f \in \mathfrak{M}^{\downarrow}, \tag{2.3}$$

where

$$H_u g(t) := \int_0^t g(s) u(s)\, ds, \qquad g \in \mathfrak{M}^+,$$

with the best constant c holds if and only if the following holds:

(i) $1 < p \le q < \infty$ *and* $A_0 + A_1 < \infty$, *where*

$$A_0 := \sup_{t>0} \left(\int_0^t U^q(\tau) w(\tau)\, d\tau \right)^{\frac{1}{q}} V^{-\frac{1}{p}}(t),$$

$$A_1 := \sup_{t>0} W_*^{\frac{1}{q}}(t) \left(\int_0^t \left(\frac{U(\tau)}{V(\tau)} \right)^{p'} v(\tau)\, d\tau \right)^{\frac{1}{p'}},$$

and in this case $c \approx A_0 + A_1$;

(ii) $\max\{q, 1\} < p < \infty$ *and* $B_0 + B_1 < \infty$, *where*

$$B_0 := \left(\int_0^\infty V^{-\frac{r}{p}}(t) \left(\int_0^t U^q(\tau) w(\tau)\, d\tau \right)^{\frac{r}{p}} U^q(t) w(t)\, dt \right)^{\frac{1}{r}},$$

$$B_1 := \left(\int_0^\infty W_*^{\frac{r}{p}}(t) \left(\int_0^t \left(\frac{U(\tau)}{V(\tau)} \right)^{p'} v(\tau)\, d\tau \right)^{\frac{r}{p'}} w(t)\, dt \right)^{\frac{1}{r}},$$

and in this case $c \approx B_0 + B_1$;

(iii) $q < p \le 1$ *and* $B_0 + C_0 < \infty$, *where*

$$C_0 : = \left(\int_0^\infty \left(\operatorname*{ess\,sup}_{\tau \in (0,t)} \frac{U^p(\tau)}{V(\tau)} \right)^{\frac{r}{p}} W_*^{\frac{r}{p}}(t) w(t)\, dt \right)^{\frac{1}{r}},$$

and in this case $c \approx B_0 + C_0$;

(iv) $p \le \min\{q, 1\} < \infty$ *and* $D_0 < \infty$, *where*

$$D_0 := \sup_{t>0} V^{-\frac{1}{p}}(t) \left(\int_0^\infty U^q(\min\{\tau, t\}) w(\tau)\, d\tau \right)^{\frac{1}{q}},$$

and in this case $c = D_0$;

(v) $p \le 1$ *and* $q = \infty$ *and* $E_0 < \infty$, *where*

$$E_0 := \operatorname*{ess\,sup}_{t>0} V^{-\frac{1}{p}}(t) \left(\operatorname*{ess\,sup}_{\tau>0} U(\min\{\tau, t\}) w(\tau) \right),$$

and in this case $c = E_0$;

(vi) $1 < p < \infty$ *and* $q = \infty$ *and* $F_0 < \infty$, *where*

$$F_0 := \operatorname*{ess\,sup}_{t>0} w(t) \left(\int_0^t \left(\int_\tau^t u(y) V^{-1}(y)\, dy \right)^{p'} v(\tau)\, d\tau \right)^{\frac{1}{p'}},$$

and in this case $c = F_0$;

(vii) $p = \infty$ *and* $0 < q < \infty$ *and* $G_0 < \infty$, *where*

$$G_0 := \left(\int_0^\infty \left(\int_0^t \frac{u(y)\, dy}{\operatorname{ess\,sup}_{\tau \in (0,y)} v(\tau)} \right)^q w(t)\, dt \right)^{\frac{1}{q}},$$

and in this case $c = G_0$;

(viii) $p = q = \infty$ *and* $H_0 < \infty$, *where*

$$H_0 := \operatorname*{ess\,sup}_{t>0} \left(\int_0^t \frac{u(y)\, dy}{\operatorname{ess\,sup}_{\tau \in (0,y)} v(\tau)} \right) w(t),$$

and in this case $c = H_0$.

For the sake of completeness we recall the definition of spaces we are going to use, and some properties of them.

