



An extension of Caputo fractional derivative operator and its applications

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Abstract

In this paper, an extension of Caputo fractional derivative operator is introduced, and the extended fractional derivatives of some elementary functions are calculated. At the same time, extensions of some hypergeometric functions and their integral representations are presented by using the extended fractional derivative operator, linear and bilinear generating relations for extended hypergeometric functions are obtained, and Mellin transforms of some extended fractional derivatives are also determined. ©2016 All rights reserved.

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1. Introduction

Special functions are used in the application of mathematics to physical and engineering problems. In recent years, many authors considered the several extensions of well known special functions (see, for example, [2, 3, 9, 12, 13]; see also the very recent work [8, 10]). In 1994, Chaudhry and Zubair [4], introduced the generalized representation of gamma function. In 1997, Chaudhry et al. [2] presented the following extension of Euler's beta function

$$B_p(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} e^{\left(\frac{-p}{t(1-t)}\right)} dt,$$

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where $Re(p) > 0, Re(x) > 0, Re(y) > 0$.

Recently, Chaudhry et al. [3] used $B_p(x, y)$ to extend the hypergeometric functions as

$$F_p(\alpha, \beta; \gamma; z) := \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \frac{B_p(\beta + n, \gamma - \beta)}{B(\beta, \gamma - \beta)} z^n,$$

where $p \geq 0, Re(\gamma) > Re(\beta) > 0$ and $|z| < 1$. The symbol $(\alpha)_n$ denotes the Pochhammer’s symbol defined by

$$(\alpha)_n := \frac{\Gamma(a + n)}{\Gamma(a)}, \quad (\alpha)_0 := 1.$$

Afterwards, in [1] Özarslan and Özergin obtained linear and bilinear generating relations for extended hypergeometric functions by defining the extension of the Riemann-Liouville fractional derivative operator as

$$\begin{aligned} \mathbf{D}_z^{\alpha,p} f(z) &:= \frac{d^m}{dz^m} \mathbf{D}_z^{\alpha-m} f(z) \\ &= \frac{d^m}{dz^m} \left\{ \frac{1}{\Gamma(-\alpha + m)} \int_0^z (z - t)^{-\alpha+m-1} e^{\left(\frac{-pz^2}{t(z-t)}\right)} f(t) dt \right\}, \end{aligned}$$

where $Re(p) > 0$ and $m - 1 < Re(\alpha) < m, m \in \mathbb{N}$. It is obvious that, these extensions given above, coincide with original ones when $p = 0$.

The above-mentioned works have largely motivated our present study. The principle aim of the paper is to present extension of the Caputo fractional derivative operator and calculating the extended fractional derivatives of some elementary functions. In the sequel, extensions of some hypergeometric functions and their integral representations are presented by using the extended fractional derivative operator, linear and bilinear generating relations for extended hypergeometric functions are obtained, and Mellin transforms of some extended fractional derivatives are also determined.

2. Extended hypergeometric functions

In this section, we introduce the extensions of Gauss hypergeometric function ${}_2F_1$, the Appell hypergeometric functions F_1, F_2 and the Lauricella hypergeometric function $F_{D,p}^3$. Throughout this paper we assume that $Re(p) > 0$ and $m \in \mathbb{N}$.

Definition 2.1. The extended Gauss hypergeometric function is

$${}_2F_1(a, b; c; z; p) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(b - m)_n} \frac{B_p(b - m + n, c - b + m)}{B(b - m, c - b + m)} \frac{z^n}{n!} \tag{2.1}$$

for all $|z| < 1$ where $m < Re(b) < Re(c)$.

Definition 2.2. The extended Appell hypergeometric function F_1 is

$$F_1(a, b, c; d; x, y; p) := \sum_{n,k=0}^{\infty} \frac{(a)_{n+k} (b)_n (c)_k}{(a - m)_{n+k}} \frac{B_p(a - m + n + k, d - a + m)}{B(a - m, d - a + m)} \frac{x^n y^k}{n! k!} \tag{2.2}$$

for all $|x| < 1, |y| < 1$ where $m < Re(a) < Re(d)$.

Definition 2.3. The extended Appell hypergeometric function F_2 is

$$F_2(a, b, c; d, e; x, y; p) := \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}(b)_n(c)_k}{(b-m)_n(e)_k} \frac{B_p(b-m+n, d-b+m)}{B(b-m, d-b+m)} \frac{x^n y^k}{n! k!} \tag{2.3}$$

$$= \sum_{n,k=0}^{\infty} \frac{(a)_{n+k}(b)_n(c)_k}{(d)_n(c-m)_k} \frac{B_p(c-m+k, e-c+m)}{B(c-m, e-c+m)} \frac{x^n y^k}{n! k!} \tag{2.4}$$

$$= \sum_{n,k=0}^{\infty} \left[\frac{(a)_{n+k}(b)_n(c)_k}{(b-m)_n(c-m)_k} \frac{B_p(b-m+n, d-b+m)}{B(b-m, d-b+m)} \frac{B_p(c-m+k, e-c+m)}{B(c-m, e-c+m)} \frac{x^n y^k}{n! k!} \right] \tag{2.5}$$

for all $|x| + |y| < 1$ where $m < Re(b) < Re(d)$ and $m < Re(c) < Re(e)$.

