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Chelyshkov collocation method for a class of mixed functional integro-differential equations

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ABSTRACT

In this study, a numerical matrix method based on Chelyshkov polynomials is presented to solve the linear functional integro-differential equations with variable coefficients under the initial-boundary conditions. This method transforms the functional equation to a matrix equation by means of collocation points. Also, using the residual function and Mean Value Theorem, an error analysis technique is developed. Some numerical examples are performed to illustrate the accuracy and applicability of the method.

1. Introduction

Functional integro-differential equations (FIDEs) have a major importance in modeling of some phenomena in many applied areas which include engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory, electrostatics, etc. [1–11]. Since the mentioned equations are usually difficult to solve analytically, numerical methods are required.

In recent years, some techniques have been used for solving FIDEs such as He's variational iteration technique [11] Homotopy perturbation method [12], He's homotopy perturbation method [13], Modified homotopy perturbation method [14], Rationalized Haar functions method [15], Chebyshev cardinal functions method [16], Differential transformation method [17], Tau method with error estimation [18], He's Variational iteration method [19], Collocation Methods [20], Adomian decomposition method [21], Adomain-Pade technique [22], Discontinuous Galerkin method [23], Legendre Multiwavelets [24], Trigonometric wavelets [25], Spectral Methods [26,27], and Meshless Method [28]. In addition, to solve the mentioned type equations, matrix methods based on Taylor, Chebyshev, Bessel, Bernoulli and Legendre polynomials have been studied by some authors [29–34].

In this paper, we deal with the mth order equations:

$$\sum_{i=0}^{n}\sum_{k=0}^{m}P_{ik}(x)y^{(k)}(\alpha_{ik}x+\tau_{ik})+F(x)+I(x)=g(x),\quad 0\leqslant x,t\leqslant b\leqslant 1, \tag{1}$$

so that

$$F(x)=\int_0^1\sum_{s=0}^{m}K_s(x,t)y^{(s)}(t)dt \quad\text{and}\quad I(x)=\int_0^{h(x)}\sum_{s=0}^{m}V_s(x,t)y^{(s)}(t)dt,$$

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under the mixed conditions

$$\sum_{k=0}^{m-1}(a_{jk}y^{(k)}(0)+b_{jk}y^{(k)}(b))=\lambda_j,\quad j=0,1,\ldots,m-1, \tag{2}$$

where $y^{(0)}(x)=y(x)$ is the unknown function, the known functions, $g(x), h(x), K_s(x,t)$ and $V_s(x,t)$ are defined on interval $0\leqslant x,t\leqslant 1$ which can be expanded to Maclaurin series and also $a_{jk}, b_{jk},$ and λ_j are real constants. The aim of this study is to find an appropriate solution expressed in the following form

$$y_N(x)=\sum_{n=0}^{N}a_nC_{Nn}(x), \tag{3}$$

so that a_n's and $C_{Nn}(x)$, $n=0,1,2,\ldots,N$, respectively, are the unknown Chelyshkov coefficients and Chelyshkov orthogonal polynomials of the degree N which can be chosen as any positive integer such that $N\geqslant m$.

The area of orthogonal polynomials is a very active research area in mathematics as well as in applications in mathematical physics, engineering and computer science. One of the latest set of orthogonal polynomials is the set of the Chelyshkov polynomials $\{C_{N0}(x), C_{N1}(x),\ldots, C_{NN}(x),\ldots\}$. Recently, these polynomials have created by Chelyshkov [35–37], which are orthogonal over the interval $[0,1]$ with respect to the weight function $w(x)=1$, and are explicitly defined by

$$C_{Nn}(x)=\sum_{j=0}^{N-n}(-1)^j\binom{N-n}{j}\binom{N+n+j+1}{N-n}x^{n+j},\quad n=0,1\ldots N.$$

This yields The Rodrigues' type representation

$$C_{Nn}(x)=\frac{1}{(N-n)!}\frac{1}{x^{n+1}}\frac{d^{N-n}}{dx^{N-n}}\left(x^{N+n+1}(1-x)^{N-n}\right),\quad n=0,1,\ldots,N,$$

and the following orthogonality relations

$$\int_0^1 C_{Np}(x)C_{Nq}(x)dx=\begin{cases}0, & p=q\\ \frac{1}{p+q+1}, & p\neq q\end{cases},\quad p,q=0,1,\ldots,N.$$

Also, it follows from this relation that

$$\int_0^1 C_{Nn}(x)dx=\int_0^1 x^n dx=\frac{1}{n+1}.$$

By using the Rodrigues' formula and the Cauchy integral formula for derivatives of an analytic function, one can obtain the integral relation

$$C_{Nn}(x)=\frac{1}{2\pi i}\frac{1}{x^{n+2}}\int_{\Omega_1}\frac{\bar{z}^{(N+2+n)}(1-z)^{N-n}}{(z-x^{-1})^{N-n+1}}dz,$$

where Ω_1 is a closed curve, which encloses the point $z=x^{-1}$.

Chelyshkov polynomials $C_{Nn}(x)$ have the analogous properties to those of the classical orthogonal polynomials. In fact, these polynomials are an example of such alternative orthogonal ones, which are not solutions of the hypergeometric type equations, but can be expressed in terms of the Jacobi ones. In addition, they can also be connected to hypergeometric functions, orthogonal exponential polynomials, and Jacobi polynomials $P_k^{(\alpha,\beta)}$ by the following relation

$$C_{Nn}(x)=(-1)^{N-n}x^{N-n}P_{N-n}^{(0,2n+1)}(2x-1),\quad n=0,1\ldots N.$$

Hence, they keep distinctively attributes of the classical orthogonal polynomials and may be facilitated to different problems on approximation. In the family of orthogonal polynomials $\{C_{Nn}(x)\}$, every member has degree N with N–n simple roots. Hence, for every N if the roots of the polynomial are chosen as node points, then an accurate numerical quadrature can be derived.

