Journal of Computational and Applied Mathematics 305 (2016) 11–17

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

The optimal control problem in the processes described by the Goursat problem for a hyperbolic equation in variable exponent Sobolev spaces with dominating mixed derivatives

R.A. Bandaliyev [a,*], V.S. Guliyev [a,b,**], I.G. Mamedov [c], A.B. Sadigov [c]

[a] *Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan*

[b] *Ahi Evran University, Department of Mathematics, Kirsehir, Turkey*

[c] *Institute of Control Systems of NAS of Azerbaijan, Baku, Azerbaijan*

ARTICLE INFO

Article history:
Received 19 September 2015
Received in revised form 7 March 2016

MSC:
primary 37D30
secondary 49B20
49K20

Keywords:
Optimal control
Pontryagin's maximum principle
Goursat problem
Variable exponent Sobolev spaces with dominating mixed derivatives

ABSTRACT

In this paper a necessary and sufficient condition, such as the Pontryagin's maximum principle for an optimal control problem with distributed parameters, is given by a hyperbolic equation of the second order with $L_{p(x)}$-coefficients. The results can be used in the theory of optimal processes for distribution Pontryagin maximum principle for various controlled processes described by hyperbolic equations of second order with discontinuous coefficients in variable exponent Sobolev spaces with dominant mixed derivatives.

1. Introduction

It is well known that various optimal control problems described by hyperbolic equations, as well as the equations of mathematical physics at various assumptions obtained some necessary and sufficient conditions of optimality. Development of optimal control theory led to its application to practical problems, such as a controlled objects, optimization of dynamical systems and others. Many of these optimal control problems, the solution of which is the subject of numerous works, described by hyperbolic equations. The problem of optimal control of systems with distributed parameters has numerous applications.

The Pontryagin maximum principle is a fundamental result of the theory of necessary optimality conditions of the first order, which initially proved (in the linear case R.V. Gamkrelidze, in the nonlinear case V.G. Boltyanskii (see [1])) for optimal control problems described by ordinary differential equations. Later works were dedicated to the conclusion of necessary

* Corresponding author.
** Corresponding author at: Ahi Evran University, Department of Mathematics, Kirsehir, Turkey.
E-mail addresses: bandaliyev.rovshan@math.ab.az (R.A. Bandaliyev), vagif@guliyev.com (V.S. Guliyev), ilgar-mammadov@rambler.ru (I.G. Mamedov), aminaga@box.az (A.B. Sadigov).

http://dx.doi.org/10.1016/j.cam.2016.03.024
0377-0427/

conditions for optimality in the more complex control problems with lumped and distributed parameters. Optimal control problems described by hyperbolic equations under Goursat conditions originates in the paper [2]. Further various aspects of the problem of optimal control processes described by Goursat–Darboux systems were investigated in [3–24] and others. Many of the processes occurring in the theory of filtration of fluids in fractured media described pseudoparabolic (hyperbolic) and parabolic equations with discontinuous coefficients. Note that some properties of the solutions of the Dirichlet problem for a parabolic equation with discontinuous coefficients in Sobolev type spaces were investigated in [25].

Correct solvability of the Goursat boundary value problem plays an important role in qualitative theory of optimal processes. Goursat problems for hyperbolic equations with discontinuous coefficients of the non-classical boundary conditions are studied in [26–29] and others. The present work is devoted to the conclusion of necessary and sufficient condition such as the maximum principle of Pontryagin for an optimal control problem with distributed parameters described by a hyperbolic equation of the second order with $L_{p(x)}$-coefficients.