Definition 2.4. The extended Lauricella hypergeometric function $F_{D,p}^3$ is

$$F_{D,p}^3(a, b, c, d; e; x, y, z; p) := \sum_{n,k,r=0}^{\infty} \frac{(a)_{n+k+r}(b)_n(c)_k(d)_r}{(a-m)_{n+k+r}} \frac{B_p(a-m+n+k+r, e-a+m)}{B(a-m, e-a+m)} \frac{x^n y^k z^r}{n! k! r!} \tag{2.6}$$

for all $\sqrt{|x|} + \sqrt{|y|} + \sqrt{|z|} < 1$ where $m < Re(a) < Re(e)$.

Note that when $p = 0$, these functions reduces to well known Gauss hypergeometric function ${}_2F_1$, Appell functions F_1, F_2 and Lauricella function F_D^3 , respectively.

3. Extended Caputo fractional derivative operator

The fractional derivative operators has gained considerable popularity and importance during the past few years. In literature point of view many fractional derivative operators already proved their importance. Very recently, fractional operator, whose derivative has singular kernel introduced by Yang et al. [14]. Motivated by above work many researchers applied new derivative in certain real world problems (see, e.g., [5, 7, 15–17]). In the sequel, we aim to extend the definition of the classical Caputo fractional derivative operator.

The classical Caputo fractional derivative is defined by

$$D^\mu f(z) := \frac{1}{\Gamma(m-\mu)} \int_0^z (z-t)^{m-\mu-1} \frac{d^m}{dt^m} f(t) dt,$$

where $m-1 < Re(\mu) < m, m \in \mathbb{N}$. We refer [6] to the reader for more information about fractional calculus.

Inspired by the same idea in [1], we introduce the *Extended Caputo Fractional Derivative* as

$$D_z^{\mu,p} f(z) := \frac{1}{\Gamma(m-\mu)} \int_0^z (z-t)^{m-\mu-1} e^{\left(\frac{-pz^2}{i(z-t)}\right)} \frac{d^m}{dt^m} f(t) dt, \tag{3.1}$$

where $Re(p) > 0$ and $m-1 < Re(\mu) < m, m \in \mathbb{N}$. In the case $p = 0$, extended Caputo fractional derivative reduces to classical Caputo Fractional derivative, and also when $\mu = m \in \mathbb{N}_0$ and $p = 0$,

$$D_z^{m,0} f(z) := f^{(m)}(z).$$

Now, we begin our investigation by calculating the extended fractional derivatives of some elementary functions.

Theorem 3.1. *Let $m-1 < Re(\mu) < m$ and $Re(\mu) < Re(\lambda)$ then*

$$D_z^{\mu,p} \left\{ z^\lambda \right\} = \frac{\Gamma(\lambda+1)B_p(\lambda-m+1, m-\mu)}{\Gamma(\lambda-\mu+1)B(\lambda-m+1, m-\mu)} z^{\lambda-\mu}.$$

Proof. With direct calculation, we get

$$\begin{aligned}
 D_z^{\mu,p} \{z^\lambda\} &= \frac{1}{\Gamma(m-\mu)} \int_0^z (z-t)^{m-\mu-1} e^{\left(\frac{-pz^2}{t(z-t)}\right)} \frac{d^m}{dt^m} t^\lambda dt \\
 &= \frac{1}{\Gamma(m-\mu)} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-m+1)} \int_0^z (z-t)^{m-\mu-1} t^{\lambda-m} e^{\left(\frac{-pz^2}{t(z-t)}\right)} dt \\
 &= \frac{z^{\lambda-\mu}}{\Gamma(m-\mu)} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-m+1)} \int_0^1 (1-u)^{m-\mu-1} u^{\lambda-m} e^{\left(\frac{-p}{u(1-u)}\right)} du \\
 &= \frac{\Gamma(\lambda+1)B_p(\lambda-m+1, m-\mu)}{\Gamma(\lambda-\mu+1)B(\lambda-m+1, m-\mu)} z^{\lambda-\mu}. \quad \square
 \end{aligned}$$

Remark 3.2. Note that, $D_z^{\mu,p} \{z^\lambda\} = 0$ for $\lambda = 0, 1, \dots, m-1$.

The next theorem expresses the extended Caputo fractional derivative of an analytic function.