In this paper, we develop a new collocation method derived from Chelyshkov polynomials and illustrate the method by the problems in some works in the literature.

2. Fundamental matrix relations

We first consider the approximate solution $y(x)$ of Eq. (1) defined by the truncated orthogonal Chelyshkov series (3), which is in the form

$$y(x)\cong y_N(x)=\sum_{n=0}^{N}a_nC_{Nn}(x).$$

Then we can convert to the finite series (3) and its derivative $y_N^{(k)}(x)$ to matrix forms

$$y_N(x) = \mathbf{C}(x)\mathbf{A} \text{ or } y_N(x) = \boldsymbol{X}(x)\mathcal{C}\mathbf{A}$$
and
$$y_N^{(k)}(x) = \mathbf{C}^{(k)}(x)\mathbf{A} \text{ or } y_N(x) = \boldsymbol{X}(x)\boldsymbol{B}^k\mathcal{C}\mathbf{A}, k = 0,1,2,\ldots,N, \tag{4}$$

where

$$\mathbf{C}(x) = [\,C_{N0}(x) \quad C_{N1}(x) \quad \cdots \quad C_{NN}(x)\,], \mathbf{C}^{(k)}(x) = [\,C_{N0}^{(k)}(x) \quad C_{N1}^{(k)}(x) \quad \cdots \quad C_{CN}^{(k)}(x)\,],$$

$$\boldsymbol{X}(x) = \begin{bmatrix} 1 & x & \cdots & x^N \end{bmatrix}^T, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \ldots & N \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}_{(N+1)x(N+1)} \quad \mathbf{A} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix},$$

if N is odd,

$$\mathcal{C} = \begin{bmatrix} \binom{N}{0}\binom{N+1}{N} & 0 & \ldots & 0 & 0 \\ -\binom{N}{1}\binom{N+2}{N} & \binom{N-1}{0}\binom{N+2}{N-1} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{N}{N-1}\binom{2N}{N} & -\binom{N-1}{N-2}\binom{2N}{N-1} & \ldots & \binom{1}{0}\binom{2N}{1} & 0 \\ -\binom{N}{N}\binom{2N+1}{N} & \binom{N-1}{N-1}\binom{2N+1}{N-1} & \ldots & -\binom{1}{1}\binom{2N+1}{1} & 1 \end{bmatrix}_{(N+1)x(N+1)},$$

if N is even,

$$\mathcal{C} = \begin{bmatrix} \binom{N}{0}\binom{N+1}{N} & 0 & \ldots & 0 & 0 \\ -\binom{N}{1}\binom{N+2}{N} & \binom{N-1}{0}\binom{N+2}{N-1} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\binom{N}{N-1}\binom{2N}{N} & \binom{N-1}{N-2}\binom{2N}{N-1} & \ldots & \binom{1}{0}\binom{2N}{1} & 0 \\ \binom{N}{N}\binom{2N+1}{N} & -\binom{N-1}{N-1}\binom{2N+1}{N-1} & \ldots & -\binom{1}{1}\binom{2N+1}{1} & 1 \end{bmatrix}_{(N+1)x(N+1)}.$$

By using the matrix forms (4), we obtain matrix relation for the differential–difference part of Eq. (1)

$$[D(x)] = \sum_{i=0}^{n}\sum_{k=0}^{m} \boldsymbol{P}_{\boldsymbol{ik}}(x)\boldsymbol{X}(x)\boldsymbol{M}(\alpha_{ik},\tau_{ik})\boldsymbol{B}^k\mathcal{C}\mathbf{A}, \tag{5}$$

where, for $\alpha_{ik} \neq 0$ and $\tau_{ik} \neq 0$ [38],

$$\boldsymbol{M}(\alpha_{ik},\tau_{ik}) = \begin{bmatrix} \binom{0}{0}\alpha_{ik}^0\tau_{ik}^0 & \binom{1}{0}\alpha_{ik}^0\tau_{ik}^1 & \binom{2}{0}\alpha_{ik}^0\tau_{ik}^2 & \ldots & \binom{N}{0}\alpha_{ik}^0\tau_{ik}^N \\ 0 & \binom{1}{1}\alpha_{ik}^1\tau_{ik}^0 & \binom{2}{1}\alpha_{ik}^1\tau_{ik}^1 & \ldots & \binom{N}{1}\alpha_{ik}^1\tau_{ik}^N \\ 0 & 0 & \binom{2}{2}\alpha_{ik}^2\tau_{ik}^0 & \ldots & \binom{N}{2}\alpha_{ik}^2\tau_{ik}^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \binom{N}{N}\alpha_{ik}^N\tau_{ik}^N \end{bmatrix},$$

and, for $\alpha_{ik} \neq 0$ and $\tau_{ik} = 0$,

$$\boldsymbol{M}(\alpha_{ik},0)=\begin{bmatrix}1&0&0&\dots&0\\0&\alpha_{ik}^1&0&\dots&0\\0&0&\alpha_{ik}^2&\dots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&0&\alpha_{ik}^N\end{bmatrix}.$$

