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A Certain Sequence of Functions Involving the Aleph Function

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Abstract: Sequences of functions play an important role in approximation theory. In this paper, we aim to establish a (presumably new) sequence of functions involving the Aleph function by using operational techniques. Some generating relations and finite summation formulas of the sequence presented here are also considered.

Keywords: Special function; Generating relation; Aleph \aleph -function; Sequence of functions; Finite summation formula.

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1 Introduction

Recently, interest has developed into study of operational techniques, due to their importance in many field of engineering and mathematical physics. The sequences of functions play an important role in approximation theory. They can be used to show that a solution to a differential equation exists. Therefore, a large body of research into the development of these sequences has been published

In the literature, there are numerous sequences of functions, which are widely used in physics and mathematics as well as in engineering. Sequences of functions are also used to solve some differential equations in a rather efficient way. Here, we introduce and investigate further computable extensions of the sequence of functions involving the Aleph function, represented with \aleph , by using operational techniques. The generating relations

and finite summation formulas in terms of the Aleph function, are written in compact and easily computable form in Sections 2 and 3. Finally, some special cases and concluding remarks are discussed in Section 4.

Throughout this paper, let \mathbb{C} , \mathbb{R} , \mathbb{R}^+ , \mathbb{Z}_0^- , and \mathbb{N} be sets of complex numbers, real numbers, positive real numbers, non-positive integers and positive integers respectively. Also $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. The Aleph function, which is a general higher transcendental function and was introduced by Südland *et al.* [26, 27], is defined by means of a Mellin-Barnes type integral in the following manner (see, e.g., [23, 24])

$$\begin{aligned} \aleph[z] &= \aleph_{p_k, q_k, \tau_k; r}^{m, n} \left[z \left| \begin{array}{l} (a_j, A_j)_{1, n}, [\tau_k (a_{jk}, A_{jk})]_{n+1, p_k; r} \\ (b_j, B_j)_{1, m}, [\tau_k (b_{jk}, B_{jk})]_{m+1, q_k; r} \end{array} \right. \right] \\ &:= \frac{1}{2\pi i} \int_L \Omega_{p_k, q_k, \tau_k; r}^{m, n}(s) z^{-s} ds \end{aligned} \quad (1)$$

where $z \in \mathbb{C} - \{0\}$, $i = \sqrt{-1}$ and

$$\begin{aligned} \Omega_{p_k, q_k, \tau_k; r}^{m, n}(s) &= \\ &= \frac{\prod_{j=1}^m \Gamma(b_j + B_j s) \prod_{j=1}^n \Gamma(1 - a_j - A_j s)}{\sum_{k=1}^r \tau_k \prod_{j=m+1}^{q_k} \Gamma(1 - b_{jk} - B_{jk} s) \prod_{j=n+1}^{p_k} \Gamma(a_{jk} + A_{jk} s)} \end{aligned} \quad (2)$$

here Γ denotes the familiar Gamma function; the integration path $L = L_{i\gamma\infty}$ ($\gamma \in \mathbb{R}$) extends from $\gamma - i\infty$ to $\gamma + i\infty$; the poles of Gamma function $\Gamma(1 - a_j - A_j s)$ ($j = 1, 2, \dots, n$) do not coincide with those of $\Gamma(b_j + B_j s)$ ($j = 1, 2, \dots, m$); the parameters $p_k, q_k \in \mathbb{N}_0$ satisfy the conditions $0 \leq n \leq p_k$, $1 \leq m \leq q_k$; $\tau_k > 0$ ($k = 1, 2, \dots, r$); the parameters $A_j, B_j, A_{jk}, B_{jk} > 0$ and $a_j, b_j, a_{jk}, b_{jk} \in \mathbb{C}$; the empty product in (2) is (as usual) understood to be unity. The existence conditions for the defining integral (1) are given below

$$\varphi_l > 0, \quad |\arg(z)| < \frac{\pi}{2} \varphi_l, \quad l = 1, 2, \dots, r \quad (3)$$


$$\varphi_l \geq 0, \quad |\arg(z)| < \frac{\pi}{2} \varphi_l \text{ and } \Re\{\zeta_l\} + 1 < 0 \quad (4)$$