Definition 2.3. Let $0 < p, \theta \le \infty$ and let ω be a non-negative measurable function on $(0, 1)$. We denote by ${}^{c}LM_{p\theta,\omega}(\mathbb{D})$, the complementary local Morrey-type space, the space of all measurable functions f on $\mathbb{D}$ with finite quasi-norm

$$\|f\|_{{}^{c}LM_{p\theta,\omega}(\mathbb{D})} = \|\omega(r) \|f\|_{L_p({}^{c}B(0,r))}\|_{L_\theta(0,1)}.$$

Definition 2.4. Let $0 < p, \theta \le \infty$ and let ω be a non-negative measurable function on $(0, 1)$. We denote by $W\,{}^{c}LM_{p\theta,\omega}(\mathbb{D})$, the weak complementary local Morrey-type space, the space of all measurable functions f on $\mathbb{D}$ with finite quasinorms

$$\|f\|_{W\,{}^{c}LM_{p\theta,\omega}(\mathbb{D})} = \|\omega(r) \|f\|_{WL_p({}^{c}B(0,r))}\|_{L_\theta(0,1)}.$$

Remark 2.5. In view of the inequalities

$$\begin{aligned} \|\omega(r)\|f\|_{L_p({}^{c}B(0,r))}\|_{L_\theta(0,1)} &\ge \|\omega(r)\|f\|_{L_p({}^{c}B(0,r))}\|_{L_\theta(0,t)} \\ &\ge \|\omega\|_{L_\theta(0,t)} \|f\|_{L_p({}^{c}B(0,t))}, \qquad t \in (0,1), \\ \|\omega(r)\|f\|_{WL_p({}^{c}B(0,r))}\|_{L_\theta(0,1)} &\ge \|\omega(r)\|f\|_{WL_p({}^{c}B(0,r))}\|_{L_\theta(0,t)} \\ &\ge \|\omega\|_{L_\theta(0,t)} \|f\|_{WL_p({}^{c}B(0,t))}, \qquad t \in (0,1), \end{aligned}$$

it is clear that

$$ {}^{^c}LM_{p\theta,\omega}(\mathbb{D}) = W\,{}^{^c}LM_{p\theta,\omega}(\mathbb{D}) = \{0\} \quad \text{when} \quad \|\omega(r)\|_{L_\theta(0,t)} = +\infty \quad \text{for all} \quad t \in (0,1). $$

Here $\{0\}$ is the set of all functions equivalent to 0 on $\mathbb{D}$.

Definition 2.6. We denote by ${}^{^c}\Omega_\theta$ the set of all non-negative measurable functions ω on $(0,1)$ such that

$$ 0 < \|\omega\|_{L_\theta(0,t)} < \infty, \quad t \in (0,1). $$

When considering ${}^{^c}LM_{p\theta,\omega}(\mathbb{D})$ and $W\,{}^{^c}LM_{p\theta,\omega}(\mathbb{D})$ we always assume that $\omega \in {}^{^c}\Omega_\theta$.

We recall that the spaces ${}^{^c}LM_{p\theta,\omega}$ coincide with some weighted Lebesgue spaces.

Theorem 2.7. ([16, 17]) *Let $1 \le p < +\infty$ and $\omega \in {}^{^c}\Omega_p$. Then*

$$ L_{p,\widetilde{\omega}(|\cdot|)}(\mathbb{D}) = {}^{^c}LM_{pp,\omega}(\mathbb{D}), $$

and norms are equivalent, where

$$ \widetilde{\omega}(\tau) := \int_0^\tau \omega(t)^p \, dt. $$

3. Local L_p-estimates of sublinear operators

Suppose that T represents a linear or a sublinear operator, which satisfies

$$ |Tf(z)| \lesssim \iint_{\mathbb{D}} \frac{|f(\zeta|}{|z-\zeta|^2} \, d\xi d\eta, \tag{3.1} $$

for any $f \in L_1(\mathbb{D})$ and $z \notin \operatorname{supp} f$ with a constant independent of f and z.

We point out that the condition (3.1), when f is defined on $\mathbb{R}^n$, was introduced by Soria and Weiss in [21]. The Soria-Weiss condition is satisfied by many interesting operators in harmonic analysis, such as the Calderón-Zygmund singular operators, Carleson's maximal operators, Hardy-Littlewood maximal operators, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Ricci-Stein's oscillatory integrals, the Bochner-Riesz means and so on (cf. [21, 23, 22, 25]).