Now let us construct the matrix form of the Fredholm integral part of Eq. (1). The kernel function $K_s(x,t)$ can be approximated by the truncated Taylor series [39,40] and the truncated Chelyshkov series, respectively

$$K_s(x,t)=\sum_{m=0}^{N}\sum_{n=0}^{N}{}^{T}k_{mn}^{s}x^m t^n \quad \text{and} \quad K_s(x,t)=\sum_{m=0}^{N}\sum_{n=0}^{N}{}^{C}k_{mn}^{s}C_m(x)C_n(t), \tag{6}$$

where

$${}^{T}k_{mn}^{s}=\frac{1}{m!n!}\frac{\partial^{m+n}K(0,0)}{\partial x^m\partial t^n}, \quad m,n=0,1,2,\dots,N, \quad s=0,1,2,\dots,m.$$

We write the relations (6) in the matrix form

$$K_s(x,t)=\mathbf{X}(x)\mathbf{K}_T^s\mathbf{X}^T(t),\mathbf{K}_T^s=[{}^{T}k_{mn}^{s}], \quad m,n=0,1,\dots,N, \tag{7}$$

and

$$K_s(x,t)=\mathbf{C}(x)\mathbf{K}_C^s\mathbf{C}^T(t), \quad \mathbf{K}_C^s=[{}^{C}k_{mn}^{s}], \quad m,n=0,1,\dots,N. \tag{8}$$

From Eqs. (4), (7) and (8), we obtain

$$\begin{aligned}&\mathbf{X}(x)\mathbf{K}_T^s\mathbf{X}^T(t)=\mathbf{C}(x)\mathbf{K}_C^s\mathbf{C}^T(t)\Rightarrow\mathbf{X}(x)\mathbf{K}_T^s\mathbf{X}^T(t)=\mathbf{X}(x)\mathcal{C}\mathbf{K}_C^s\mathcal{C}^T\mathbf{X}^T(t),\\&\mathbf{K}_T^s=\mathcal{C}\mathbf{K}_C^s\mathcal{C}^T \text{ or } \mathbf{K}_C^s=(\mathcal{C})^{-1}\mathbf{K}_T^s(\mathcal{C}^T)^{-1}.\end{aligned} \tag{9}$$

By substituting the matrix forms (4) and (9) into the integral part $F(x)$ in Eq. (1), we have the matrix relation as

$$\int_0^1\sum_{s=0}^{m}K_s(x,t)y^{(s)}(t)dt=\int_0^1\sum_{s=0}^{m}\mathbf{C}(x)\mathbf{K}_C^s\mathbf{C}^T(t)\mathbf{X}(t)\mathbf{B}^s\mathcal{C}\boldsymbol{A}dt=\sum_{s=0}^{m}\mathbf{C}(x)\mathbf{K}_C^s\mathbf{Q}\mathbf{B}^s\mathcal{C}\boldsymbol{A}, \tag{10}$$

so that

$$\mathbf{Q}=\mathcal{C}^T\int_0^1\mathbf{X}^T(t)\mathbf{X}(t)dt\mathcal{C}=\mathcal{C}^T\mathbf{H}\mathcal{C};$$

where

$$\mathbf{H}=\int_0^1\mathbf{X}^T(t)\mathbf{X}(t)dt=[h_{rs}]; \quad h_{rs}=\frac{1}{r+s+1}, \quad r,s=0,1,2,\dots,N.$$

Putting the matrix form (10) into the equation $F(x)$, we have the matrix relation

$$[F(x)]=\sum_{s=0}^{m}\mathbf{X}(x)\mathcal{C}\mathbf{K}_C^s\mathbf{Q}\mathcal{C}\boldsymbol{A}.$$

Similarly, we can obtain the following matrix relation for Eq. $I(x)$

$$[I(x)]=\sum_{s=0}^{m}\mathbf{X}(x)\mathcal{C}\mathbf{V}_C^s\mathbf{Q}(x)\mathcal{C}\boldsymbol{A}, \tag{11}$$

where

$$\mathbf{Q}(x)=\mathcal{C}^T\int_0^{h(x)}\mathbf{X}^T(t)\mathbf{X}(t)dt\mathcal{C}=\mathcal{C}^T\mathbf{H}(x)\mathcal{C},$$

and

$$\mathbf{H}(x)=\int_0^{h(x)}\mathbf{X}^T(t)\mathbf{X}(t)dt=[h_{rs}(x)]; \quad h_{rs}(x)=\frac{h^{r+s+1}(x)}{r+s+1}, \quad r,s=0,1,2,\dots,N.$$

Finally we can get the matrix relations for conditions by means of the relation (4)

$$\sum_{k=0}^{m-1}(a_{jk}\mathbf{X}(0)+b_{jk}\mathbf{X}(b))\mathbf{B}^{k}\mathcal{C}\mathbf{A}=[\lambda_{j}],\quad j=0,1,\ldots,m-1. \tag{12}$$

3. Method of solution

For constructing the main matrix equation, we first substitute the matrix relations (5), (10) and (11) into Eq. (1) and then use collocation points denoted by

$$x_p=\frac{1}{N}p,\quad p=0,1,\ldots,N.$$

We obtain a system of matrix equations as

$$\left\{\sum_{i=0}^{n}\sum_{k=0}^{m}\boldsymbol{P}_{\boldsymbol{ik}}(x_p)\boldsymbol{X}(x_p)\boldsymbol{M}(\alpha_{ik},\tau_{ik})\boldsymbol{B}^{k}\mathcal{C}-\sum_{s=0}^{m}\mathbf{C}(x_p)\mathbf{K}_{C}^{s}\boldsymbol{Q}\mathcal{C}-\sum_{s=0}^{m}\mathbf{X}(x_p)\mathcal{C}\mathbf{V}_{C}^{s}\mathbf{Q}(x_p)\mathcal{C}\right\}\mathbf{A}=\mathbf{G}(x_p),$$