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where

$$\varphi_l = \sum_{j=1}^n A_j + \sum_{j=1}^m B_j - \tau_l \left(\sum_{j=n+1}^{p_l} A_{jl} + \sum_{j=m+1}^{q_l} B_{jl} \right) \quad (5)$$

$$\varsigma_l = \sum_{j=1}^m b_j - \sum_{j=1}^n a_j + \tau_l \left(\sum_{j=m+1}^{q_l} b_{jl} - \sum_{j=n+1}^{p_l} a_{jl} \right) + \frac{1}{2}(p_l - q_l). \quad (6)$$

Remark 1. Setting $\tau_k = 1$ ($k = 1, 2, \dots, r$) in (1.1) yields the I-function [25], whose further special case when $r = 1$ reduces to the familiar Fox’s H-function (see [21, 22]).

For our purpose, we also required some known functions and earlier works. In 1971, Mittal [12] gives the Rodrigues formula for the generalized Lagurre polynomials defined as

$$T_{kn}^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} \exp(p_k(x)) D^n [x^{\alpha+n} \exp(-p_k(x))] \quad (7)$$

where $p_k(x)$ is a polynomial in x of degree k and $D \equiv \frac{d}{dx}$.

Mittal [13] also proved the following relation for (7)

$$T_{kn}^{(\alpha+s-1)}(x) = \frac{1}{n!} x^{-\alpha-n} \exp(p_k(x)) T_s^n [x^\alpha \exp(-p_k(x))] \quad (8)$$

where s is constant and $T_s \equiv x(s + xD)$.

In this sequel, in 1979, Srivastava and Singh [19] studied a sequence of functions

$$V_n^{(\alpha)}(x; a, k, s) = \frac{1}{n!} x^{-\alpha} \exp\{p_k(x)\} \theta^n [x^\alpha \exp\{-p_k(x)\}], \quad (9)$$

by employing the operator $\theta \equiv x^\alpha(s + xD)$, where a and s are constants.

In this paper, a new sequence of functions $\left\{ V_n^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; a)}(x; a, k, s) \right\}_{n=0}^\infty$ is introduced as

$$V_n^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; a)}(x; a, k, s) = \frac{1}{n!} x^{-\alpha} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)] \times (T_x^{a, s})^n \left\{ x^\alpha \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [-p_k(x)] \right\}, \quad (10)$$

where $T_x^{a, s} \equiv x^\alpha(s + xD)$, a and s are constants, k is a finite and non-negative integer, $p_k(x)$ is a polynomial in x of degree k and $\aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu}$ is the Aleph function of one variable given in equation (1). Then, some generating relations and finite summation formulas for sequence of functions (10) have been obtained.

The following properties of the differential operator $T_x^{a, s} \equiv x^\alpha(s + xD)$ (Mittal [14], Patil and Thakare [15], Sri-

vastava and Singh [19]) are essential for our investigations:

$$\exp(tT_x^{a, s}) \left(x^\beta f(x) \right) = x^\beta (1 - ax^\alpha t)^{-\left(\frac{\beta+s}{a}\right)} f\left(x(1 - ax^\alpha t)^{-1/a}\right), \quad (11)$$

$$\sum_{n=0}^\infty \frac{t^n}{n!} (T_x^{a, s})^n (x^{\alpha-an} f(x)) = x^\alpha (1 + at)^{-1 + \left(\frac{\alpha+s}{a}\right)} f\left(x(1 + at)^{1/a}\right), \quad (12)$$

$$(T_x^{a, s})^n (xuv) = x \sum_{m=0}^n \binom{n}{m} (T_x^{a, s})^{n-m}(v) (T_x^{a, 1})^m(u), \quad (13)$$

$$(T_x^{a, s})^n = x^n a (s + xD)(s + a + xD) \cdots (s + (n-1)a + xD), \quad (14)$$

$$(1 + xD)(1 + a + xD) \cdots (1 + (m-1)a + xD) x^{\beta-1} = a^m \left(\frac{\beta}{a}\right)_m x^{\beta-1}, \quad (15)$$

$$(1 - at)^{-\frac{a}{a}} = (1 - at)^{-\frac{\beta}{a}} \sum_{m=0}^\infty \left(\frac{\alpha - \beta}{a}\right)_m \frac{(at)^m}{m!}. \quad (16)$$