Theorem 3.3. *If $f(z)$ is an analytic function on the disk $|z| < \rho$ and has a power series expansion $f(z) = \sum_{n=0}^\infty a_n z^n$, then*

$$D_z^{\mu,p} \{f(z)\} = \sum_{n=0}^\infty a_n D_z^{\mu,p} \{z^n\}$$

where $m-1 < Re(\mu) < m$.

Proof. Using the power series expansion of f , we get

$$D_z^{\mu,p} \left\{ f(z) \right\} = \frac{1}{\Gamma(m-\mu)} \int_0^z (z-t)^{m-\mu-1} e^{\left(\frac{-pz^2}{t(z-t)}\right)} \sum_{n=0}^\infty a_n \frac{d^m}{dt^m} t^n dt.$$

Since the power series converges uniformly and the integral converges absolutely, then the order of the integration and the summation can be changed. So we get,

$$\begin{aligned}
 D_z^{\mu,p} \left\{ f(z) \right\} &= \sum_{n=0}^\infty a_n \left(\frac{1}{\Gamma(m-\mu)} \int_0^z (z-t)^{m-\mu-1} e^{\left(\frac{-pz^2}{t(z-t)}\right)} \frac{d^m}{dt^m} t^n dt \right) \\
 &= \sum_{n=0}^\infty a_n D_z^{\mu,p} \{z^n\}. \quad \square
 \end{aligned}$$

The proof of the following theorem is obvious from Theorem 3.1 and 3.3.

Theorem 3.4. *If $f(z)$ is an analytic function on the disk $|z| < \rho$ and has a power series expansion $f(z) = \sum_{n=0}^\infty a_n z^n$, then*

$$\begin{aligned}
 D_z^{\mu,p} \left\{ z^{\lambda-1} f(z) \right\} &= \sum_{n=0}^\infty a_n D_z^{\mu,p} \left\{ z^{\lambda+n-1} \right\} \\
 &= \frac{\Gamma(\lambda) z^{\lambda-\mu-1}}{\Gamma(\lambda-\mu)} \sum_{n=0}^\infty a_n \frac{(\lambda)_n}{(\lambda-\mu)_n} \frac{B_p(\lambda-m+n, m-\mu)}{B(\lambda-m+n, m-\mu)} z^n \\
 &= \frac{\Gamma(\lambda) z^{\lambda-\mu-1}}{\Gamma(\lambda-\mu)} \sum_{n=0}^\infty a_n \frac{(\lambda)_n}{(\lambda-m)_n} \frac{B_p(\lambda-m+n, m-\mu)}{B(\lambda-m, m-\mu)} z^n,
 \end{aligned}$$

where $m-1 < Re(\mu) < m < Re(\lambda)$.

The following theorems will be useful for finding the generating function relations.

Theorem 3.5. Let $m - 1 < \operatorname{Re}(\lambda - \mu) < m < \operatorname{Re}(\lambda)$, then

$$\begin{aligned}
 D_z^{\lambda-\mu,p} \left\{ z^{\lambda-1} (1-z)^{-\alpha} \right\} &= \frac{\Gamma(\lambda) z^{\mu-1}}{\Gamma(\mu)} \sum_{n=0}^{\infty} \frac{(\alpha)_n (\lambda)_n}{(\lambda-m)_n} \frac{B_p(\lambda-m+n, \mu-\lambda+m)}{B(\lambda-m, \mu-\lambda+m)} \frac{z^n}{n!} \\
 &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} {}_2F_1(\alpha, \lambda; \mu; z; p)
 \end{aligned} \tag{3.2}$$

for $|z| < 1$.

Proof. If we use the power series expansion of $(1-z)^{-\alpha}$ and (2.1), we get

$$\begin{aligned}
 D_z^{\lambda-\mu,p} \left\{ z^{\lambda-1} (1-z)^{-\alpha} \right\} &= D_z^{\lambda-\mu,p} \left\{ z^{\lambda-1} \sum_{n=0}^{\infty} (\alpha)_n \frac{z^n}{n!} \right\} \\
 &= \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} D_z^{\lambda-\mu,p} \left\{ z^{\lambda+n-1} \right\} \\
 &= \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \frac{\Gamma(\lambda+n)}{\Gamma(\mu+n)} \frac{B_p(\lambda-m+n, m-\lambda+\mu)}{B(\lambda-m+n, m-\lambda+\mu)} z^{\mu+n-1} \\
 &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \sum_{n=0}^{\infty} \frac{(\alpha)_n (\lambda)_n}{(\mu)_n} \frac{B_p(\lambda-m+n, m-\lambda+\mu)}{B(\lambda-m+n, m-\lambda+\mu)} \frac{z^n}{n!} \\
 &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \sum_{n=0}^{\infty} \frac{(\alpha)_n (\lambda)_n}{(\lambda-m)_n} \frac{B_p(\lambda-m+n, \mu-\lambda+m)}{B(\lambda-m, \mu-\lambda+m)} \frac{z^n}{n!} \\
 &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} {}_2F_1(\alpha, \lambda; \mu; z; p). \quad \square
 \end{aligned}$$