or briefly the fundamental matrix equations

$$\left\{\sum_{i=0}^{n}\sum_{k=0}^{m}\boldsymbol{P}_{\boldsymbol{ik}}\boldsymbol{X}\boldsymbol{M}(\alpha_{ik},\tau_{ik})\boldsymbol{B}^{k}\mathcal{C}-\sum_{s=0}^{m}\boldsymbol{X}\mathcal{C}\mathbf{K}_{C}^{s}\boldsymbol{Q}\mathcal{C}-\sum_{s=0}^{m}\overline{\mathbf{X}}\,\overline{\mathcal{C}}\,\overline{\mathbf{V}_{C}^{s}}\,\overline{\mathbf{Q}}\,\overline{\overline{\mathcal{C}}}\right\}\mathbf{A}=\mathbf{G}, \tag{13}$$

where

$$\mathbf{P}_{ik}=\begin{bmatrix}P_{ik}(x_0)&0&\cdots&0\\0&P_{ik}(x_1)&\cdots&0\\\vdots&\vdots&\ddots&0\\0&0&\cdots&P_{ik}(x_N)\end{bmatrix},\mathbf{G}=\begin{bmatrix}g(x_0)\\g(x_1)\\\vdots\\g(x_N)\end{bmatrix},\mathbf{X}=\begin{bmatrix}\mathbf{X}(x_0)\\\mathbf{X}(x_1)\\\vdots\\\mathbf{X}(x_N)\end{bmatrix}=\begin{bmatrix}1&x_0&\cdots&x_0^N\\1&x_1&\cdots&x_1^N\\\vdots&\vdots&\ddots&\vdots\\1&x_N&\cdots&x_N^N\end{bmatrix},$$

$$\overline{\mathbf{X}}=\begin{bmatrix}X(x_0)&0&\cdots&0\\0&X(x_1)&\cdots&0\\\vdots&\vdots&\ddots&0\\0&0&\cdots&X(x_N)\end{bmatrix},\overline{\mathcal{C}}=\begin{bmatrix}\mathcal{C}&0&\cdots&0\\0&\mathcal{C}&\cdots&0\\\vdots&\vdots&\ddots&0\\0&0&\cdots&\mathcal{C}\end{bmatrix},\overline{\mathbf{V}_{C}^{s}}=\begin{bmatrix}\mathbf{V}_{C}^{s}&0&\cdots&0\\0&\mathbf{V}_{C}^{s}&\cdots&0\\\vdots&\vdots&\ddots&0\\0&0&\cdots&\mathbf{V}_{C}^{s}\end{bmatrix},$$

$$\overline{\mathbf{Q}}=\begin{bmatrix}\mathbf{Q}(x_0)&0&\cdots&0\\0&\mathbf{Q}(x_1)&\cdots&0\\\vdots&\vdots&\ddots&0\\0&0&\cdots&\mathbf{Q}(x_N)\end{bmatrix}\quad \textit{and}\quad \overline{\overline{\mathcal{C}}}=\begin{bmatrix}\mathcal{C}\\\mathcal{C}\\\vdots\\\mathcal{C}\end{bmatrix}.$$

Eq. (13) corresponds to a system of $(N+1)$ algebraic equations for the $(N+1)$ unknown coefficients $a_0, a_1, \ldots, a_N$. Briefly, we can write Eq. (13) in the form

$$\mathbf{WA}=\mathbf{G}\text{ or }[\mathbf{W};\mathbf{G}], \tag{14}$$

so that

$$\mathbf{W}=[w_{p,q}]=\sum_{i=0}^{n}\sum_{k=0}^{m}\boldsymbol{P}_{\boldsymbol{ik}}\boldsymbol{XMB}^{k}\mathcal{C}-\sum_{s=0}^{m}\boldsymbol{X}\mathcal{C}\mathbf{K}_{C}^{s}\boldsymbol{Q}\mathcal{C}-\sum_{s=0}^{m}\overline{\boldsymbol{X}\mathcal{C}\mathbf{V}_{\mathcal{C}}\boldsymbol{Q}}\overline{\overline{\mathcal{C}}},\quad p,q=0,1,2,\ldots,N.$$

On the other hand, the matrix forms for the conditions (2) are

$$\mathbf{U}_{p}\mathbf{A}=[\beta_p]\quad\text{or}\quad[\mathbf{U}_i;\lambda_i], \tag{15}$$

where

$$\mathbf{U}_i=[u_{\mathrm{i}\ 0}\quad u_{\mathrm{i}\ 1}\quad u_{\mathrm{i}\ 2}\quad\ldots\quad u_{\mathrm{i}\ N}],\quad \mathrm{i}=0,1,\ldots,\mathrm{m}-1,$$
$$[\widetilde{\mathbf{U}_i}]=[\mathbf{U}_i;\lambda_i].$$

Consequently, to obtain the solution of Eq. (1) under the conditions (2), by replacing the last rows in matrices (14) by the m rows of the matrix (15), we have the new augmented matrix

$$\tilde{\mathbf{W}}A=\tilde{\mathbf{G}}\quad\text{or}\quad[\tilde{\mathbf{W}};\tilde{\mathbf{G}}]. \tag{16}$$

