Guliyev et al. *Journal of Inequalities and Applications* (2015) 2015:71
DOI 10.1186/s13660-015-0584-9

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

The $L_{p_1r_1} \times L_{p_2r_2} \times \cdots \times L_{p_kr_k}$ boundedness of rough multilinear fractional integral operators in the Lorentz spaces

Vagif S Guliyev[1,2], Ismail Ekincioglu[3*] and Sh A Nazirova[4]

*Correspondence: ismail.ekincioglu@dpu.edu.tr
[3]Department of Mathematics, Dumlupınar University, Kütahya, Turkey
Full list of author information is available at the end of the article

Abstract

In this paper, we prove the O'Neil inequality for the k-linear convolution operator in the Lorentz spaces. As an application, we obtain the necessary and sufficient conditions on the parameters for the boundedness of the k-sublinear fractional maximal operator $M_{\Omega,\alpha}(\mathbf{f})$ and the k-linear fractional integral operator $I_{\Omega,\alpha}(\mathbf{f})$ with rough kernels from the spaces $L_{p_1r_1} \times L_{p_2r_2} \times \cdots \times L_{p_kr_k}$ to L_{qs}, where $n/(n+\alpha) \le p < q < \infty$, $0 < r \le s < \infty$, p is the harmonic mean of $p_1, p_2, \dots, p_k > 1$ and r is the harmonic mean of $r_1, r_2, \dots, r_k > 0$.

MSC: Primary 42B20; 42B25; 42B35; secondary 47G10

Keywords: O'Neil inequality; k-linear convolution; rearrangement estimate; k-sublinear fractional maximal function; k-linear fractional integral; harmonic mean; Lorentz space

1 Introduction

Fractional maximal and fractional integral operators are two important operators in harmonic analysis and partial differential equations. Multilinear maximal operator and multilinear fractional integral operator and related topics have been areas of research of many mathematicians such as Coifman and Grafakos [1], Grafakos [2, 3], Grafakos and Kalton [4], Kenig and Stein [5], Ding and Lu [6], Guliyev and Nazirova [7, 8], Ragusa [9] and others.

Let $k \ge 2$ be an integer and θ_j ($j = 1, 2, \dots, k$) be fixed, distinct and nonzero real numbers, and let $\mathbf{f} = (f_1, \dots, f_k)$. The k-linear convolution operator $\mathbf{f} \otimes g$ is defined by

$$(\mathbf{f} \otimes g)(x) = \int_{\mathbb{R}^n} f_1(x - \theta_1 y) \cdots f_k(x - \theta_k y) g(y)\, dy.$$

Let $\Omega \in L_s(S^{n-1})$, $s \ge 1$ and Ω be homogeneous of degree zero on $\mathbb{R}^n$, and let $0 < \alpha < n$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$. The k-sublinear fractional maximal function with rough kernel is defined by

$$M_{\Omega,\alpha}(\mathbf{f})(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y|<r} \left|\Omega(y)\right| \left|f_1(x - \theta_1 y) \dots f_k(x - \theta_k y)\right| dy,$$

and the k-linear fractional integral with rough kernel is defined by

$$I_{\Omega,\alpha}(\mathbf{f})(x) = \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^{n-\alpha}} f_1(x-\theta_1 y)\cdots f_k(x-\theta_k y)\,dy.$$

This paper consists of four sections. In Section 2, some lemmas needed to facilitate the proofs of our theorems and the O'Neil inequality for rearrangements of the k-linear convolution operator $\mathbf{f} \otimes g$ proved in [7] are given. In Section 3, we prove the O'Neil inequality for the k-linear convolution operator in the Lorentz spaces. Finally, in Section 4, we obtain rearrangement estimates for the multilinear fractional maximal function and multilinear fractional integral with rough kernels. We prove the boundedness of the multilinear fractional maximal operator $M_{\Omega,\alpha}$ and the multilinear fractional integral operator $I_{\Omega,\alpha}$ with rough kernels from the spaces $L_{p_1 r_1} \times L_{p_2 r_2} \times \cdots \times L_{p_k r_k}$ to L_{qs}, $n/(n+\alpha) \le p < q < \infty$, $0 < r \le s \le \infty$, where p and r are the harmonic means of $p_1, p_2, \ldots, p_k > 1$ and $r_1, r_2, \ldots, r_k > 0$, respectively. We show that the conditions on the parameters ensuring the boundedness cannot be weakened.

2 Preliminaries

We need the following two generalized Hardy inequalities (see [10]) which are to be used in the proof of Theorem 3.1.

We denote by $\mathfrak{M}(\mathbb{R}^n)$ the set of all extended real-valued measurable functions on $\mathbb{R}^n$. When v is a non-negative measurable function on $(0,\infty)$, we say that v is a weight. We denote $W(t) = \int_0^t w(\tau)\,d\tau$, $V(t) = \int_0^t v(\tau)\,d\tau$ and $U(r,t) = \int_t^r u(\tau)\,d\tau$. For simplicity we suppose that $0 < V(t) < \infty$, $0 < W(t) < \infty$ for all $t > 0$ and $V(\infty) = \infty$, $W(\infty) = \infty$.