In this paper the optimal control problem for a hyperbolic equation of second order with $L_{p(x)}$-coefficients with nonclassical Goursat boundary value problem is investigated. The statement of optimal control problem is studied by using a new version of the increment method that essentially uses the concept of the adjoint equation of the integral form. The method also includes the case where the coefficients of the equation are non-smooth functions from $L_{p(x)}$. In the paper it is shown that such an optimal control problem can be investigated with the help of a new concept of the adjoint equation, which can be regarded as an auxiliary equation for determination of Lagrange multipliers. In the future, we can consider a variety of classes of optimal control problem described by loaded integro-differential equations for various non-local boundary conditions. These optimal control problems actually describe more complex control processes, which are very important in the theory of optimal processes.

2. Preliminaries

Let $\mathbb{R}^2$ be the two-dimensional Euclidean space of points $x = (x_1, x_2)$, $|x| = \left(\sum_{i=1}^2 x_i^2\right)^{1/2}$ and let $G = G_1 \times G_2 = \left(x_1^0,\ h_1\right) \times \left(x_2^0, h_2\right)$ be a rectangle in $\mathbb{R}^2$ and h_i $(i = 1, 2)$ are fixed real numbers. By $\mathcal{P}(G)$ we denote the set of Lebesgue measurable functions such that $p : G \mapsto [1, \infty)$. The functions $p \in \mathcal{P}(G)$ are called variable exponents on G. We define $\underline{p} = \operatorname{ess\,inf}_{x\in G} p(x)$ and $\overline{p} = \operatorname{ess\,sup}_{x\in G} p(x)$. We denote $r_1(x_1) = \lim_{x_2\to x_2^0+0} p(x_1, x_2)$ and $r_2(x_2) = \lim_{x_1\to x_1^0+0} p(x_1, x_2)$. Let $q(x)$ be the dual variable exponent function of p defined by $\frac{1}{p(x)}+\frac{1}{q(x)} = 1$. Assume $\frac{1}{r_1(x_1)}+\frac{1}{s_1(x_1)} = 1$ and $\frac{1}{r_2(x_2)}+\frac{1}{s_2(x_2)} = 1$, where $x \in G$. Obviously, $\operatorname{ess\,sup}_{x\in G} q(x) = \overline{q} = \frac{\underline{p}}{\underline{p}-1}$ and $\operatorname{ess\,inf}_{x\in G} q(x) = \underline{q} = \frac{\overline{p}}{\overline{p}-1}$.

Definition 1 (*[30,31]*). Let $p \in \mathcal{P}(G)$. By $L_{p(x)}(G)$ we denote the space of Lebesgue measurable functions f on G such that for some $\lambda_0 > 0$

$$\int_G \left(\frac{|f(x)|}{\lambda_0}\right)^{p(x)} dx < \infty.$$

Note that the functional

$$\|f\|_{L_{p(x)}(G)} = \|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 :\ \int_G \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}$$

is defined norm in $L_{p(x)}(G)$ and the spaces $L_{p(x)}(G)$ is a Banach function spaces (see [30,31]).

Definition 2. Let $p \in \mathcal{P}(G)$. By $SW^{(1,1)}_{p(x)}(G)$ we define the variable exponent Sobolev spaces of function with dominating mixed derivatives as

$$SW^{(1,1)}_{p(x)}(G) := \left\{u :\ D_1^{i_1} D_2^{i_2} u(x) \in L_{p(x)}(G),\ i_k = 0, 1,\ k = 1, 2\right\}.$$

It is obvious that the expression

$$\|u\|_{SW^{(1,1)}_{p(\cdot)}(G)} = \sum_{i_1=0}^{1}\sum_{i_2=0}^{1} \left\|D_1^{i_1} D_2^{i_2} u\right\|_{L_{p(\cdot)}(G)} < \infty$$

defines a norm in $SW^{(1,1)}_{p(x)}(G)$.