If rank $\tilde{\mathbf{W}} = \text{rank}\ [\tilde{\mathbf{W}}; \tilde{\mathbf{G}}] = N+1$, then we can write $\mathbf{A} = (\tilde{\mathbf{W}})^{-1}\tilde{\mathbf{G}}$. Thus, the matrix $\mathbf{A}$ (thereby the coefficients $a_0, a_1, \ldots, a_N$) is uniquely determined. Also, the Eq. (1) under the conditions (2) has a unique solution. This solution is given by the truncated Chelyshkov series (3). Thus, we get the Chelyshkov polynomial solution

$$y_N(x) = \sum_{n=0}^{N} a_n C_{Nn}(x).$$

4. Error analysis based on residual function

In this section, error estimation is presented for the method. This error estimation is based on the residual function and we improve the approximate solution (3) by using this error estimation. The residual error estimation was presented in some works in the literature. For the problem (1), (2), we modify the error estimation considered in [41–45]. We can easily check the accuracy of this solution as follows:

Since the truncated Chelyshkov series (3) is approximate solution of (1) and (2), when the function $y_N(x)$ and its derivatives $y_N^{(k)}(x)$ are substituted in Eq. (1), the resulting equation must be satisfied approximately for $x \in [0, 1]$, $i = 0, 1, \ldots, N$; that is:

$$R_N(x) = \sum_{i=0}^{n}\sum_{k=0}^{m} P_{ik}(x) y_N^{(k)}(\alpha_{ik}x + \tau_{ik}) + \int_0^1 \sum_{s=0}^{m} K_s(x,t) y_N^{(s)}(t)dt + \int_0^{h(x)} \sum_{s=0}^{m} V_s(x,t) y_N^{(s)}(t)dt - g(x), \tag{17}$$

and the mixed conditions

$$\sum_{k=0}^{m-1} (a_{jk} y_N^{(k)}(0) + b_{jk} y_N^{(k)}(b)) = \lambda_j, \quad j = 0, 1, \ldots, m-1. \tag{18}$$

Here, $R_N(x)$ is the residual function associated with $y_N(x)$.

On the other hand, by means of the residual function defined by $R_N(x)$ and the mean value of the function $|R_N(x)|$ on the interval $[0, b]$ the accuracy of the solution can be controlled and the error can be estimated. If $R_N(x) \to 0$ when N is sufficiently large enough, then error decreases. Also by using the Mean-Value Theorem, we can estimate the upper bound mean error $\overline{R_N}$ as follows [46]:

$$\left| \int_0^b R_N(x)dx \right| \leqslant \int_0^b |R_N(x)|dx,$$

and

$$\int_0^b R_N(x)dx = bR_N(c) \Rightarrow \left| \int_0^b R_N(x)dx \right| = b|R_N(c)|,$$

$$\Rightarrow |R_N(c)| \leqslant \frac{\int_0^b |R_N(x)|dx}{b} = \overline{R_N}, \tag{19}$$

$$\Rightarrow |R_N(c)| \leqslant \overline{R_N}, (0 < c < b).$$

In addition, the error function $e_N(x)$ can be defined as

$$e_N(x) = y(x) - y_N(x), \tag{20}$$

where $y(x)$ is the exact solution of problem (1), (2). Then using (1), (2), (17), (18) and (20), we obtain the error differential equation

$$\sum_{i=0}^{n}\sum_{k=0}^{m} P_{ik}(x) e_N^{(k)}(\alpha_{ik}x + \tau_{ik}) + \int_0^1 \sum_{s=0}^{m} K_s(x,t) e_N^{(s)}(t)dt + \int_0^{h(x)} \sum_{s=0}^{m} V_s(x,t) e_N^{(s)}(t)dt = -R_N(x), \tag{21}$$

with the homogeneous conditions

$$\sum_{k=0}^{m-1} \left(a_{jk} e_N^{(k)}(0) + b_{jk} e_N^{(k)}(b) \right) = 0, \quad j = 0, 1, \ldots, m-1.$$

Solving the error problem (21) by the method given in Section 3, we obtain the approximation $e_{N,M}(x)$ to $e_N(x)$. Consequently, we have the improved approximate solution

$$y_{N,M}(x) = y_N(x) + e_{N,M}(x).$$

Note that if the exact solution of the problem is not known, then we can estimate the error function by $e_{N,M}(x)$.

5. Illustrative examples

In this section, we consider a problem to demonstrate the effectiveness of the method, the error estimation and the residual correction.

Example 1. Firstly, we consider the following second order linear Fredholm integro-differential equation with variable coefficient:

$$\left(\frac{x}{2}-1\right)y''(0.3x)+x^2y'(x+1)+y(x+2)-y(x-1)=\int_0^1(4xt-2x)y''(t)dt+7x-11+2x^3-2x^2,\quad 0\leqslant x\leqslant 1,\tag{22}$$

with the initial conditions $y(0) = 4$ and $y'(0) = -4$. The exact solution of problem is $y(x) = (x-2)^2$. By applying the suggested method for $m = 2$, $p_{00}(x) = 1$, $p_{10}(x) = -1$, $p_{01}(x) = x^2$, $p_{02}(x) = x/2-1$, $K_0(x,t) = 4xt-2x$, $g(x) = 7x-11+2x^3-2x^2$ and $N = 2$ we write the fundamental matrix relation from (13) as

$$\{\mathbf{P}_{00}\boldsymbol{X}\boldsymbol{M}(1,2)\mathcal{C}+\mathbf{P}_{10}\boldsymbol{X}\boldsymbol{M}(1,-1)\mathcal{C}+\mathbf{P}_{01}\boldsymbol{X}\boldsymbol{M}(1,1)\boldsymbol{B}\mathcal{C}+\mathbf{P}_{02}\boldsymbol{X}\boldsymbol{M}(0.3,0)\boldsymbol{B}^2\mathcal{C}-\boldsymbol{X}\mathcal{C}\mathbf{K}_c^0\boldsymbol{Q}\boldsymbol{B}^2\mathcal{C}\}\mathbf{A}=\mathbf{G},$$