Theorem 3.1. *Assume that T is a sublinear operator satisfying condition* (3.1)*.*

(i) *Let $1 < p < \infty$ and T be bounded on $L_p(\mathbb{D})$. If f is such that*

$$ \int_0^\tau t^{\frac{2}{p'}-1} \|f\|_{L_p({}^cB(0,t))} \, dt < \infty \quad \text{for all} \quad \tau \in (0,1), \tag{3.2} $$

then for any $\tau \in (0,1)$ the following inequality holds with constant $c > 0$ independent of f and τ:

$$ \|Tf\|_{L_p({}^cB(0,\tau))} \le c\tau^{-\frac{2}{p'}} \int_0^\tau t^{\frac{2}{p'}-1} \|f\|_{L_p({}^cB(0,t))} \, dt. \tag{3.3} $$

(ii) *Let $1 \le p < \infty$ and T be bounded from $L_p(\mathbb{D})$ to $WL_p(\mathbb{D})$. If f satisfies condition* (3.2)*, then for any $\tau \in (0,1)$ the following inequality holds with constant $c > 0$ independent of f and τ:*

$$ \|Tf\|_{WL_p({}^cB(0,\tau))} \le c\tau^{-\frac{2}{p'}} \int_0^\tau t^{\frac{2}{p'}-1} \|f\|_{L_p({}^cB(0,t))} \, dt. \tag{3.4} $$

Proof. Let $1 \le p < \infty$. Applying Hölder's inequality, in view of the monotonicity of $\|f\|_{L_p({}^cB(0,t))}$, we get that

$$\begin{aligned}\iint_{B(0,\tau)} |f(\zeta)|\, d\xi d\eta &= \sum_{n=0}^{\infty} \iint_{B(0,2^{-n}\tau)\setminus B(0,2^{-n-1}\tau)} |f(\zeta)|\, d\xi d\eta \\ &\le \sum_{n=0}^{\infty} |B(0,2^{-n}\tau)|^{\frac{1}{p'}} \left(\iint_{B(0,2^{-n}\tau)\setminus B(0,2^{-n-1}\tau)} |f(\zeta)|^p\, d\xi d\eta \right)^{\frac{1}{p}} \\ &\lesssim \sum_{n=0}^{\infty} (2^{-n}\tau)^{\frac{2}{p'}} \|f\|_{L_p({}^cB(0,2^{-n-1}\tau))} \\ &\approx \sum_{n=0}^{\infty} \|f\|_{L_p({}^cB(0,2^{-n-1}\tau))} \int_{2^{-n-2}\tau}^{2^{-n-1}\tau} t^{\frac{2}{p'}-1}\, dt \\ &\lesssim \sum_{n=0}^{\infty} \int_{2^{-n-2}\tau}^{2^{-n-1}\tau} t^{\frac{2}{p'}-1} \|f\|_{L_p({}^cB(0,t))}\, dt \\ &= \int_0^{\tau/2} t^{\frac{2}{p'}-1} \|f\|_{L_p({}^cB(0,t))}\, dt \quad \text{for all} \quad \tau \in (0,1). \end{aligned} \tag{3.5}$$

Hence, if f satisfies condition (3.2), then $f \in L_1^{\text{loc}}(\mathbb{D})$. On the other hand, condition (3.2) implies that $f \in L_p({}^cB(0,\tau))$, $\tau \in (0,1)$. Consequently, if f satisfies condition (3.2), then $f \in L_1(\mathbb{D})$.

Assume that T is bounded on $L_p(\mathbb{D})$ for $1 < p < \infty$ or T is bounded from $L_p(\mathbb{D})$ to $WL_p(\mathbb{D})$ for $1 \le p < \infty$. First we prove that in both cases $Tf(z)$ exists for a.a. $z \in \mathbb{D}$ and for any f satisfying condition (3.2).

Let $\tau \in (0,1)$. We write $f = f_1 + f_2$ with $f_1 = f\chi_{{}^cB(0,\tau/2)}$ and $f_2 = f\chi_{B(0,\tau/2)}$. By condition (3.2) it is clear that $f \in L_p({}^cB(0,\tau/2))$, so that $f_1 \in L_p(\mathbb{D})$. Consequently, the $L_p(\mathbb{D})$-boundedness of T in the case (i) or the boundedness of T from $L_p(\mathbb{D})$ to $WL_p(\mathbb{D})$ in the case (ii) implies the existence of $Tf_1(z)$ for a.a. $z \in \mathbb{D}$.

Now we prove existence of $Tf_2(z)$ for all $z \in {}^cB(0,\tau)$. Since $z \in {}^cB(0,\tau)$, $\zeta \in B(0,\tau/2)$ implies $|z-\zeta| \ge |z| - |\zeta| \ge (1/2)|z|$, noting that $f_2 \in L_1(\mathbb{D})$, in view of condition (3.1), we obtain that

$$|Tf_2(z)| \lesssim |z|^{-2} \iint_{B(0,\tau)} |f(\zeta)|\, d\xi d\eta, \quad z \in {}^cB(0,\tau). \tag{3.6}$$

This proves the existence of $Tf_2(z)$ for all $z \in {}^cB(0,\tau)$. Sublinearity of T implies that $|Tf(z)| \le |Tf_1(z)| + |Tf_2(z)|$, and the existence of $Tf(z)$ for a.e. $z \in {}^cB(0,\tau)$ follows from the existence of $Tf_1(z)$ and $Tf_2(z)$ for a.e. $z \in {}^cB(0,\tau)$. Since $\mathbb{D}\setminus\{0\} = \bigcup_{\tau\in(0,1)}({}^cB(0,\tau))$, we get the existence of $Tf(z)$ for a.e. $z \in \mathbb{D}$.