Lemma 1. *Let $p \in \mathcal{P}(G)$ and $1 < \underline{p} \le \overline{p} < \infty$. Then the space $SW^{(1,1)}_{p(x)}(G)$ is complete.*

Proof. Let $\{u_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $SW^{(1,1)}_{p(x)}(G)$. Then $\left\{D_1^{i_1} D_2^{i_2} u_n\right\}$ is a Cauchy sequence in $L_{p(x)}(G)$ for all $0 \le i_1, i_2 \le 1$. By the completeness of $L_{p(x)}(G)$ (see [30]) there exists a $g_{i_1,i_2} \in L_{p(x)}(G)$ such that $\left\|D_1^{i_1} D_2^{i_2} u_n - g_{i_1,i_2}\right\|_{L_{p(x)}(G)} \to 0$ as $n \to \infty$

and for all $0 \le i_1, i_2 \le 1$. Applying Hölder inequality in variable exponent Lebesgue spaces (see [30,31]), for $\varphi \in C_c^\infty(G)$, we get

$$\int_G \left(D_1^{i_1} D_2^{i_2} u_n(x) - g_{i_1,i_2}(x)\right) D_1^{i_1} D_2^{i_2} \varphi(x)\, dx \le \left(\frac{1}{\underline{p}} + \frac{1}{\overline{p}}\right) \left\| D_1^{i_1} D_2^{i_2} u_n - g_{i_1,i_2} \right\|_{L_{p(x)}(G)} \left\| D_1^{i_1} D_2^{i_2} \varphi \right\|_{L_{q(x)}(G)}.$$

Since $\left\| D_1^{i_1} D_2^{i_2} u_n - g_{i_1,i_2} \right\|_{L_{p(x)}(G)} \to 0$ and $D_1^{i_1} D_2^{i_2} \varphi(x)$ are bounded for any $\varphi \in C_c^\infty(G)$, by the Lebesgue dominated convergence theorem in variable Lebesgue spaces (see [30]), we have

$$\lim_{n\to\infty} \int_G u_n(x)\, D_1^{i_1} D_2^{i_2} \varphi(x)\, dx = \int_G g_{i_1,i_2}(x)\, D_1^{i_1} D_2^{i_2} \varphi(x)\, dx.$$

Therefore, for all $\varphi \in C_c^\infty(G)$, we have

$$\begin{aligned} \int_G u(x) D_1^{i_1} D_2^{i_2} \varphi(x)\, dx &= \lim_{n\to\infty} \int_G u_n(x) D_1^{i_1} D_2^{i_2} \varphi(x)\, dx \\ &= (-1)^{i_1+i_2} \lim_{n\to\infty} \int_G D_1^{i_1} D_2^{i_2} u_n(x) \varphi(x)\, dx = (-1)^{i_1+i_2} \int_G g_{i_1,i_2}(x)\varphi(x)\, dx. \end{aligned}$$

This shows $D_1^{i_1} D_2^{i_2} u$ exists weakly and $g_{i_1,i_2} = D_1^{i_1} D_2^{i_2} u$. Thus $u \in SW_{p(x)}^{(1,1)}(G)$ and $u_n \to u$ as $n \to \infty$, which completes the proof.

3. Problem statement and main result

Let the controlled object is described by the equation

$$\left(V_{1,1}u\right)(x) \equiv D_1 D_2 u(x) + a_{1,0}(x) D_1 u(x) + a_{0,1}(x) D_2 u(x) + a_{0,0}(x) u(x) = \varphi\left(x, \nu(x)\right), \tag{3.1}$$

the following non-classical Goursat conditions (see [26])

$$\begin{cases} V_{0,0}u \equiv u\left(x_1^0, x_2^0\right) = \varphi_{0,0} \\ \left(V_{1,0}u\right)(x_1) \equiv D_1 u\left(x_1, x_2^0\right) = \varphi_{1,0}\left(x_1\right) \\ \left(V_{0,1}u\right)(x_2) \equiv D_2 u\left(x_1^0, x_2\right) = \varphi_{0,1}\left(x_2\right), \end{cases} \tag{3.2}$$