Theorem 3.6. Let $m - 1 < \operatorname{Re}(\lambda - \mu) < m < \operatorname{Re}(\lambda)$, then

$$\begin{aligned}
 D_z^{\lambda-\mu,p} \left\{ z^{\lambda-1} (1-az)^{-\alpha} (1-bz)^{-\beta} \right\} \\
 &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \sum_{n,k=0}^{\infty} \frac{(\lambda)_{n+k} (\alpha)_n (\beta)_k}{(\lambda-m)_{n+k}} \frac{B_p(\lambda-m+n+k, \mu-\lambda+m)}{B(\lambda-m, \mu-\lambda+m)} \frac{(az)^n (bz)^k}{n! k!} \\
 &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_1(\lambda, \alpha, \beta; \mu; az; bz; p)
 \end{aligned} \tag{3.3}$$

for $|az| < 1$ and $|bz| < 1$.

Proof. Using the power series expansion of $(1-az)^{-\alpha}$, $(1-bz)^{-\beta}$ and (2.2), we get

$$\begin{aligned}
 D_z^{\lambda-\mu,p} \left\{ z^{\lambda-1} (1-az)^{-\alpha} (1-bz)^{-\beta} \right\} &= D_z^{\lambda-\mu,p} \left\{ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\alpha)_n (\beta)_k}{n! k!} a^n b^k z^{\lambda+n+k-1} \right\} \\
 &= \sum_{n,k=0}^{\infty} \frac{(\alpha)_n (\beta)_k}{n! k!} a^n b^k D_z^{\lambda-\mu,p} \left\{ z^{\lambda+n+k-1} \right\} \\
 &= \sum_{n,k=0}^{\infty} \frac{(\alpha)_n (\beta)_k}{n! k!} a^n b^k \frac{\Gamma(\lambda+n+k) B_p(\lambda-m+n+k, m-\lambda+\mu)}{\Gamma(\lambda-m+n+k) \Gamma(m-\lambda+\mu)} z^{\mu+n+k-1} \\
 &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \sum_{n,k=0}^{\infty} \frac{(\lambda)_{n+k} (\alpha)_n (\beta)_k}{(\lambda-m)_{n+k}} \frac{B_p(\lambda-m+n+k, m-\lambda+\mu)}{B(\lambda-m, m-\lambda+\mu)} \frac{(az)^n (bz)^k}{n! k!} \\
 &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_1(\lambda, \alpha, \beta; \mu; az; bz; p). \quad \square
 \end{aligned}$$

Theorem 3.7. Let $m - 1 < \operatorname{Re}(\lambda - \mu) < m < \operatorname{Re}(\lambda)$, then

$$\begin{aligned}
 & D_z^{\lambda-\mu,p} \left\{ z^{\lambda-1} (1-az)^{-\alpha} (1-bz)^{-\beta} (1-cz)^{-\gamma} \right\} \\
 &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \sum_{n,k,r=0}^{\infty} \frac{(\lambda)_{n+k+r} (\alpha)_n (\beta)_k (\gamma)_r}{(\lambda-m)_{n+k+r}} \frac{B_p(\lambda-m+n+k+r, \mu-\lambda+m)}{B(\lambda-m, \mu-\lambda+m)} \frac{(az)^n (bz)^k (cz)^r}{n! k! r!} \\
 &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_{D,p}^3(\lambda, \alpha, \beta, \gamma; \mu; az; bz; cz; p)
 \end{aligned} \tag{3.4}$$

for $|az| < 1, |bz| < 1$ and $|cz| < 1$.

Proof. Using the power series expansion of $(1-az)^{-\alpha}, (1-bz)^{-\beta}, (1-cz)^{-\gamma}$ and (2.6), we get

$$\begin{aligned}
 & D_z^{\lambda-\mu,p} \left\{ z^{\lambda-1} (1-az)^{-\alpha} (1-bz)^{-\beta} (1-cz)^{-\gamma} \right\} \\
 &= D_z^{\lambda-\mu,p} \left\{ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{(\alpha)_n (\beta)_k (\gamma)_r}{n! k! r!} a^n b^k c^r z^{\lambda+n+k+r-1} \right\} \\
 &= \sum_{n,k,r=0}^{\infty} \frac{(\alpha)_n (\beta)_k (\gamma)_r}{n! k! r!} a^n b^k c^r D_z^{\lambda-\mu,p} \left\{ z^{\lambda+n+k+r-1} \right\} \\
 &= \sum_{n,k,r=0}^{\infty} \frac{(\alpha)_n (\beta)_k (\gamma)_r}{n! k! r!} a^n b^k c^r \frac{\Gamma(\lambda+n+k+r) B_p(\lambda-m+n+k+r, m-\lambda+\mu)}{\Gamma(\lambda-m+n+k+r) \Gamma(m-\lambda+\mu)} z^{\mu+n+k+r-1} \\
 &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \sum_{n,k,r=0}^{\infty} \frac{(\lambda)_{n+k+r} (\alpha)_n (\beta)_k (\gamma)_r}{(\lambda-m)_{n+k+r}} \frac{B_p(\lambda-m+n+k+r, m-\lambda+\mu)}{B(\lambda-m, m-\lambda+\mu)} \frac{(az)^n (bz)^k (cz)^r}{n! k! r!} \\
 &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_{D,p}^3(\lambda, \alpha, \beta, \gamma; \mu; az; bz; cz; p). \quad \square
 \end{aligned}$$