(i) Let $1 < p < \infty$ and T is bounded on $L_p(\mathbb{D})$. To prove (3.3), we note that

$$\|Tf\|_{L_p({}^cB(0,\tau))} \le \|Tf_1\|_{L_p({}^cB(0,\tau))} + \|Tf_2\|_{L_p({}^cB(0,\tau))}. \tag{3.7}$$

The boundedness of T on $L_p(\mathbb{D})$ implies that

$$\|Tf_1\|_{L_p({}^cB(0,\tau))} \le \|Tf_1\|_{L_p(\mathbb{D})} \lesssim \|f_1\|_{L_p(\mathbb{D})} \approx \|f\|_{L_p({}^cB(0,\tau/2))}. \tag{3.8}$$

Since

$$\|f\|_{L_p({}^cB(0,\tau/2))} \lesssim \tau^{-\frac{2}{p'}} \int_0^{\tau/2} t^{\frac{2}{p'}-1} \|f\|_{L_p({}^cB(0,t))}\, dt, \tag{3.9}$$

by inequality (3.8), we get that

$$\|Tf_1\|_{L_p({}^{c}B(0,\tau))} \lesssim \tau^{-\frac{2}{p'}} \int_0^{\tau} t^{\frac{2}{p'}-1} \|f\|_{L_p({}^{c}B(0,t))}\, dt. \tag{3.10}$$

On the other hand, by (3.6), we have that

$$\begin{aligned}
\|Tf_2\|_{L_p({}^{c}B(0,\tau))} &= \left(\iint_{{}^{c}B(0,\tau)} |Tf_2(z)|^p \, dxdy \right)^{\frac{1}{p}} \\
&\lesssim \left(\iint_{{}^{c}B(0,\tau)} \left(|z|^{-2} \iint_{B(0,\tau)} |f(\zeta)| \, d\xi d\eta \right)^p dxdy \right)^{\frac{1}{p}} \\
&= \left(\iint_{{}^{c}B(0,\tau)} |z|^{-2p} \, dxdy \right)^{\frac{1}{p}} \left(\iint_{B(0,\tau)} |f(\zeta)| \, d\xi d\eta \right) \\
&\lesssim \tau^{-\frac{2}{p'}} \left(\iint_{B(0,\tau)} |f(\zeta)| \, d\xi d\eta \right).
\end{aligned}$$

By inequality (3.5), we arrive at

$$\|Tf_2\|_{L_p({}^{c}B(0,\tau))} \lesssim \tau^{-\frac{2}{p'}} \int_0^{\tau} t^{\frac{2}{p'}-1} \|f\|_{L_p({}^{c}B(0,t))}\, dt. \tag{3.11}$$

Combining inequalities (3.7), (3.10) and (3.11), we get (3.3).

(ii) Let $1 \le p < \infty$ and T be bounded from $L_p(\mathbb{D})$ to $WL_p(\mathbb{D})$. To prove (3.4), we note that

$$\|Tf\|_{WL_p({}^{c}B(0,\tau))} \le \|Tf_1\|_{WL_p({}^{c}B(0,\tau))} + \|Tf_2\|_{WL_p({}^{c}B(0,\tau))}. \tag{3.12}$$

The boundedness of T from $L_p(\mathbb{D})$ to $WL_p(\mathbb{D})$ implies that

$$\|Tf_1\|_{WL_p({}^{c}B(0,\tau))} \le \|Tf_1\|_{WL_p(\mathbb{D})} \lesssim \|f_1\|_{L_p(\mathbb{D})} \approx \|f\|_{L_p({}^{c}B(0,\tau/2))}. \tag{3.13}$$

By inequalities (3.13) and (3.9), we get that

$$\|Tf_1\|_{WL_p({}^{c}B(0,\tau))} \lesssim \tau^{-\frac{2}{p'}} \int_0^{\tau} t^{\frac{2}{p'}-1} \|f\|_{L_p({}^{c}B(0,t))}\, dt. \tag{3.14}$$

On the other hand, by (3.6), we have that

$$\begin{aligned}
\|Tf_2\|_{WL_p({}^{c}B(0,\tau))} &\le \||z|^{-2}\|_{WL_p({}^{c}B(0,\tau))} \iint_{B(0,\tau)} |f(\zeta)| \, d\xi d\eta \\
&\le \tau^{-\frac{2}{p'}} \int_0^{\tau} t^{\frac{2}{p'}-1} \|f\|_{L_p({}^{c}B(0,t))}\, dt.
\end{aligned} \tag{3.15}$$

Combining inequalities (3.12), (3.14) and (3.15), we get (3.4).