2 Generating Relations

First generating relation:

$$\sum_{n=0}^\infty V_n^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; a)}(x; a, k, s) x^{-an} t^n = (1 - at)^{-\left(\frac{\alpha+s}{a}\right)} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)] \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} \left[-p_k \left(x(1 - at)^{-1/a} \right) \right]. \quad (17)$$

Second generating relation:

$$\sum_{n=0}^\infty V_n^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; a-an)}(x; a, k, s) x^{-an} t^n = (1 + at)^{-1 + \left(\frac{\alpha+s}{a}\right)} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)] \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} \left[-p_k \left(x(1 + at)^{1/a} \right) \right]. \quad (18)$$

Third generating relation:

$$\sum_{m=0}^\infty \binom{m+n}{n} V_{m+n}^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; a)}(x; a, k, s) x^{-am} t^m = (1 - at)^{-\left(\frac{\alpha+s}{a}\right)} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)] V_n^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; a)} \left(x(1 - at)^{-1/a}; a, k, s \right) \times \frac{\aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)]}{\aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} \left[x(1 - at)^{-1/a} \right]}. \quad (19)$$

Proof of first generating relation:

From (10), we have

$$\sum_{n=0}^{\infty} V_n^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; \alpha)}(x; a, k, s) t^n = x^{-\alpha} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)] \exp(t T_x^{a, s}) \left\{ x^\alpha \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [-p_k(x)] \right\}. \tag{20}$$

Using operational technique (11) in (20), we get

$$\sum_{n=0}^{\infty} V_n^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; \alpha)}(x; a, k, s) t^n = (1 - ax^a t)^{-\left(\frac{\alpha+s}{a}\right)} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)] \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} \times \left[-p_k \left(x(1 - ax^a t)^{-1/a} \right) \right]. \tag{21}$$

Replacing t by tx^{-a} , (17) is obtained.

Proof of second generating relation:

From (10), we obtain

$$\sum_{n=0}^{\infty} x^{-an} V_n^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; \alpha - an)}(x; a, k, s) t^n = x^{-a} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)] \sum_{n=0}^{\infty} \frac{t^n}{n!} (T_x^{a, s})^n \times \left\{ x^{a-an} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [-p_k(x)] \right\}. \tag{22}$$

Applying operational technique (12) in (22), we have

$$\sum_{n=0}^{\infty} x^{-an} V_n^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; \alpha - an)}(x; a, k, s) t^n = (1 + at)^{-1 + \left(\frac{\alpha+s}{a}\right)} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)] \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} \times \left[-p_k \left(x(1 + at)^{1/a} \right) \right], \tag{23}$$

which yields the desired result.

Proof of third generating relation:

We can write the following equation from (10)

$$(T_x^{a, s})^n \left[x^\alpha \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [-p_k(x)] \right] = n! x^\alpha \frac{V_n^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; \alpha)}(x; a, k, s)}{\aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)]}. \tag{24}$$

Thus we have

$$\exp(t T_x^{a, s}) \left\{ (T_x^{a, s})^n \left[x^\alpha \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [-p_k(x)] \right] \right\} = n! \exp(t T_x^{a, s}) \left[x^\alpha \frac{V_n^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; \alpha)}(x; a, k, s)}{\aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)]} \right], \tag{25}$$

or

$$\sum_{m=0}^{\infty} \frac{t^m}{m!} (T_x^{a, s})^{m+n} \left\{ x^\alpha \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [-p_k(x)] \right\} = n! \exp(t T_x^{a, s}) \left\{ x^\alpha \frac{V_n^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; \alpha)}(x; a, k, s)}{\aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)]} \right\}. \tag{26}$$