where $\varphi\left(x, \nu(x)\right) \in L_{p(x)}(G)$, $a_{0,0}(x) \in L_{p(x)}(G)$, $a_{1,0}(x) \in L_{(\infty, r_2(x_2))}(G)$, $a_{0,1}(x) \in L_{(r_1(x_1), \infty)}(G)$, $\varphi_{0,0} \in \mathbb{R}$, $\varphi_{1,0}\left(x_1\right) \in L_{r_1(x_1)}(G_1)$, $\varphi_{0,1}\left(x_2\right) \in L_{r_2(x_2)}(G_2)$ and $D_k = \frac{\partial}{\partial x_k}$ $(k = 1, 2)$ is the generalized differential operator in the sense of Sobolev. Let $\nu(x) = (\nu_1(x), \dots, \nu_m(x))$-$m$-dimensional control vector function and $\varphi\left(x, \nu(x)\right)$ be given function defined on $G \times \mathbb{R}^m$ and satisfying Caratheodory condition on $G \times \mathbb{R}^m$:

(1) $\varphi(x, \nu(x))$ is measurable by x in G for all $\nu \in \mathbb{R}^m$;
(2) $\varphi(x, \nu(x))$ is continuous by ν in $\mathbb{R}^m$ for almost all $x \in G$;
(3) for any $\delta > 0$ there exists $\varphi_\delta^0(x) \in L_{p(x)}(G)$ such that $|\varphi\left(x, \nu(x)\right)| \le \varphi_\delta^0(x)$ for almost all $x \in G$ and for any function $\nu(x)$.

Since the coefficients of Eq. (3.1) are non-smooth, we mean the solution of problem (3.1)–(3.2) in the generalized sense. Let the vector function $\nu(x)$ be measurable and bounded on G and for almost every $x \in G$ it takes its value from the given set $\Omega \subset \mathbb{R}^m$. Then the vector function is called admissible controls. The set of all admissible controls is denoted by Ω_∂.

Now consider the following optimal control problem: Find an admissible control $\nu(x)$ from Ω_∂, for which the solution of the problem (3.1)–(3.2) $u \in SW_{p(x)}^{1,1}(G)$ gives the minimizing of the multi-point functional

$$F(\nu) = \sum_{k=1}^{N} \left[\alpha_k u\left(x_1^0, x_2^{(k)}\right) + \beta_k u\left(x_1^{(k)}, x_2^0\right)\right] \to \min, \tag{3.3}$$

where $\left(x_1^{(k)}, x_2^{(k)}\right) \in \overline{G}$ given fixed points, $\alpha_k, \beta_k \in \mathbb{R}$ given real numbers and N a positive integer.

To obtain the necessary and sufficient conditions for optimality first we find the increment of the functional (3.3). Let $\nu(x)$ and $\nu(x) + \Delta\nu(x)$ be different admissible controls, and $u(x)$ and $u(x) + \Delta u(x)$ respectively solve the problem (3.1)–(3.2) in the space $SW_{p(x)}^{1,1}(G)$. Then the increment of the functional (3.3) is of the form

$$\Delta F(\nu) = \sum_{k=1}^{N} \left[\alpha_k \Delta u\left(x_1^0, x_2^{(k)}\right) + \beta_k \Delta u\left(x_1^{(k)}, x_2^0\right)\right]. \tag{3.4}$$

Obviously, in this case the function $\Delta u \in SW_{p(x)}^{1,1}(G)$ is the solution of the equation

$$V_{1,1}\Delta u(x) = \Delta\varphi(x), \tag{3.5}$$

satisfying trivial conditions

$$\begin{cases} V_{0,0}\Delta u = 0 \\ \left(V_{1,0}\Delta u\right)(x_1) = 0 \\ \left(V_{0,1}\Delta u\right)(x_2) = 0, \end{cases} \tag{3.6}$$

where $\Delta\varphi(x) = \varphi(x, \nu(x) + \Delta\nu(x)) - \varphi(x, \nu(x))$. The operator $V = \left(V_{1,1}, V_{0,0}, V_{1,0}, V_{0,1}\right) : SW^{1,1}_{p(x)}(G) \mapsto E_{p(x)} = L_{p(x)}(G) \times \mathbb{R} \times L_{r_1(x_1)}(G_1) \times L_{r_2(x_2)}(G_2)$ generated by the problem (3.1)–(3.2) is bounded by the above mentioned assumptions.