The proof is completed. □

4. Boundedness in complementary local Morrey-type spaces

The following statements hold true.

Theorem 4.1. *Let $0 < \theta_1, \theta_2 \le \infty$ and $\omega_i \in {}^{c}\Omega_{\theta_i}$, $i = 1, 2$. Assume that T is a sublinear operator satisfying condition* (3.1).

(i) *Let $1 < p < \infty$ and T be bounded on $L_p(\mathbb{D})$. If*

(a) $1<\theta_1\le\theta_2<\infty$, *and*

$$\sup_{0<t<1}\left(\int_0^t \omega_2^{\theta_2}(\tau)\,d\tau\right)^{\frac{1}{\theta_2}}\left(\int_0^t \omega_1^{\theta_1}(\tau)\,d\tau\right)^{-\frac{1}{\theta_1}}<\infty, \tag{4.1}$$

$$\sup_{0<t<1}\left(\int_t^1 \omega_2^{\theta_2}(\tau)\tau^{-\frac{2\theta_2}{p'}}\,d\tau\right)^{\frac{1}{\theta_2}}\times$$
$$\times\left(\int_0^t \tau^{\frac{2\theta_1'}{p'}}\left(\int_0^\tau \omega_1^{\theta_1}(s)\,ds\right)^{-\theta_1'}\omega_1^{\theta_1}(\tau)\,d\tau\right)^{\frac{1}{\theta_1'}}<\infty; \tag{4.2}$$

(b) $\max\{\theta_2,1\}<\theta_1<\infty$, $1/r=1/\theta_2-1/\theta_1$, *and*

$$\left(\int_0^1\left(\int_0^t \omega_1^{\theta_1}(\tau)\,d\tau\right)^{-\frac{r}{\theta_1}}\left(\int_0^t \omega_2^{\theta_2}(\tau)\,d\tau\right)^{\frac{r}{\theta_1}}\omega_2^{\theta_2}(t)\,dt\right)^{\frac{1}{r}}<\infty, \tag{4.3}$$

$$\left(\int_0^1\left(\int_t^1 \omega_2^{\theta_2}(\tau)\tau^{-\frac{2\theta_2}{p'}}\,d\tau\right)^{\frac{r}{\theta_1}}\times\right.$$
$$\left.\times\left(\int_0^t \tau^{\frac{2\theta_1'}{p'}}\left(\int_0^\tau \omega_1^{\theta_1}(s)\,ds\right)^{-\theta_1'}\omega_1^{\theta_1}(\tau)\,d\tau\right)^{-\frac{r}{\theta_1'}}\omega_2^{\theta_2}(t)t^{-\frac{2\theta_2}{p'}}\,dt\right)^{\frac{1}{r}}<\infty; \tag{4.4}$$

(c) $\theta_2<\theta_1\le 1$, $1/r=1/\theta_2-1/\theta_1$, (4.3) *holds and*

$$\left(\int_0^1\left(\operatorname*{ess\,sup}_{\tau\in(0,t)}\tau^{\frac{2\theta_1}{p'}}\left(\int_0^\tau \omega_1^{\theta_1}(s)\,ds\right)^{-1}\right)^{\frac{r}{\theta_1}}\times\right.$$
$$\left.\times\left(\int_t^1 \omega_2^{\theta_2}(\tau)\tau^{-\frac{2\theta_2}{p'}}\,d\tau\right)^{\frac{r}{\theta_1}}\omega_2^{\theta_2}(t)t^{-\frac{2\theta_2}{p'}}\,dt\right)^{\frac{1}{r}}<\infty; \tag{4.5}$$

(d) $\theta_1\le\min\{\theta_2,1\}<\infty$ *and*

$$\sup_{0<t<1}\left(\int_0^t \omega_1^{\theta_1}(s)\,ds\right)^{-\frac{1}{\theta_1}}\times$$
$$\times\left(\int_0^1\left(\min\left\{\tau^{\frac{2}{p'}},t^{\frac{2}{p'}}\right\}\right)^{\theta_2}\omega_2^{\theta_2}(\tau)\tau^{-\frac{2\theta_2}{p'}}\,d\tau\right)^{\frac{1}{\theta_2}}<\infty; \tag{4.6}$$