Theorem 3.8. Let $m - 1 < \operatorname{Re}(\lambda - \mu) < m < \operatorname{Re}(\lambda)$ and $m < \operatorname{Re}(\beta) < \operatorname{Re}(\gamma)$, then

$$\begin{aligned}
 & D_z^{\lambda-\mu,p} \left\{ z^{\lambda-1} (1-z)^{-\alpha} {}_2F_1\left(\alpha, \beta; \gamma; \frac{x}{1-z}\right) \right\} \\
 &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left[\frac{(\alpha)_{n+k} (\beta)_n (\lambda)_k}{(\beta-m)_n (\lambda-m)_k} \right. \\
 &\quad \left. \frac{B_p(\beta-m+n, \gamma-\beta+m)}{B(\beta-m, \gamma-\beta+m)} \frac{B_p(\lambda-m+k, \mu-\lambda+m)}{B(\lambda-m, \mu-\lambda+m)} \frac{x^n z^k}{n! k!} \right] \\
 &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_2(\alpha, \beta, \lambda; \gamma, \mu; x, z; p)
 \end{aligned} \tag{3.5}$$

for $|x| + |z| < 1$.

Proof. Using the power series expansion of $(1-z)^{-\alpha}, F_p$ and (2.5), we get

$$\begin{aligned}
 & D_z^{\lambda-\mu,p} \left\{ z^{\lambda-1} (1-z)^{-\alpha} {}_2F_1\left(\alpha, \beta; \gamma; \frac{x}{1-z}\right) \right\} \\
 &= D_z^{\lambda-\mu,p} \left\{ z^{\lambda-1} (1-z)^{-\alpha} \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\beta-m)_n n!} \frac{B_p(\beta-m+n, \gamma-\beta+m)}{B(\beta-m, \gamma-\beta+m)} \left(\frac{x}{1-z}\right)^n \right\} \\
 &= D_z^{\lambda-\mu,p} \left\{ z^{\lambda-1} (1-z)^{-\alpha-n} \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\beta-m)_n} \frac{B_p(\beta-m+n, \gamma-\beta+m)}{B(\beta-m, \gamma-\beta+m)} \frac{x^n}{n!} \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\beta-m)_n} \frac{B_p(\beta-m+n, \gamma-\beta+m)}{B(\beta-m, \gamma-\beta+m)} \frac{x^n}{n!} D_z^{\lambda-\mu,p} \left\{ z^{\lambda-1}(1-z)^{-\alpha-n} \right\} \\
 &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left[\frac{(\alpha)_{n+k}(\beta)_n(\lambda)_k}{(\beta-m)_n(\lambda-m)_k} \frac{B_p(\beta-m+n, \gamma-\beta+m)}{B(\beta-m, \gamma-\beta+m)} \frac{B_p(\lambda-m+k, \mu-\lambda+m)}{B(\lambda-m, \mu-\lambda+m)} \frac{x^n z^k}{n!k!} \right] \\
 &= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_2(\alpha, \beta, \lambda; \gamma, \mu; x, z; p). \quad \square
 \end{aligned}$$

4. Generating functions

In this section, we use the equalities (3.2), (3.3) and (3.5) for obtaining linear and bilinear generating relations for the extended hypergeometric function ${}_2F_1$.

Theorem 4.1. *Let $m - 1 < Re(\lambda - \mu) < m < Re(\lambda)$, then*

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} {}_2F_1(\alpha + n, \lambda; \mu; z; p)t^n = (1-t)^{-\alpha} {}_2F_1\left(\alpha, \lambda; \mu; \frac{z}{1-t}; p\right), \tag{4.1}$$

where $|z| < \min\{1, |1-t|\}$.