The integral representation of the functions in the space $SW^{(1,1)}_{p(x)}(G)$

$$u(x) = u\left(x_1^0, x_2^0\right) + \int_{x_1^0}^{x_1} u_{\alpha_1}\left(\alpha_1, x_2^0\right) d\alpha_1 + \int_{x_2^0}^{x_2} u_{\alpha_2}\left(x_1^0, \alpha_2\right) d\alpha_2 + \int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2} u_{\alpha_1\alpha_2}\left(\alpha_1, \alpha_2\right) d\alpha_1 d\alpha_2 \tag{3.7}$$

holds.

Remark 1. Note that in the case $p(x) = p = const$ the integral representation (3.7) was obtained in [32]. The proof of integral representation (3.7) in the variable exponent case is similar to the constant exponent case.

Next, we show that the operator V has an adjoint operator $V^\star = \left(\omega_{1,1}, \omega_{0,0}, \omega_{1,0}, \omega_{0,1}\right)$, which acts in the spaces $E_{q(x)}(G) = L_{q(x)}(G) \times \mathbb{R} \times L_{s_1(x_1)}(G_1) \times L_{s_2(x_2)}(G_2)$ and satisfy the condition (3.5). Then, by definition, we have

$$\begin{aligned} f(Vu) &= \iint_G f_{1,1}(x)\left(V_{1,1}u\right)(x)dx + f_{0,0}\left(V_{0,0}u\right) + \int_{x_1^0}^{h_1} f_{1,0}(x_1)\left(V_{1,0}u\right)(x_1)\,dx_1 \\ &\quad + \int_{x_2^0}^{h_2} f_{0,1}(x_2)\left(V_{0,1}u\right)(x_2)\,dx_2 = \iint_G f_{1,1}(x)\left[D_1D_2u(x) + a_{1,0}(x)D_1u(x) + a_{0,1}(x)D_2u(x)\right. \\ &\quad \left. + a_{0,0}(x)u(x)\right]dx + f_{0,0}u\left(x_1^0, x_2^0\right) + \int_{x_1^0}^{h_1} f_{1,0}(x_1)D_1u\left(x_1, x_2^0\right)dx_1 + \int_{x_2^0}^{h_2} f_{0,1}(x_2)D_2u\left(x_1^0, x_2\right)dx_2 \\ &= \iint_G f_{1,1}(x)\left\{D_1D_2u(x) + a_{1,0}(x)\left[D_1u\left(x_1, x_2^0\right) + \int_{x_2^0}^{x_2} u_{x_1\alpha_2}(x_1, \alpha_2)\,d\alpha_2\right]\right. \\ &\quad + a_{0,1}(x)\left[D_2u\left(x_1^0, x_2\right) + \int_{x_1^0}^{x_1} u_{\alpha_1x_2}(\alpha_1, x_2)\,d\alpha_1\right] + a_{0,0}(x)\left[u\left(x_1^0, x_2^0\right) + \int_{x_1^0}^{x_1} u_{\alpha_1}\left(\alpha_1, x_2^0\right)d\alpha_1\right. \\ &\quad \left.\left. \times \int_{x_2^0}^{x_2} u_{\alpha_2}\left(x_1^0, \alpha_2\right)d\alpha_2 + \int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2} u_{\alpha_1\alpha_2}(\alpha_1, \alpha_2)\,d\alpha_1d\alpha_2\right]\right\}dx + f_{0,0}u\left(x_1^0, x_2^0\right) \\ &\quad + \int_{x_1^0}^{h_1} f_{1,0}(x_1)D_1u\left(x_1, x_2^0\right)dx_1 + \int_{x_2^0}^{h_2} f_{0,1}(x_2)D_2u\left(x_1^0, x_2\right)dx_2 = u\left(x_1^0, x_2^0\right)\cdot\omega_{0,0}f \\ &\quad + \int_{x_1^0}^{h_1}\left(\omega_{1,0}f\right)(x_1)D_1u\left(x_1, x_2^0\right)dx_1 + \int_{x_2^0}^{h_2}\left(\omega_{0,1}f\right)(x_2)D_2u\left(x_1^0, x_2\right)dx_2 \\ &\quad + \iint_G\left(\omega_{1,1}f\right)(x)\,D_1D_2u(x)\,dx = \left(V^\star f\right)(u), \end{aligned} \tag{3.8}$$