(e) $\theta_1=\infty$, $0<\theta_2<\infty$, *and*

$$\left(\int_0^1\left(\int_0^t \frac{y^{\frac{2}{p'}-1}dy}{\operatorname{ess\,sup}_{\tau\in(0,y)}\omega_1(\tau)}\right)^{\theta_2}\omega_2^{\theta_2}(t)\;t^{-\frac{2\theta_2}{p'}}\,dt\right)^{\frac{1}{\theta_2}}<\infty; \tag{4.7}$$

(f) $\theta_1\le 1$, $\theta_2=\infty$, *and*

$$\operatorname*{ess\,sup}_{0<t<1}\left(\int_0^t \omega_1^{\theta_1}(\tau)\,d\tau\right)^{-\frac{1}{\theta_1}}\left(\operatorname*{ess\,sup}_{0<\tau<1}\min\left\{\tau^{\frac{2}{p'}},t^{\frac{2}{p'}}\right\}\omega_2(\tau)\tau^{-\frac{2}{p'}}\right)<\infty; \tag{4.8}$$

(g) $1<\theta_1<\infty$, $\theta_2=\infty$, *and*

$$\operatorname*{ess\,sup}_{0<t<1}\omega_2(t)t^{-\frac{2}{p'}}\times$$
$$\times\left(\int_0^t\left(\int_\tau^t y^{\frac{2}{p'}-1}\left(\int_0^y \omega_1^{\theta_1}(s)\,ds\right)^{-1}dy\right)^{\theta_1'}\omega_1^{\theta_1}(\tau)\,d\tau\right)^{\frac{1}{\theta_1'}}<\infty; \tag{4.9}$$

(h) $\theta_1 = \theta_2 = \infty$, *and*

$$\operatorname*{ess\,sup}_{0<t<1} \left(\int_0^t \frac{y^{\frac{2}{p'}-1} dy}{\operatorname{ess\,sup}_{\tau \in (0,y)} \omega_1(\tau)} \right) \omega_2(t) t^{-\frac{2}{p'}} < \infty, \tag{4.10}$$

then there exists a constant $c > 0$ such that the inequality

$$\|Tf\|_{{}^{c}LM_{p\theta_2,\omega_2}(\mathbb{D})} \le c \|f\|_{{}^{c}LM_{p\theta_1,\omega_1}(\mathbb{D})} \tag{4.11}$$

holds for all $f \in {}^{c}LM_{p\theta_1,\omega_1}(\mathbb{D})$.

(ii) *Let $1 \le p < \infty$ and T be bounded from $L_p(\mathbb{D})$ to $WL_p(\mathbb{D})$. If conditions (a) - (h) hold, then*

$$\|Tf\|_{W\,{}^{c}LM_{p\theta_2,\omega_2}(\mathbb{D})} \le c \|f\|_{{}^{c}LM_{p\theta_1,\omega_1}(\mathbb{D})} \tag{4.12}$$

holds for all $f \in {}^{c}LM_{p\theta_1,\omega_1}(\mathbb{D})$ with constant $c > 0$ independent of f.

Proof.

(i) Let $1 < p < \infty$, T be bounded on $L_p(\mathbb{D})$ and conditions (a) - (h) hold. Assume that $f \in {}^{c}LM_{p\theta_1,\omega_1}(\mathbb{D})$. In view of Theorem 3.1, we have that

$$\begin{aligned} \|Tf\|_{{}^{c}LM_{p\theta_2,\omega_2}(\mathbb{D})} &= \left\| \omega_2(\tau) \|Tf\|_{L_p({}^{c}B(0,\tau))} \right\|_{L_{\theta_2}(0,1)} \\ &\lesssim \left\| \omega_2(\tau) \tau^{-\frac{2}{p'}} \int_0^\tau t^{\frac{2}{p'}-1} \|f\|_{L_p({}^{c}B(0,t))} \, dt \right\|_{L_{\theta_2}(0,1)}. \end{aligned}$$

Since conditions (a) - (h) hold, applying Theorem 2.2, we arrive at

$$\begin{aligned} \|Tf\|_{{}^{c}LM_{p\theta_2,\omega_2}(\mathbb{D})} &\le c \left\| \omega_1(t) \|f\|_{L_p({}^{c}B(0,t))} \right\|_{L_{\theta_1}(0,1)} \\ &= c \, \|f\|_{{}^{c}LM_{p\theta_1,\omega_1}(\mathbb{D})}. \end{aligned}$$