Proof. Taking the identity

$$[(1-z) - t]^{-\alpha} = (1-t)^{-\alpha} \left(1 - \frac{z}{1-t}\right)^{-\alpha}$$

in [11] and expanding the left hand side, we get

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} (1-z)^{-\alpha} \left(\frac{t}{1-z}\right)^n = (1-t)^{-\alpha} \left(1 - \frac{z}{1-t}\right)^{-\alpha},$$

when $|t| < |1-z|$. If we multiply the both sides with $z^{\lambda-1}$ and apply the extended Caputo fractional derivative operator $D_z^{\lambda-\mu,p}$, we get

$$D_z^{\lambda-\mu,p} \left\{ \sum_{n=0}^{\infty} \frac{(\alpha)_n t^n}{n!} z^{\lambda-1} (1-z)^{-\alpha-n} \right\} = D_z^{\lambda-\mu,p} \left\{ (1-t)^{-\alpha} z^{\lambda-1} \left(1 - \frac{z}{1-t}\right)^{-\alpha} \right\}.$$

Since $|t| < |1-z|$ and $Re(\lambda) > Re(\mu) > 0$, it is possible to change the order of the summation and the derivative as

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} D_z^{\lambda-\mu,p} \left\{ z^{\lambda-1} (1-z)^{-\alpha-n} \right\} t^n = (1-t)^{-\alpha} D_z^{\lambda-\mu,p} \left\{ z^{\lambda-1} \left(1 - \frac{z}{1-t}\right)^{-\alpha} \right\}.$$

So we get the result after using Theorem 3.5 on both sides. □

Theorem 4.2. *Let $m - 1 < Re(\lambda - \mu) < m < Re(\lambda)$, then*

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} {}_2F_1(\beta - n, \lambda; \mu; z; p)t^n = (1-t)^{-\alpha} F_1\left(\beta, \alpha, \lambda; \mu; z; \frac{zt}{1-t}; p\right),$$

where $|t| < \frac{1}{1+|z|}$.

Proof. Taking the identity

$$[1 - (1 - z)t]^{-\alpha} = (1 - t)^{-\alpha} \left(1 + \frac{zt}{1 - t}\right)^{-\alpha}$$

in [11] and expanding the left hand side, we get

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} (1 - z)^n t^n = (1 - t)^{-\alpha} \left(1 - \frac{-zt}{1 - t}\right)^{-\alpha},$$

when $|t| < |1 - z|$. If we multiply the both sides with $z^{\lambda-1}(1 - z)^{-\beta}$ and apply the extended Caputo fractional derivative operator $D_z^{\lambda-\mu,p}$, we get

$$D_z^{\lambda-\mu,p} \left\{ \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} z^{\lambda-1}(1 - z)^{-\beta} (1 - z)^n t^n \right\} = D_z^{\lambda-\mu,p} \left\{ (1 - t)^{-\alpha} z^{\lambda-1}(1 - z)^{-\beta} \left(1 - \frac{-zt}{1 - t}\right)^{-\alpha} \right\}.$$

Since $|zt| < |1 - t|$ and $Re(\lambda) > Re(\mu) > 0$, it is possible to change the order of the summation and the derivative as

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} D_z^{\lambda-\mu,p} \left\{ z^{\lambda-1}(1 - z)^{-\beta+n} \right\} t^n = (1 - t)^{-\alpha} D_z^{\lambda-\mu,p} \left\{ z^{\lambda-1}(1 - z)^{-\beta} \left(1 - \frac{-zt}{1 - t}\right)^{-\alpha} \right\}.$$

So we get the result after using Theorem 3.5 and Theorem 3.6. □

Theorem 4.3. Let $m - 1 < Re(\beta - \gamma) < m < Re(\beta)$ and $m < Re(\lambda) < Re(\mu)$, then

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} {}_2F_1(\alpha + n, \lambda; \mu; z; p) {}_2F_1(-n, \beta; \gamma; u; p) = F_2 \left(\alpha, \lambda, \beta; \mu, \gamma; z, \frac{ut}{1 - t}; p \right).$$

Proof. If we take $t \rightarrow (1 - u)t$ in (4.1) and then multiply the both sides with $u^{\beta-1}$, we get

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} {}_2F_1(\alpha + n, \lambda; \mu; z; p) u^{\beta-1} (1 - u)^n t^n = u^{\beta-1} [1 - (1 - u)t]^{-\alpha} {}_2F_1 \left(\alpha, \lambda; \mu; \frac{z}{1 - (1 - u)t}; p \right).$$

Applying the fractional derivative $D_u^{\beta-\gamma}$ to both sides and changing the order we find

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} {}_2F_1(\alpha + n, \lambda; \mu; z; p) D_u^{\beta-\gamma} \left\{ u^{\beta-1} (1 - u)^n \right\} t^n \\ = D_u^{\beta-\gamma} \left\{ u^{\beta-1} [1 - (1 - u)t]^{-\alpha} {}_2F_1 \left(\alpha, \lambda; \mu; \frac{z}{1 - (1 - u)t}; p \right) \right\} \end{aligned}$$

when $|z| < 1$, $\left| \frac{1-u}{1-z} t \right| < 1$ and $\left| \frac{z}{1-t} \right| + \left| \frac{ut}{1-t} \right| < 1$. If we write the equality like