where $f = \left(f_{1,1}(x), f_{0,0}, f_{1,0}(x_1), f_{0,1}(x_2)\right) \in E_{q(x)}(G)$ is an arbitrary linear bounded functional on $E_{p(x)}(G)$, $u \in SW^{1,1}_{p(x)}(G)$ and $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. Expressions for the $\omega_{i,j}f$ $(i, j = 0, 1)$ are given as follows:

$$\omega_{0,0}f \equiv \iint_G f_{1,1}(x)a_{0,0}(x)dx + f_{0,0},$$

$$\left(\omega_{1,0}f\right)(x_1) \equiv \int_{x_1}^{h_1}\int_{x_2^0}^{h_2} a_{0,0}(\tau_1, x_2)f_{1,1}(\tau_1, x_2)\,d\tau_1dx_2 + \int_{x_2^0}^{h_2} a_{1,0}(x_1, x_2)f_{1,1}(x_1, x_2)\,dx_2 + f_{1,0}(x_1),$$

$$\left(\omega_{0,1}f\right)(x_2) \equiv \int_{x_1^0}^{h_1}\int_{x_2}^{h_2} a_{0,0}(x_1, \tau_2)f_{1,1}(x_1, \tau_2)\,dx_1d\tau_2 + \int_{x_1^0}^{h_1} a_{0,1}(x_1, x_2)f_{1,1}(x_1, x_2)\,dx_1 + f_{0,1}(x_2),$$

$$\begin{aligned}\left(\omega_{1,1}f\right)(x_1, x_2) &\equiv f_{1,1}(x_1, x_2) + \int_{x_2}^{h_2} a_{1,0}(x_1, \tau_2)f_{1,1}(x_1, \tau_2)\,d\tau_2 + \int_{x_1}^{h_1} a_{0,1}(\tau_1, x_2)f_{1,1}(\tau_1, x_2)\,d\tau_1 \\ &\quad + \int_{x_1}^{h_1}\int_{x_2}^{h_2} a_{0,0}(\tau_1, \tau_2)f_{1,1}(\tau_1, \tau_2)\,d\tau_1d\tau_2.\end{aligned}$$

Now in (3.8) instead of $u(x)$ substitute the solution of the problem (3.5)–(3.6); i.e. we replace a function $u(x)$ by $\Delta u(x)$. Then the equality

$$f(V\Delta u) = \iint_G f_{1,1}(x)\Delta\varphi(x)dx = \iint_G \left(\omega_{1,1}f\right)(x)D_1D_2\Delta u(x)dx \equiv \left(V^{\star}f\right)(\Delta u) \tag{3.9}$$

holds for all $f \in E_{q(x)}(G)$. In other words

$$-\iint_G f_{1,1}(x)\Delta\varphi(x)dx + \iint_G \left(\omega_{1,1}f\right)(x)D_1D_2\Delta u(x)dx = 0. \tag{3.10}$$