Lemma 2.1 [11] *Let $0 < r \le s < \infty$ and let v, w be weights. Then the inequality*

$$\left(\int_0^\infty \bigl(g(t)\bigr)^s w(t)\,dt\right)^{1/s} \le C\left(\int_0^\infty \bigl(g(t)\bigr)^r v(t)\,dt\right)^{1/r} \tag{2.1}$$

holds for all non-negative and non-increasing g on $(0,\infty)$ if and only if

$$A_1 \equiv \sup_{t>0} W^{1/s}(t) V^{-1/r}(t) < \infty,$$

and the best constant C in (2.1) equals A_1.

Lemma 2.2 [11, 12] *Let $r, s \in (0,\infty)$ and let v, w be weights.*

(i) *Let $1 < r \le s < \infty$. Then the inequality*

$$\left(\int_0^\infty \left(\frac{1}{t}\int_0^t g(\tau)\,d\tau\right)^s w(t)\,dt\right)^{1/s} \le C\left(\int_0^\infty \bigl(g(t)\bigr)^r v(t)\,dt\right)^{1/r} \tag{2.2}$$

holds for all non-negative and non-increasing g on $(0,\infty)$ if and only if $A_1 < \infty$,

$$A_2 \equiv \sup_{t>0}\left(\int_t^\infty \frac{w(\tau)}{\tau^s}\,d\tau\right)^{1/s}\left(\int_0^t \frac{v(\tau)\tau^{r'}}{V^{r'}(\tau)}\,d\tau\right)^{1/r'} < \infty,$$

and the best constant C in (2.2) satisfies $C \approx A_1 + A_2$.

(ii) *Let* $0 < r \leq 1$, $r \leq s$. *Then* (2.2) *holds if and only if* $A_1 < \infty$,

$$A_3 \equiv \sup_{t>0} t\left(\int_t^\infty \frac{w(\tau)}{\tau^s}\,d\tau\right)^{1/s} V^{-1/r}(t) < \infty,$$

and the best constant C in (2.2) *satisfies* $C \approx A_1 + A_3$.

Lemma 2.3 [13] *Let* $r, s \in (0,\infty)$ *and let* u, v, w *be weight functions.*

(i) *Let* $1 < r \leq s < \infty$. *Then the inequality*

$$\left(\int_0^\infty \left(\int_t^\infty g(\tau)u(\tau)\,d\tau\right)^s w(t)\,dt\right)^{1/s} \leq C\left(\int_0^\infty \bigl(g(t)\bigr)^r v(t)\,dt\right)^{1/r} \tag{2.3}$$

holds for all non-negative and non-increasing g on $(0,\infty)$ *if and only if*

$$A_4 \equiv \sup_{t>0}\left(\int_0^t U^s(t,\tau)w(\tau)\,d\tau\right)^{1/s} V^{-1/r}(t) < \infty,$$

also

$$A_5 \equiv \sup_{t>0} W^{1/s}(t)\left(\int_t^\infty U^{r'}(\tau,t)V^{-r'}(\tau)v(\tau)\,d\tau\right)^{1/r'} < \infty,$$

and the best constant C in (2.3) *satisfies* $C \approx A_4 + A_5$.

(ii) *Let* $0 < r \leq 1$, $r \leq s$. *Then* (2.3) *holds if and only if* $A_4 < \infty$ *and the best constant C in* (2.3) *equals* A_4.

Lemma 2.4 [13] *Let* $r \in (0,\infty)$ *and let* u, v, w *be weight functions.*

(i) *Let* $1 < r < \infty$. *Then the inequality*

$$\sup_{t>0}\left(\int_t^\infty g(\tau)u(\tau)\,d\tau\right)w(t) \leq C\left(\int_0^\infty \bigl(g(t)\bigr)^r v(t)\,dt\right)^{1/r} \tag{2.4}$$

holds for all non-negative and non-increasing g on $(0,\infty)$ *if and only if*

$$A_6 \equiv \sup_{t>0} w(t)\left(\int_t^\infty U^{r'}(\tau,t)V^{-r'}(\tau)v(\tau)\,d\tau\right)^{1/r'} < \infty,$$

and the best constant C in (2.4) *equals* A_6.

(ii) *Let* $0 < r \leq 1$ *and* $r \leq s$. *Then* (2.4) *holds if and only if*

$$A_7 \equiv \sup_{t>0}\sup_{0<\tau<t} U(\tau,t)w(\tau)V^{-1/r}(t) < \infty,$$

and the best constant C in (2.4) *equals* A_7.