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} {}_2F_1(\alpha + n, \lambda; \mu; z; p) D_u^{\beta-\gamma} \left\{ u^{\beta-1} (1 - u)^n \right\} t^n \\ = D_u^{\beta-\gamma} \left\{ u^{\beta-1} \left[1 - \frac{-ut}{1 - t} \right]^{-\alpha} {}_2F_1 \left(\alpha, \lambda; \mu; \frac{z}{1 - \frac{-ut}{1-t}}; p \right) \right\} \end{aligned}$$

and using Theorem 3.5 and Theorem 3.8 we get the desired result. □

5. Further results and observations

In this section, we apply the extended Caputo fractional derivative operator (3.1) to familiar functions e^z and ${}_2F_1(a, b; c; z)$. We also obtain the Mellin transforms of some extended Caputo fractional derivatives and we give the integral representations of extended hypergeometric functions.

Theorem 5.1. *The extended Caputo fractional derivative of $f(z) = e^z$ is*

$$D_z^{\mu,p}\{e^z\} = \frac{z^{m-\mu}}{\Gamma(m-\mu)} \sum_{n=0}^{\infty} \frac{z^n}{n!} B_p(m-\mu, n+1)$$

for all z .

Proof. Using the power series expansion of e^z and Theorem 3.3, we get

$$\begin{aligned} D_z^{\mu,p}\{e^z\} &= \sum_{n=0}^{\infty} \frac{1}{n!} D_z^{\mu,p}\{z^n\} \\ &= \sum_{n=m}^{\infty} \frac{\Gamma(n+1)B_p(n-m+1, m-\mu)}{\Gamma(n-\mu+1)B(n-m+1, m-\mu)} \frac{z^{n-\mu}}{n!} \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(n+m+1)B_p(n+1, m-\mu)}{\Gamma(n+m-\mu+1)B(n+1, m-\mu)} \frac{z^{n+m-\mu}}{(n+m)!} \\ &= \frac{z^{m-\mu}}{\Gamma(m-\mu)} \sum_{n=0}^{\infty} \frac{z^n}{n!} B_p(m-\mu, n+1). \end{aligned}$$

□

Theorem 5.2. *The extended Caputo fractional derivative of ${}_2F_1(a, b; c; z)$ is*

$$D_z^{\mu,p}\left\{{}_2F_1(a, b; c; z)\right\} = \frac{(a)_m(b)_m}{(c)_m} \frac{z^{m-\mu}}{\Gamma(1-\mu+m)} \sum_{n=0}^{\infty} \frac{(a+m)_n(b+m)_n}{(c+m)_n(1-\mu+m)_n} \frac{B_p(m-\mu, n+1)}{B(m-\mu, n+1)} \frac{z^n}{n!}$$

for $|z| < 1$.

Proof. Using the power series expansion of ${}_2F_1(a, b; c; z)$ and making similar calculations, we get

$$\begin{aligned} D_z^{\mu,p}\left\{{}_2F_1(a, b; c; z)\right\} &= D_z^{\mu,p}\left\{\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}\right\} \\ &= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} D_z^{\mu,p}\{z^n\} \\ &= \sum_{n=m}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} \frac{\Gamma(n+1)B_p(m-\mu, n-m+1)}{\Gamma(n-\mu+1)B(m-\mu, n-m+1)} z^{n-\mu} \\ &= \sum_{n=0}^{\infty} \frac{(a)_{n+m}(b)_{n+m}}{(c)_{n+m}(n+m)!} \frac{\Gamma(n+m+1)B_p(m-\mu, n+1)}{\Gamma(n+m-\mu+1)B(m-\mu, n+1)} z^{n+m-\mu} \\ &= \frac{(a)_m(b)_m}{(c)_m} \frac{z^{m-\mu}}{\Gamma(1-\mu+m)} \sum_{n=0}^{\infty} \frac{(a+m)_n(b+m)_n}{(c+m)_n(1-\mu+m)_n} \frac{B_p(m-\mu, n+1)}{B(m-\mu, n+1)} \frac{z^n}{n!}. \end{aligned}$$

□

The following two theorems are about the Mellin transforms of extended Caputo fractional derivatives of two functions.