Therefore the function $\Delta u(x)$ as an element of $SW^{1,1}_{p(x)}(G)$ satisfies the condition (3.6). Using the integral representation (3.7), we have

$$\alpha_k\Delta u\left(x_1^0, x_2^{(k)}\right) + \beta_k\Delta u\left(x_1^{(k)}, x_2^0\right) = \iint_G B_k(x)D_1D_2\Delta u(x)dx,$$

where $B_k(x) = \alpha_k\theta\left(x_1 - x_1^0\right)\theta\left(x_2 - x_2^{(k)}\right) + \beta_k\theta\left(x_1 - x_1^{(k)}\right)\theta\left(x_2 - x_2^0\right)$; and $\theta(t) = \begin{cases} 1, & t > 0 \\ 0, & t \le 0 \end{cases}$ is the Heaviside function. Therefore, the increment (3.4) of the functional (3.3) can be represented as

$$\Delta F(\nu) = \iint_G \sum_{k=1}^{N} B_k(x)D_1D_2\Delta u(x)dx,$$

or

$$\Delta F(\nu) = \iint_G B(x)D_1D_2\Delta u(x)dx, \tag{3.11}$$

and

$$B(x) = \sum_{k=1}^{N} B_k(x).$$

By (3.10), the increment (3.11) can be represented in the form

$$\Delta F(\nu) = \iint_G \left[B(x) + \left(\omega_{1,1}f\right)(x)\right]D_1D_2\Delta u(x)dx - \iint_G f_{1,1}(x)\Delta\varphi(x)\,dx. \tag{3.12}$$

Since $\omega_{1,1}$ depends only on one element f, equality (3.12) holds for all $f_{1,1} \in L_{q(x)}(G)$. For the integro-differential expression (3.12) we consider the equation

$$\left(\omega_{1,1}f_{1,1}\right)(x) + B(x) = 0, \quad x \in G, \tag{3.13}$$

is said to be adjoint equation for the optimal control problem (3.1)–(3.3). As the function of $f_{1,1}(x)$ we take the solution of Eq. (3.13) in $L_{q(x)}(G)$. Then equality (3.12) has the simple form

$$\Delta F(\nu) = -\iint_G f_{1,1}(x)\Delta\varphi(x)\,dx.$$

Now, for a fixed $(\tau_1, \tau_2) \in G$ consider the following needle variation of admissible control $\nu(x)$:

$$\Delta\nu_\varepsilon(x) = \begin{cases} \widehat{\nu} - \nu(x), & x \in G_\varepsilon \\ 0, & x \in G \setminus G_\varepsilon, \end{cases}$$

where $\widehat{\nu} \in \Omega_\partial$, $\varepsilon > 0$ is sufficiently small parameter and $G_\varepsilon = \left(\tau_1 - \frac{\varepsilon}{2}, \tau_1 + \frac{\varepsilon}{2}\right) \times \left(\tau_2 - \frac{\varepsilon}{2}, \tau_2 + \frac{\varepsilon}{2}\right) \subset G$. A control $\nu_\varepsilon(x)$ defined by the equality $\nu_\varepsilon(x) = \nu(x) + \Delta\nu_\varepsilon(x)$ is an admissible control for all sufficiently small $\varepsilon > 0$ and all the $\widehat{\nu} \in \Omega_\partial$, where $(\tau_1, \tau_2) \in G$ is some fixed point, called a needle perturbation given by control $\nu(x)$. It is obvious that

$$\begin{aligned} F(\nu_\varepsilon) - F(\nu) &= -\iint_{G_\varepsilon} f_{1,1}(x)\left[\varphi\left(x, \nu(x) + \Delta\nu_\varepsilon(x)\right) - \varphi\left(x, \nu(x)\right)\right]dx \\ &= -\iint_{G_\varepsilon} f_{1,1}(x)\left[\varphi\left(x, \widehat{\nu}(x)\right) - \varphi\left(x, \nu(x)\right)\right]dx. \end{aligned} \tag{3.14}$$

Since the optimal control problem is linear, it follows from (3.14) following theorem.