(ii) Let $1 \le p < \infty$, T be bounded from $L_p(\mathbb{D})$ to $WL_p(\mathbb{D})$ and conditions (a) - (h) hold. Assume that $f \in {}^{c}LM_{p\theta_1,\omega_1}(\mathbb{D})$. In view of Theorem 3.1 and Theorem 2.2, we arrive at

$$\begin{aligned} \|Tf\|_{W\,{}^{c}LM_{p\theta_2,\omega_2}(\mathbb{D})} &\lesssim \left\| \omega_2(\tau) \tau^{-\frac{2}{p'}} \int_0^\tau t^{\frac{2}{p'}-1} \|f\|_{L_p({}^{c}B(0,t))} \, dt \right\|_{L_{\theta_2}(0,1)} \\ &= c \, \|f\|_{{}^{c}LM_{p\theta_1,\omega_1}(\mathbb{D})}. \end{aligned}$$

□

Corollary 4.2. *Let $1 < p < \infty$ and $\omega_i \in {}^{c}\Omega_p$, $i = 1, 2$. Assume that T is a sublinear operator satisfying condition* (3.1)*, bounded on $L_p(\mathbb{D})$.*

If conditions

$$\sup_{0<t<1} \left(\int_0^t \omega_2^p(\tau) \, d\tau \right)^{\frac{1}{p}} \left(\int_0^t \omega_1^p(\tau) \, d\tau \right)^{-\frac{1}{p}} < \infty \tag{4.13}$$

and

$$\sup_{0<t<1} \left(\int_t^1 \tau^{2(1-p)} \omega_2^p(\tau) \, d\tau \right)^{\frac{1}{p}} \left(\int_0^t \left(\int_0^\tau \omega_1^p(s) \, ds \right)^{-p'} \tau^2 \omega_1^p(\tau) \, d\tau \right)^{\frac{1}{p'}} < \infty \tag{4.14}$$

hold, then

$$\|Tf\|_{L_{p,\widetilde{\omega}_2(|\cdot|)}(\mathbb{D})} \le c \|f\|_{L_{p,\widetilde{\omega}_1(|\cdot|)}(\mathbb{D})},$$

with constant $c > 0$ independent of f. Here

$$\widetilde{\omega}_i(t) := \int_0^t \omega_i(\tau)^p \, d\tau, \quad i = 1, 2. \tag{4.15}$$

Proof. The statement follows from Theorems 4.1 and 2.7 when $\theta_1 = \theta_2 = p$. □

5. Main results

As it is mentioned in the introduction, the operators Π_k, $k \in \mathbb{N}$ are bounded on $L_p(\mathbb{D})$, $1 < p < \infty$. Our main result in this paper is to extend these results to complementary local Morrey-type spaces.

Theorem 5.1. *Let $k \in \mathbb{N}$, $1 < p < \infty$, $0 < \theta_1, \theta_2 \le \infty$ and $\omega_i \in {}^{c}\Omega_{\theta_i}$, $i = 1, 2$. If (a) - (h) hold then there exists a constant $c > 0$ such that the inequality*

$$\|\Pi_k f\|_{{}^{c}LM_{p\theta_2,\omega_2}(\mathbb{D})} \le c_2 \|f\|_{{}^{c}LM_{p\theta_1,\omega_1}(\mathbb{D})}$$

holds for all $f \in {}^{c}LM_{p\theta_1,\omega_1}(\mathbb{D})$.

Proof. First we will show that the operator Π_k satisfies condition (3.1). The inequality

$$\left| \frac{z-\zeta}{1-z\bar{\zeta}} \right| < 1 \quad \text{if} \quad |z| < 1 \quad \text{and} \quad |\zeta| < 1 \tag{5.1}$$

