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A NEW STUDY ON THE ABSOLUTE SUMMABILITY FACTORS OF FOURIER SERIES

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Abstract. In this paper, we establish a new theorem on $\mid A, p_n \mid_k$ summability factors of Fourier series using matrix transformation, which generalizes a main theorem of Bor [6] on $\left|\bar{N}, p_n\right|_k$ summability factors of Fourier series. Also some new results have been obtained.

1. Introduction and Preliminaries

Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ... \tag{1.1}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \tag{1.2}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.3}$$

defines the sequence (σ_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [10]). The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|\Delta \sigma_{n-1}\right|^k < \infty. \tag{1.4}$$

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The series $\sum a_n$ is said to be summable $|A|_k$, $k \geq 1$, if (see[14])

$$\sum_{n=1}^{\infty} n^{k-1} \ \left|\bar{\Delta} A_n(s)\right|^k < \infty, \tag{1.5}$$

and also it is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [13])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty \tag{1.6}$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s). \tag{1.7}$$

If we set $p_n = 1$ for all n, $|A, p_n|_k$ summability is the same as $|A|_k$ summability. Also, if we take $a_{nv} = p_v/P_n$, then $|A|_k$ summability is the same as $|R, p_n|_k$ summability (see [3]). In the special case, when we take $a_{nv} = p_v/P_n$, then $|A, p_n|_k$ summability is the same as $\left|\bar{N}, p_n\right|_k$ summability.
A sequence (λ_n) is said to be convex if $\Delta^2\lambda_n \geq 0$ for every positive integer n, where $\Delta^2\lambda_n = \Delta(\Delta\lambda_n)$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.
Let $f(t)$ be a periodic function with period 2π, and integrable (L) over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)dt = 0 \tag{1.8}$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n cosnt + b_n sinnt) \quad = \quad \sum_{n=1}^{\infty} C_n(t). \tag{1.9}$$

2. KNOWN RESULTS

Many works dealing with $\left|\bar{N}, p_n\right|_k$ summability factors of Fourier series have been done
(see [2], [4]-[9], [11]-[12]). Among them, in [5], Bor has proved the following theorem concerning the $\left|\bar{N}, p_n\right|_k$ summability factors of Fourier series.
Theorem 2.1. If (λ_n) is a convex sequence such that $\sum p_n\lambda_n < \infty$, where (p_n) is a sequence of positive numbers such that $P_n \to \infty$ as $n \to \infty$ and $\sum_{v=1}^{n} P_v C_v(t) = O(P_n)$, then the series $\sum C_n(t) P_n \lambda_n$ is summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$.

Later on, Bor [6] has proved the following theorem, the conditions on the sequence (λ_n) is more general than in Theorem 2.1.
Theorem 2.2. If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n\lambda_n < \infty$, where (p_n) is a sequence of positive numbers such that $P_n \to \infty$ as $n \to \infty$ and $\sum_{v=1}^{n} P_v C_v(t) = O(P_n)$, then the series $\sum C_n(t) P_n \lambda_n$ is summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$.

3. MAIN RESULT

The aim of this paper is to generalize Theorem 2.2 for absolute matrix summability.
Before stating the main theorem, we must first introduce some further notations. Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \tag{3.1}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{3.2}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{3.3}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{3.4}$$

Now, we shall prove the following theorem.
Theorem 3.1. If $A = (a_{nv})$ is a positive normal matrix such that

$$\bar{a}_{no} = 1, \ n = 0, 1, ..., \tag{3.5}$$

$$a_{n-1,v} \geq a_{nv}, \quad \text{for} \quad n \geq v + 1, \tag{3.6}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right) \tag{3.7}$$

and all the conditions of Theorem 2.2 are satisfied, then the series $\sum C_n(t) P_n \lambda_n$ is summable $|A, p_n|_k$, $k \geq 1$.
It should be noted that, if we take $a_{nv} = \frac{p_v}{P_n}$ in above theorem, then we get Theorem 2.2.
We need the following lemma for the proof of our theorem.
Lemma 3.2 ([6]). If (λ_n) is a non-negative and non-increasing sequence such that $\sum p_n \lambda_n$ is convergent, where (p_n) is a sequence of positive numbers such that $P_n \to \infty$ as $n \to \infty$, then $P_n \lambda_n = O(1)$ as $n \to \infty$ and $\sum P_n \Delta \lambda_n < \infty$.