Applying operational technique (11), the above equation can then be written as

$$\sum_{m=0}^{\infty} \frac{t^m}{m!} (T_x^{a, s})^{m+n} \left[x^\alpha \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [-p_k(x)] \right] = n! x^\alpha (1 - ax^a t)^{-\left(\frac{\alpha+s}{a}\right)} \times \frac{V_n^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; \alpha)} \left(x(1 - ax^a t)^{-1/a}; a, k, s \right)}{\aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} \left[p_k \left(x(1 - ax^a t)^{-1/a} \right) \right]}. \tag{27}$$

Using (24) in (27), we obtain

$$\sum_{m=0}^{\infty} \frac{t^m (m+n)!}{m! n!} x^\alpha \frac{V_{m+n}^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; \alpha)}(x; a, k, s)}{\aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [-p_k(x)]} = x^\alpha (1 - ax^a t)^{-\left(\frac{\alpha+s}{a}\right)} \times \frac{V_n^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; \alpha)} \left(x(1 - ax^a t)^{-1/a}; a, k, s \right)}{\aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} \left[-p_k \left(x(1 - ax^a t)^{-1/a} \right) \right]}, \tag{28}$$

or

$$\sum_{m=0}^{\infty} \binom{m+n}{n} V_{m+n}^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; \alpha)}(x; a, k, s) t^m = (1 - ax^a t)^{-\left(\frac{\alpha+s}{a}\right)} \times \frac{\aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)] V_n^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; \alpha)} \left(x(1 - ax^a t)^{-1/a}; a, k, s \right)}{\aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} \left[p_k \left(x(1 - ax^a t)^{-1/a} \right) \right]}. \tag{29}$$

Replacing t by tx^{-a} , this gives result (19).

Remark 2. If we give some suitable parametric replacement in (17), (18) and (19), then we can see the known results (see [1–3, 7–13, 15, 17–20]).

3 Finite Summation Formulas

First finite summation formula:

$$V_n^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; \alpha)}(x; a, k, s) = \sum_{m=0}^n \frac{1}{m!} (ax^a)^m \left(\frac{\alpha}{a}\right)_m V_{n-m}^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; 0)}(x; a, k, s). \tag{30}$$

Second finite summation formula:

$$V_n^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; \alpha)}(x; a, k, s) = \sum_{m=0}^n \frac{1}{m!} (ax^a)^m \left(\frac{\alpha - \beta}{a}\right)_m V_{n-m}^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; \beta)}(x; a, k, s). \tag{31}$$

Proof of first finite summation formula:

From (10), we obtain

$$V_n^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; \alpha)}(x; a, k, s) = \frac{1}{n!} x^{-\alpha} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)] (T_x^{a, s})^n \left\{ x x^{\alpha-1} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [-p_k(x)] \right\}. \tag{32}$$

Using operational techniques (13), (14) and (15), we get

$$\begin{aligned} V_n^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; \alpha)}(x; a, k, s) &= \frac{1}{n!} x^{1-\alpha} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)] \\ &\times \sum_{m=0}^n \binom{n}{m} (T_x^{a, s})^{n-m} \left\{ \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [-p_k(x)] \right\} (T_x^{a, 1})^m (x^{\alpha-1}) \\ &= \frac{1}{n!} x^{1-\alpha} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)] \\ &\times \sum_{m=0}^n \frac{n!}{m!(n-m)!} (T_x^{a, s})^{n-m} \left\{ \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [-p_k(x)] \right\} \\ &\times x^{ma} [(1+xD)(1+a+xD)\dots(1+(m-1)a+xD)] \\ &\times (x^{\alpha-1}) = \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)] \sum_{m=0}^n \frac{x^{ma} a^m \left(\frac{\alpha}{a}\right)_m}{m!(n-m)!} (T_x^{a, s})^{n-m} \\ &\times \left\{ \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [-p_k(x)] \right\}. \tag{33} \end{aligned}$$

On the other hand, taking $\alpha = 0$ and replacing n by $n - m$ in (32), we find

$$V_{n-m}^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; 0)}(x; a, k, s) = \frac{1}{(n-m)!} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)] (T_x^{a, s})^{n-m} \left\{ \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [-p_k(x)] \right\} \tag{34}$$

or

$$(T_x^{a, s})^{n-m} \left\{ \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [-p_k(x)] \right\} = (n-m)! \frac{V_{n-m}^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; 0)}(x; a, k, s)}{\aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)]}. \tag{35}$$

Thus, using (35) in (33), we have the required result (30).