yields that

$$\begin{aligned}
|\Pi_k f(z)| &\le |T_{-k,k} f(z)| + |P_k f(z)| \\
&\le \frac{k}{\pi} \iint_{\mathbb{D}} \frac{|f(\zeta)|}{|\zeta - z|^2}\, d\xi d\eta + \frac{k}{\pi} \iint_{\mathbb{D}} \left| \frac{\zeta - z + \overline{\zeta - z}}{1 - z\bar{\zeta}} \bar{\zeta} - 1 \right|^{k-1} \frac{|f(\zeta)|}{|1 - z\bar{\zeta}|^2}\, d\xi d\eta \\
&\le \frac{k}{\pi} \iint_{\mathbb{D}} \frac{|f(\zeta)|}{|\zeta - z|^2}\, d\xi d\eta + \frac{k}{\pi} \iint_{\mathbb{D}} \left(2 \left| \frac{z-\zeta}{1 - z\bar{\zeta}} \right| |\zeta| + 1 \right)^{k-1} \frac{|f(\zeta)|}{|1 - z\bar{\zeta}|^2}\, d\xi d\eta \\
&\le \frac{k}{\pi} \iint_{\mathbb{D}} \frac{|f(\zeta)|}{|\zeta - z|^2}\, d\xi d\eta + \frac{k 3^{k-1}}{\pi} \iint_{\mathbb{D}} \frac{|f(\zeta)|}{|1 - z\bar{\zeta}|^2}\, d\xi d\eta \\
&\le \frac{k}{\pi} \iint_{\mathbb{D}} \frac{|f(\zeta)|}{|\zeta - z|^2}\, d\xi d\eta + \frac{k 3^{k-1}}{\pi} \iint_{\mathbb{D}} \frac{|f(\zeta)|}{|\zeta - z|^2}\, d\xi d\eta \\
&\lesssim \iint_{\mathbb{D}} \frac{|f(\zeta)|}{|\zeta - z|^2}\, d\xi d\eta
\end{aligned}$$

for any $f \in {}^{c}LM_{p\theta_1,\omega_1}(\mathbb{D})$ and $z \notin \operatorname{supp} f$.

Since the operator Π_k is bounded on $L_p(\mathbb{D})$, by Theorem 4.1, we get that

$$\|\Pi_k f\|_{{}^{c}LM_{p\theta_2,\omega_2}(\mathbb{D})} \le c \|f\|_{{}^{c}LM_{p\theta_1,\omega_1}(\mathbb{D})}.$$

The proof is completed. □

In view of Theorem 2.7, by Theorem 5.1, we immediately get the following statement:

Corollary 5.2. *Let $k \in \mathbb{N}$, $1 < p < \infty$ and $\omega_i \in {}^{c}\Omega_p$, $i = 1, 2$. If conditions* (4.13) *and* (4.14) *hold then*

$$\|\Pi_k f\|_{L_{p,\widetilde{\omega}_2(|\cdot|)}(\mathbb{D})} \le c \|f\|_{L_{p,\widetilde{\omega}_1(|\cdot|)}(\mathbb{D})},$$

with constant $c > 0$ independent of f, where $\widetilde{\omega}_i$, $i = 1, 2$ are defined by (4.15).

Proof. Since the operator Π_k is bounded in $L_p(\mathbb{D})$ and satisfies condition (3.1), the statement of Corollary 5.2 follows from Theorem 4.1. □

Since the function $\widetilde{T}_1 f$ is the solution of the Schwarz BVP

$$g_{\bar{z}} = f \text{ in } \mathbb{D}, \operatorname{Re} g = 0 \text{ on } \partial\mathbb{D}, \operatorname{Im} g(0) = 0, \tag{5.2}$$

when $f \in L^1(\mathbb{D})$, by Theorem 5.1 and Corollary 5.2, respectively, we get the following a priori estimates for the derivative of the solution of (5.2).

Theorem 5.3. *Let $1 < p < \infty$, $0 < \theta_1, \theta_2 \le \infty$ and $\omega_i \in {}^{c}\Omega_{\theta_i}$, $i = 1, 2$. If (a) - (h) hold, then for the solution of* (5.2) *the inequality*

$$\|\partial_z g\|_{{}^{c}LM_{p\theta_2,\omega_2}(\mathbb{D})} \le c\|f\|_{{}^{c}LM_{p\theta_1,\omega_1}(\mathbb{D})}$$

holds for all $f \in {}^{c}LM_{p\theta_1,\omega_1}(\mathbb{D})$ with a constant $c > 0$ independent of f.

Corollary 5.4. *Let $1 < p < \infty$ and $\omega_i \in {}^{c}\Omega_p$, $i = 1, 2$. If conditions* (4.13) *and* (4.14) *hold, then for the solution of* (5.2) *the inequality*

$$\|\partial_z g\|_{L_{p,\widetilde{\omega}_2(|\cdot|)}(\mathbb{D})} \le c\|f\|_{L_{p,\widetilde{\omega}_1(|\cdot|)}(\mathbb{D})},$$

holds for all $f \in L_{p,\widetilde{\omega}_1(|\cdot|)}(\mathbb{D})$ with a constant $c > 0$ independent of f, where $\widetilde{\omega}_i$, $i = 1, 2$ are defined by (4.15).

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