where

$$\mathbf{P}_{00}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad \mathbf{P}_{10}=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{bmatrix},\quad \mathbf{P}_{01}=\begin{bmatrix}0&0&0\\0&1/4&0\\0&0&1\end{bmatrix},\quad \mathbf{P}_{02}=\begin{bmatrix}-1&0&0\\0&-3/4&0\\0&0&-1/2\end{bmatrix},$$

$$\mathbf{X}=\begin{bmatrix}\mathbf{X}(x_0)\\\mathbf{X}(x_1)\\\mathbf{X}(x_2)\end{bmatrix}=\begin{bmatrix}1&0&0\\1&1/2&1/4\\1&1&1\end{bmatrix},\quad \boldsymbol{M}(1,2)=\begin{bmatrix}1&2&4\\0&1&4\\0&0&1\end{bmatrix},\quad \boldsymbol{M}(1,-1)=\begin{bmatrix}1&-1&1\\0&1&-2\\0&0&1\end{bmatrix},$$

$$\boldsymbol{M}(1,1)=\begin{bmatrix}1&1&1\\0&1&2\\0&0&1\end{bmatrix},\quad \boldsymbol{M}(0.3,0)=\begin{bmatrix}1&0&0\\0&0.3&0\\0&0&0.09\end{bmatrix},\quad \boldsymbol{Q}=\begin{bmatrix}1&0&0\\0&1/3&0\\0&0&1/5\end{bmatrix},\quad \boldsymbol{B}=\begin{bmatrix}0&1&0\\0&0&2\\0&0&0\end{bmatrix},$$

$$\boldsymbol{B}^2=\begin{bmatrix}0&0&2\\0&0&0\\0&0&0\end{bmatrix},\quad \mathcal{C}=\begin{bmatrix}3&0&0\\-12&4&0\\10&-5&1\end{bmatrix},\quad \boldsymbol{K}_c^0=\begin{bmatrix}0&0&0\\-1/6&-1/4&5/12\\-5/6&-5/4&25/12\end{bmatrix}\quad\text{and}\quad \mathbf{G}=\begin{bmatrix}g(x_0)\\g(x_1)\\g(x_2)\end{bmatrix}=\begin{bmatrix}-11\\-8\\-35/4\end{bmatrix}.$$

From (16) the augmented matrix for this fundamental matrix equation is computed as follows

$$[\tilde{\mathbf{W}};\tilde{\mathbf{G}}]=\begin{bmatrix}-26&7&1&;&-11\\3&0&0&;&4\\-12&4&0&;&-4\end{bmatrix}.$$

By solving this system, the Chelyshkov coefficient matrix is obtained as

$$\boldsymbol{A}=[4/3\quad 3\quad 8/3]^T.$$

Hence, the approximation solution of the problem for $N = 2$ is found as

$$y_2(x)=x^2-4x+4,$$

which is the exact solution of the problem.

Example 2 (*see* [47]). Now, we deal with the linear Volterra–Fredholm integro-differential equation in the following form;

$$x^2y''(x)+xy'(x)-xy(x)=e^x-\sin(x)+\frac{x\cos(x)}{2}+\int_0^1\sin(x)e^{-t}y(t)dt-\frac{1}{2}\int_0^x\cos(x)e^{-t}y(t)dt,\quad 0\leqslant x,t\leqslant 1,\tag{23}$$

under the initial conditions

$$y(0)=1\quad\text{and}\quad y'(0)=1,$$

which has the exact solution $y(x) = e^x$.

From Eq. (13), the fundamental matrix equation of the equation is written as

$$\{\boldsymbol{P}_{00}\boldsymbol{X}\boldsymbol{M}(1,0)\mathcal{C}+\boldsymbol{P}_{01}\boldsymbol{X}\boldsymbol{M}(1,0)\boldsymbol{B}\mathcal{C}+\boldsymbol{P}_{02}\boldsymbol{X}\boldsymbol{M}(1,0)\boldsymbol{B}^2\mathcal{C}-\boldsymbol{X}\mathcal{C}\mathbf{K}_c^0\boldsymbol{Q}\mathcal{C}-\overline{\mathbf{X}}\overline{\mathcal{C}}\overline{\mathbf{V}_c^0}\overline{\mathbf{Q}}\overline{\overline{\mathcal{C}}}\}\mathbf{A}=\mathbf{G}.$$

Thus, following the method given in Section 3, we obtain approximate solutions to the problem in terms of Chelyshkov polynomials for $N = 5, 7$, respectively,

$$y_5(x) = 1 + x + 0.5x^2 + 0.16673336270x^3 + 0.04061157874x^4 + 0.01065008157x^5,$$
$$y_7(t) = 1 + x + 0.5x^2 + 0.1666666296x^3 + 0.04165767115x^4 + 0.008371211118x^5 + 0.001071244931x^6 + 0.0008529403702x^7,$$

and

$$y_{10}(x) = 1 + x + 0.5x^2 + 0.1666676166x^3 + 0.04165356044x^4 + 0.008421162555x^5 + 0.001071244931x^6 + 0.0008529403702x^7 - 0.0007445540389x^8 + 0.0004828059816x^9 - 0.0001229984119x^{10}.$$

Table 1 gives a comparison of numerical results of the uncorrected and corrected approximate solutions obtained by the presented method for $N = 5, 7$ with the exact solutions of Eq. (23). In addition, Fig. 1 shows the comparison between exact and approximate and improved approximate solutions for various values of N, M. Finally, using (19), we construct the upper bound mean error $\overline{R_N}$ in $[0,1]$ for approximate solutions obtained by the presented method for $N = 5, 7$ and we show these mean errors in Table 2. As seen from Table 2, it is said that the obtained solutes improve when N, M are increased.