Theorem 1. *Let $f_{1,1}(x) \in L_{q(x)}(G)$ be a solution of the adjoint equation (3.13). Then for the optimality of the admissible control $\nu(x)$, necessary and sufficient that for almost all $x \in G$ satisfy the Pontryagin maximum condition*

$$\max_{\widehat{\nu}\in\Omega_\partial} H\left(x, f_{1,1}(x), \widehat{\nu}\right) = H\left(x, f_{1,1}(x), \nu\right),$$

where $H\left(x, f_{1,1}(x), \nu\right) = f_{1,1}(x) \cdot \varphi(x, \nu)$ is the Hamilton–Pontryagin function.

Proof. Suppose that a control $\nu(x_1, x_2) \in \Omega_\partial$ gives the minimum value of the functional (3.3). Then by (3.14), we have

$$-\iint_{G_\varepsilon} \left[H\left(x_1, x_2, f_{1,1}(x_1, x_2), \widehat{\nu}\right) - H\left(x_1, x_2, f_{1,1}(x_1, x_2), \nu(x_1, x_2)\right)\right] dx_1 dx_2 \geq 0. \quad (3.15)$$

Dividing both sides of (3.15) by ε^2 and passing to the limit as $\varepsilon \to +0$, for almost all $(\tau_1, \tau_2) \in G$ and using analog of Lebesgue differentiation theorem in $L_{p(x)}$ (see [30]) for all $\nu \in \Omega_\partial$, we get

$$H\left(\tau_1, \tau_2, f_{1,1}(\tau_1, \tau_2), \nu(\tau_1, \tau_2)\right) - H\left(\tau_1, \tau_2, f_{1,1}(\tau_1, \tau_2), \widehat{\nu}\right) \geq 0. \quad (3.16)$$

Thus, for optimal control of $\nu(x_1, x_2) \in \Omega_\partial$ it is necessary to satisfy the condition (3.16). Besides, the equality

$$\Delta F(\nu) = -\iint_G \Delta H\left(x_1, x_2, f_{1,1}(x_1, x_2), \nu(x_1, x_2)\right) dx_1 dx_2$$

show that this condition is also sufficient for optimal control of $\nu(x_1, x_2)$, where $\Delta H\left(x_1, x_2, f_{1,1}, \nu\right) = H\left(x_1, x_2, f_{1,1}, \nu + \Delta\nu\right) - H\left(x_1, x_2, f_{1,1}, \nu\right)$.

This completes the proof.

Remark 2. Theorem 1 shows that the solution to the optimal control problem (3.1)–(3.3), it is sufficient to find a solution $f_{1,1}(x) \in L_{q(x)}(G)$ of the integral equation (3.13). Then the optimal control $\nu(x)$ can be found as element of the Ω_∂, which gives the maximum value to the functional $H\left(x, f_{1,1}(x), \nu(x)\right)$ in Ω_∂ with respect to the function ν.

Example. It is obvious that Eq. (3.1) generalizes the vibrating string equation and the telegraph equation. Indeed, if we take $a_{0,0}(x) = -k,\ k = const \geq 0$ and $a_{1,0}(x) = a_{0,1}(x) \equiv 0$ in the right hand side of Eq. (3.1), we get

$$D_1 D_2 u(x) - k\, u(x) = \varphi(x, \nu(x)). \quad (3.17)$$

It is well known that (3.17) is a controlled process described by the telegraph equation. The telegraph equation arises in modeling of filtering and radio. Let $k = 0$. Then the adjoint equation (3.13) for the optimal control problem (3.1)–(3.3) takes the more simple form

$$f_{1,1}(x) + B(x) = 0, \quad x \in G.$$

Acknowledgments

The authors would like to express their gratitude to the referee for his/her very valuable comments and suggestions. The research of R. Bandaliyev and V. Guliyev was partially supported by the grant of Presidium Azerbaijan National Academy of Science 2015.

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