Theorem 5.3. *Let $Re(\lambda) > m - 1$ and $Re(s) > 0$, then*

$$\mathfrak{M} \left[D_z^{\mu,p} \{ z^\lambda \} : s \right] = \frac{\Gamma(\lambda + 1)\Gamma(s)}{\Gamma(\lambda - m + 1)\Gamma(m - \mu)} B(m - \mu + s, \lambda - m + s + 1) z^{\lambda - \mu}.$$

Proof. Using the definition of Mellin transform we get

$$\begin{aligned} \mathfrak{M} \left[D_z^{\mu,p} \{ z^\lambda \} : s \right] &= \int_0^\infty p^{s-1} D_z^{\mu,p} \{ z^\lambda \} dp \\ &= \int_0^\infty p^{s-1} \frac{\Gamma(\lambda + 1)B_p(m - \mu, \lambda - m + 1)}{\Gamma(\lambda - \mu + 1)B(m - \mu, \lambda - m + 1)} z^{\lambda - \mu} dp \\ &= \frac{\Gamma(\lambda + 1)z^{\lambda - \mu}}{\Gamma(\lambda - \mu + 1)B(m - \mu, \lambda - m + 1)} \int_0^\infty p^{s-1} B_p(m - \mu, \lambda - m + 1) dp. \end{aligned}$$

From the equality

$$\int_0^\infty b^{s-1} B_p(x, y) db = \Gamma(s)B(x + s, y + s), \quad Re(s) > 0, Re(x + s) > 0, Re(y + s) > 0$$

in [2, page 21] we get the result. □

Theorem 5.4. *Let $Re(s) > 0$ and $|z| < 1$, then*

$$\mathfrak{M} [D_z^{\mu,p} \{ (1 - z)^{-\alpha} \} : s] = \frac{\Gamma(s)}{\Gamma(m - \mu)} \sum_{n=0}^\infty \frac{B(m - \mu + s, n + s + 1)}{\Gamma(n + 1)} (\alpha)_n z^n.$$

Proof. With using the power series expansion of $(1 - z)^{-\alpha}$ and taking $\lambda = n$ in Theorem 5.3, we get

$$\begin{aligned} \mathfrak{M} \left[D_z^{\mu,p} \{ (1 - z)^{-\alpha} \} : s \right] &= \mathfrak{M} \left[D_z^{\mu,p} \left\{ \sum_{n=0}^\infty \frac{(\alpha)_n}{n!} z^n \right\} : s \right] \\ &= \sum_{n=0}^\infty \frac{(\alpha)_n}{n!} \mathfrak{M} [D_z^{\mu,p} \{ z^n \} : s] \\ &= \frac{\Gamma(s)}{\Gamma(m - \mu)} \sum_{n=m}^\infty \frac{B(m - \mu + s, n - m + s + 1)}{\Gamma(n - m + 1)} (\alpha)_n z^n \\ &= \frac{\Gamma(s)}{\Gamma(m - \mu)} \sum_{n=0}^\infty B(m - \mu + s, n + s + 1) \frac{(\alpha)_{n+m} z^n}{n!}. \end{aligned} \quad \square$$

Theorem 5.5. *The following integral representations are valid*

$${}_2F_1(a, b; c; z; p) = \frac{1}{B(b - m, c - b + m)} \int_0^1 \left\{ t^{b-m-1} (1 - t)^{c-b+m-1} e_2^{\left(\frac{-p}{i(1-t)}\right)} F_1(a, b; b - m; zt) \right\} dt, \quad (5.1)$$

$$\begin{aligned} F_1(a, b, c; d; x, y; p) &= \frac{1}{B(a - m, d - a + m)} \int_0^1 \left\{ t^{a-m-1} (1 - t)^{d-a+m-1} e^{\left(\frac{-p}{i(1-t)}\right)} F_1(a, b, c; a - m; xt, yt) \right\} dt, \end{aligned} \quad (5.2)$$

$$\begin{aligned} F_2(a, b, c; d, e; x, y; p) &= \frac{1}{B(b - m, d - b + m)} \int_0^1 \left\{ t^{b-m-1} (1 - t)^{d-b+m-1} e^{\left(\frac{-p}{i(1-t)}\right)} F_2(a, b, c; b - m, e; xt, y) \right\} dt, \end{aligned} \quad (5.3)$$

$$F_2(a, b, c; d, e; x, y; p) = \frac{1}{B(c-m, e-c+m)} \int_0^1 \left\{ t^{c-m-1} (1-t)^{e-c+m-1} e^{\left(\frac{-p}{t(1-t)}\right)} F_2(a, b, c; d, b-m; x, yt) \right\} dt, \quad (5.4)$$

$$F_2(a, b, c; d, e; x, y; p) = \frac{1}{B(b-m, e-b+m)B(c-m, e-c+m)} \int_0^1 \int_0^1 \left\{ t^{b-m-1} u^{c-m-1} (1-t)^{d-b+m-1} (1-u)^{e-c+m-1} e^{\left(-\frac{p}{t(1-t)} - \frac{p}{u(1-u)}\right)} F_2(a, b, c; b-m, c-m; xt, y\tau) \right\} dt du. \quad (5.5)$$

Proof. The integral representations (5.1)-(5.5) can be obtained directly by replacing the function B_p with its integral representation in (2.1)-(2.5), respectively. \square

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