Example 3 (*see [48]*). Lastly, let us consider the first order linear Volterra integro-differential equation with variable coefficient

$$y'(x) = y(x) - 2y'(x - 0.5) + (x - x^2)y(0.5x - 1) + \int_0^x xe^{-t}y(t)dt + \int_0^{x/2}(x^2 - 2t - 2)y'(t)dt, \quad 0 \leqslant x \leqslant 1 y(0) = 1. \tag{24}$$

By following operations of our method for $N = 4, 7$, we find the following approximate solutions;

$$y_4(x) = 1 + 0.9999803578x + 0.4994695509x^2 + 0.1662082613x^3 + 0.0432204606x^4 + 0.0095478611x^5,$$

and

$$y_7(t) = 1 + 1.0000467750x + 0.50001120160x^2 + 0.166400850x^3 + 0.0413807480x^4 + 0.008748071270x^5 + 0.0018775830x^6 - 0.000185510x^7.$$

Table 1
Comparison of the values of exact and approximate solutions of the problem (23) for x values.

x_i	Exact solution	Present method		Improved present method	
	$y(x_i) = e^{x_i}$	$N = 5, y_5(x_i)$	$N = 7, y_7(x_i)$	$M = 7, y_{5,7}(x_i)$	$M = 9, y_{7,9}(x_i)$
0	1	1	1	1	1
0.2	1,22140275816017	1,221402253453690	1,22140275148372	1,2214027579986	1,221402758157
0.4	1,491824697641270	1,491819648463820	1,49182463942173	1,4918246981768	1,491824697610
0.6	1,82211880039051	1,822105817290790	1,82211859842912	1,8221187981833	1,822118800353
0.8	2,22554092849247	2,22549180308316	2,22554048115809	2,2255409343879	2,225540928405
1	2,71828182845905	2,71799502301000	2,71828051076880	2,7182812727449	2,718281826965

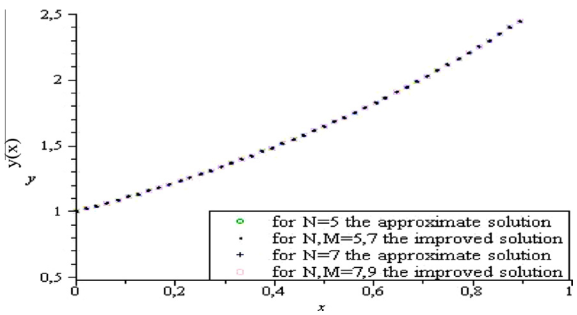

Fig. 1. Comparison of exact and numerical solutions for the problem (23).

Table 2
The upper bound mean errors of the problem (23).

$\overline{R}_{N,M}$	$\overline{R_5}$	$\overline{R_7}$	$\overline{R}_{5,7}$	$\overline{R}_{7,9}$
The values of $\overline{R}_{N,M}$	$2,6013898 \times 10^{-3}$	$1,15440313 \times 10^{-5}$	$9,9172273 \times 10^{-6}$	$3,56366703 \times 10^{-8}$

Table 3
Comparison of the values of exact and approximate solutions of the problem (24) for x values.

x_i	Exact solution	Present method		Improved present method	
	$y(x_i) = e^{x_i}$	$N = 4, y_4(x_i)$	$N = 7, y_7(x_i)$	$M = 8, y_{4,8}(x_i)$	$M = 9, y_{7,9}(x_i)$
0	1	1	1	1	1
0.2	1,2214027581602	1.221110092859520	1,221410136397	1.22140271274681	1.221402755695040
0.4	1,4918246976413	1.491554240333120	1,491826470705	1.49182480275464	1.491824705298540
0.6	1,8221188003905	1.822568572731520	1,822100282740	1.82211915384715	1.822118822217320
0.8	2,2255409284925	2.227281567976320	2,225511236626	2.22554125456388	2.225540945438840
1	2,7182818284591	2.720714051600000	2,718279711321	2.71828164988813	2.718281811628550

Table 4a
Comparisons of the actual absolute errors and the estimated absolute errors of the problem (24).

x_i	Absolute errors		Estimated absolute errors	
	$\|e_4(x_i)\|$	$\|e_7(x_i)\|$	$\|e_{4,8}(x_i)\|$	$\|e_{7,9}(x_i)\|$
0	0	0	0	0
0.2	$2,92665301 \times 10^{-4}$	$7,37823657 \times 10^{-6}$	$4,54133600 \times 10^{-8}$	$2,465130 \times 10^{-9}$
0.4	$2,70457308 \times 10^{-4}$	$1,77306389 \times 10^{-6}$	$1,05113370 \times 10^{-7}$	$7,657270 \times 10^{-9}$
0.6	$4,49772341 \times 10^{-4}$	$1,851765008 \times 10^{-5}$	$3,53456640 \times 10^{-7}$	$2,1826810 \times 10^{-9}$
0.8	$1,74063948 \times 10^{-3}$	$2,969186637 \times 10^{-5}$	$3,26071410 \times 10^{-7}$	$1,6946370 \times 10^{-9}$
1	$2,43222314 \times 10^{-3}$	$2,1171383500 \times 10^{-6}$	$1,78570920 \times 10^{-7}$	$1,68305000 \times 10^{-8}$

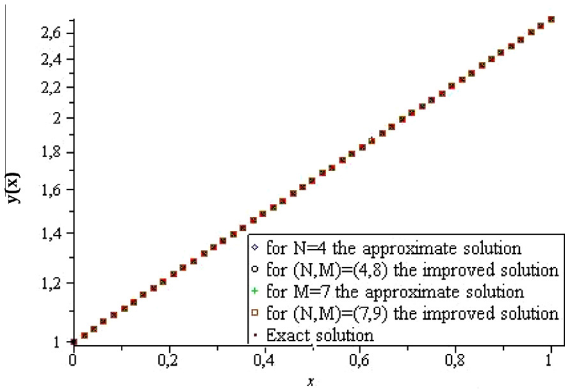

Fig. 2. Comparison of the exact and approximate solutions $y_N(x)$ for $N = 4$, 7 and $M = 8,9$ of the problem (24).