4. PROOF OF THEOREM 3.1

Let $T_n(t)$ denotes the A-transform of the series $\sum C_n(t) P_n \lambda_n$. Then, by (12) and (13), we have

$$\bar{\Delta} I_n(t) \quad = \quad \sum_{v=1}^{n} \hat{a}_{nv} C_v(t) P_v \lambda_v.$$

Applying Abel's transformation to this sum, we get that

$$\begin{aligned}
\bar{\Delta} I_n(t) &= \sum_{v=1}^{n} \hat{a}_{nv} C_v(t) P_v \lambda_v \\
&= \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv}\lambda_v) \sum_{r=1}^{v} P_r C_r(t) + \hat{a}_{nn}\lambda_n \sum_{v=1}^{n} P_v C_v(t) \\
&= O(1) \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv}\lambda_v) P_v + O(1) a_{nn}\lambda_n P_n \\
&= O(1) \left\{ \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv})\lambda_v P_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \Delta\lambda_v P_v + a_{nn}\lambda_n P_n \right\} \\
&= O(1) \left\{ I_{n,1}(t) + I_{n,2}(t) + I_{n,3}(t) \right\}.
\end{aligned}$$

Since

$$|I_{n,1}(t) + I_{n,2}(t) + I_{n,3}(t)|^k \leq 3^k (|I_{n,1}(t)|^k + |I_{n,2}(t)|^k + |I_{n,3}(t)|^k),$$

to complete the proof of Theorem 3.1, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |I_{n,r}(t)|^k < \infty, \quad \text{for} \quad r = 1, 2, 3. \tag{3.8}$$

First, by applying Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} |I_{n,1}(t)|^k &= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} \left\{ \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \lambda_v P_v \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} \left\{ \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \lambda_v^k P_v^k \right\} \times \left\{ \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \right\}^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} a_{nn}^{k-1} \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \lambda_v^k P_v^k \\
&= O(1) \sum_{v=1}^{m} \lambda_v^k P_v^k \sum_{n=v+1}^{m+1} |\Delta_v(\hat{a}_{nv})| \\
&= O(1) \sum_{v=1}^{m} \lambda_v^k P_v^k a_{vv} \\
&= O(1) \sum_{v=1}^{m} \lambda_v^k P_v^k \frac{p_v}{P_v} \\
&= O(1) \sum_{v=1}^{m} (P_v \lambda_v)^{k-1} p_v \lambda_v \\
&= O(1) \sum_{v=1}^{m} p_v \lambda_v = O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.2.
Now, using Hölder's inequality, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|I_{n,2}(t)|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|^k P_v\Delta\lambda_v\right\}\\
&\quad\times\left\{\sum_{v=1}^{n-1}P_v\Delta\lambda_v\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|^{k-1}|\hat{a}_{n,v+1}|P_v\Delta\lambda_v\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|P_v\Delta\lambda_v\\
&= O(1)\sum_{v=1}^{m}P_v\Delta\lambda_v\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}P_v\Delta\lambda_v = O(1)\quad\text{as}\quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.2.
Finally, since $P_n\lambda_n = O(1)$ as $n\to\infty$, as in $I_{n,1}(t)$, we have that

$$\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|I_{n,3}(t)|^k &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}a_{nn}^k\lambda_n^kP_n^k\\
&= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\frac{p_n}{P_n}\right)^{k}\lambda_n^kP_n^k\\
&= O(1)\sum_{n=1}^{m}(P_n\lambda_n)^{k-1}p_n\lambda_n\\
&= O(1)\quad\text{as}\quad m\to\infty.
\end{aligned}$$

This completes the proof of Theorem 3.1.

4. CONCLUSIONS

Corollary 5.1. If we take $p_n = 1$ for all values of n in Theorem 3.1, then we get a result for dealing with $|A|_k$ summability factors of Fourier series.
Corollary 5.2. If we take $a_{nv} = \frac{p_v}{P_n}$ in Theorem 3.1, then we get Theorem 2.2.
Corollary 5.3. If we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n in Theorem 3.1, then we get a result concerning the $|C,1|_k$ summability factors of Fourier series.

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