Proof of second finite summation formula:

From (10), we have

$$\sum_{n=0}^{\infty} V_n^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; \alpha)}(x; a, k, s) t^n = x^{-\alpha} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)] \exp(tT_x^{a, s}) \left\{ x^{\alpha} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [-p_k(x)] \right\}. \tag{36}$$

Applying operational technique (11) in (36) we obtain

$$\sum_{n=0}^{\infty} V_n^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; \alpha)}(x; a, k, s) t^n = (1 - ax^a t)^{-\left(\frac{\alpha+s}{a}\right)} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} \left[p_k(x) \right] \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} \left[-p_k(x) \left(x(1 - ax^a t)^{-1/a} \right) \right]. \tag{37}$$

Using operational technique (16), (37) reduces to

$$\begin{aligned} &\sum_{n=0}^{\infty} V_n^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; \alpha)}(x; a, k, s) t^n \\ &= (1 - ax^a t)^{-\left(\frac{\beta+s}{a}\right)} \sum_{m=0}^{\infty} \left(\frac{\alpha - \beta}{a}\right)_m \frac{(ax^a t)^m}{m!} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} \\ &\quad \times [p_k(x)] \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} \left[-p_k(x) \left(x(1 - ax^a t)^{-1/a} \right) \right] \\ &= x^{-\beta} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)] \sum_{m=0}^{\infty} \left(\frac{\alpha - \beta}{a}\right)_m \frac{(ax^a t)^m}{m!} \\ &\quad \times \exp(tT_x^{a, s}) \left\{ x^{\beta} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [-p_k(x)] \right\} \\ &= x^{-\beta} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)] \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\frac{\alpha - \beta}{a}\right)_m \frac{(ax^a)^m}{m!n!} \\ &\quad \times (T_x^{a, s})^n \left\{ x^{\beta} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [-p_k(x)] \right\} t^{n+m} \\ &= x^{-\beta} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)] \sum_{n=0}^{\infty} \sum_{m=0}^n \left(\frac{\alpha - \beta}{a}\right)_m \frac{(ax^a)^m}{m!(n-m)!} \\ &\quad \times (T_x^{a, s})^{n-m} \left\{ x^{\beta} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [-p_k(x)] \right\} t^n. \tag{38} \end{aligned}$$

Now equating the coefficients of t^n , we get

$$V_n^{(\lambda, \mu; \delta_r, \sigma_r, \tau_r; l; \alpha)}(x; a, k, s) = \sum_{m=0}^n \left(\frac{\alpha - \beta}{a}\right)_m \frac{(ax^a)^m}{m!} \times \frac{1}{(n-m)!} x^{-\beta} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [p_k(x)] (T_x^{a, s})^{n-m} \times \left\{ x^{\beta} \aleph_{\delta_r, \sigma_r, \tau_r; l}^{\lambda, \mu} [-p_k(x)] \right\}. \tag{39}$$

Using (10) in (39), we have result (31).

4 Special Cases

Here we consider some interesting special cases of the results given in Section 2 and 3.

1. If we select $\tau_k = 1$ ($k = 1, 2, \dots, r$), all the results established in equations (17), (18), (19), (30) and (31) can be reduced in to the results given by Mehar Chand [1].
2. The Aleph function can easily be reduced to the Fox's H-function by assigning suitable values to the parameters. All the results established in equation

(17), (18), (19), (30) and (31) are then reduced to the results given by Praveen Agarwal, Mehar Chand and Saket Dwivedi [2].

3. All the results established in equations (17), (18), (19), (30) and (31) can be reduced in to the known results given in [3, 4, 6].

It is further noted that a number of other special cases of our main results, as illustrated in Sections 2 and 3, can also be obtained. In this paper, we have studied a new sequence of functions involving the Aleph function by using operational techniques, and we have established some generating relations and finite summation formulas of the sequence. Moreover, in view of close relationships of the Aleph function with other special functions, it does not seem difficult to construct various known and new sequences.

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