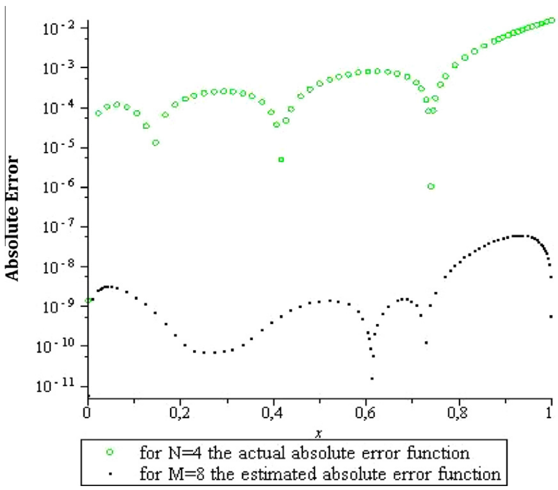

Fig. 3a. Comparison of the actual absolute error functions $|e_4(x)|$ and the estimated absolute error functions $|e_{4,8}(x)|$ for $(N, M) = (4, 8)$ of the problem (24).

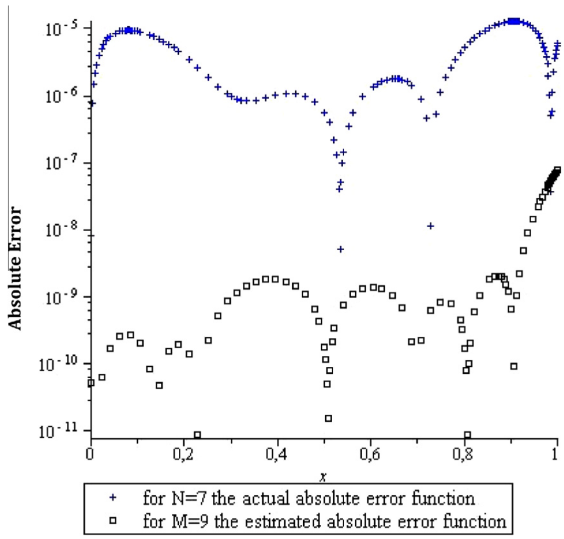

Fig. 3b. Comparison of the actual absolute error functions $|e_7(x)|$ and the estimated absolute error functions $|e_{7,9}(x)|$ for $(N,M) = (7,9)$ of the problem (24).

Table 4b
Comparison of the actual absolute errors of the problem (24).

x_i	Laguerre series method [48]		Present method	
	$e_4(x_i)$	$e_7(x_i)$	$e_4(x_i)$	$e_7(x_i)$
0	1.7764e−015	1.9984e−015	0	0
0.2	3.4609e−004	6.0220e−007	2.9267e−004	7.3782e−006
0.4	5.8635e−004	1.5190e−006	2.7046e−004	1.7731e−006
0.6	2.2284e−004	1.1574e−006	4.4977e−004	1.8518e−005
0.8	8.6266e−004	1.1334e−006	1.7406e−003	2.9692e−005
1	1.8766e−003	3.5020e−006	2.4322e−003	2.1171e−006

The error problems for solutions $N = 4, 7$, respectively, become

$$e_4'(x) = e_4(x) - 2e_4'(x-0.5) + (x - x^2)e_4(0.5x-1) + \int_0^x xe^{-t}e_4(t)dt + \int_0^{x/2}(x^2 - 2t - 2)e_4'(t)dt, \quad 0 \leqslant x \leqslant 1$$
$$e_4(0) = 0,$$

and

$$e_7'(x) = e_7(x) - 2e_7'(x-0.5) + (x - x^2)e_7(0.5x-1) + \int_0^x xe^{-t}e_7(t)dt + \int_0^{x/2}(x^2 - 2t - 2)e_7'(t)dt, \quad 0 \leqslant x \leqslant 1$$
$$e_7(0) = 0.$$

By solving these problems for $M = 8, 9$ with the method introduced in Section 3, respectively, we obtain the approximations in Table 3. The some numerical values of the approximate solutions, the improved solutions and exact solution are compared in Table 3. In Table 4a, the actual absolute errors are compared with the absolute estimated by the presented method for $N = 4, 7$ and $M = 8, 9$ and also the absolute and improved absolute error functions are compared in Figs. 3a and 3b. In Table 4b, we compare the absolute errors obtained by the present method, Laguerre series Method [48]. In addition, Fig. 2 shows exact, approximate and improved approximate solutions.

It is seen from Tables 3 and 4 and Figs. 3a and 3b that errors decrease when N and M are increased.

6. Conclusions

In this paper, we have presented the Chelyshkov collocation scheme, based on Sezer's Matrix Method for the solution of the linear mth order mixed functional integro-differential equations with variable coefficients and by using residual function and Mean Value Theorem, we also form a criterion to calculate the upper bound mean error in the given interval. In addition, we compared the numerical values of the approximate solutions obtained by the method in tables and figures. This comparison shows that the suggested method is quite effective. Moreover, the method can be developed for linear and nonlinear problems with the variable coefficients by means of any collocation points.

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