Lemma 2.5 [13] *Let* $r \in (0,\infty)$ *and let* u, v, w *be weight functions.*

(i) *Let* $1 < r < \infty$. *Then the inequality*

$$\sup_{t>0}\left(\int_0^t k(t,\tau)g(\tau)u(\tau)\,d\tau\right)w(t) \leq C\left(\int_0^\infty \bigl(g(t)\bigr)^r v(t)\,dt\right)^{1/r} \tag{2.5}$$

holds for all non-negative and non-increasing g on $(0,\infty)$ *if and only if*

$$A_8 \equiv \sup_{t>0} w(t)\left(\int_0^t \left(\int_s^t k(t,\tau)V^{-1}(\tau)\,d\tau\right)^{r'} v(s)\,ds\right)^{1/r'} < \infty,$$

and the best constant C in (2.5) *equals* A_8.

(ii) *Let* $0 < r \le 1$, $r \le s$. *Then* (2.5) *holds if and only if*

$$A_9 \equiv \sup_{t>0}\sup_{\tau>0} K\bigl(t,\min(\tau,t)\bigr)w(\tau)V^{-1/r}(t) < \infty,$$

and the best constant C in (2.5) *equals* A_9.

Let g be a measurable function on $\mathbb{R}^n$. The distribution function of g is defined by the equality

$$\lambda_g(t) = \bigl|\bigl\{x \in \mathbb{R}^n : \bigl|g(x)\bigr| > t\bigr\}\bigr|, \quad t \ge 0.$$

We shall denote by $L_0(\mathbb{R}^n)$ the class of all measurable functions g on $\mathbb{R}^n$, which are finite almost everywhere and such that $\lambda_g(t) < \infty$ for all $t > 0$ (see [14]). If a function g belongs to $L_0(\mathbb{R}^n)$, then its non-increasing rearrangement is defined to be the function g^* which is non-increasing on $(0,\infty)$ equi-measurable with $|g(x)|$:

$$\bigl|\bigl\{t > 0 : g^*(t) > \tau\bigr\}\bigr| = \lambda_g(\tau)$$

for all $\tau \ge 0$. Moreover, by the Hardy-Littlewood theorem (see [15], p.44) and for every $f_1, f_2 \in L_0(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \bigl|f_1(x)f_2(x)\bigr|\,dx \le \int_0^\infty f_1^*(t)f_2^*(t)\,dt.$$

Equi-measurable rearrangements of functions play an important role in various fields of mathematics. We give some of the main important properties (see, for example, [15]):

(1) if $0 < t < t + \tau$, then

$$(g+h)^*(t+\tau) \le g^*(t) + h^*(\tau),$$

(2) if $0 < p < \infty$, then

$$\int_{\mathbb{R}^n} \bigl|g(x)\bigr|^p\,dx = \int_0^\infty \bigl(g^*(t)\bigr)^p\,dt,$$

(3) for any $t > 0$ and for any set E,

$$\sup_{|E|=t} \int_E \bigl|g(x)\bigr|\,dx = \int_0^t g^*(\tau)\,d\tau.$$

We denote by $WL_p(\mathbb{R}^n)$ the weak L_p space of all measurable functions g with finite norm

$$\|f\|_{WL_p} = \sup_{t>0} t^{1/p} f^*(t) < \infty, \quad 1 \le p < \infty.$$

The function $g^{**} : (0,\infty) \to [0,\infty]$ is defined as $g^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds$.

Definition 2.6 If $0 < p, q < \infty$, then the Lorentz space $L_{pq}(\mathbb{R}^n)$ is the set of all classes of measurable functions f with the finite quasi-norm

$$\|f\|_{pq} \equiv \|f\|_{L_{pq}} = \left(\int_0^\infty \left(t^{1/p} f^*(t) \right)^q \frac{dt}{t} \right)^{1/q}.$$

If $0 < p \le \infty$, $q = \infty$, then $L_{p\infty}(\mathbb{R}^n) = WL_p(\mathbb{R}^n)$.

If $1 \le q \le p$ or $p = q = \infty$, then the functional $\|f\|_{pq}$ is a norm (see [16]). If $p = q = \infty$, then the space $L_{\infty\infty}(\mathbb{R}^n)$ is denoted by $L_\infty(\mathbb{R}^n)$.

In the case $1 < p, q < \infty$ we define

$$\|f\|_{(pq)} = \left(\int_0^\infty \left(t^{1/p} f^{**}(t) \right)^q \frac{dt}{t} \right)^{1/q}$$

(with the usual modification if $0 < p \le \infty$, $q = \infty$) which is a norm on $L_{pq}(\mathbb{R}^n)$ for $1 < p < \infty$, $1 \le q \le \infty$ or $p = q = \infty$. If $1 < p \le \infty$ and $1 \le q \le \infty$, then

$$\|f\|_{pq} \le \|f\|_{(pq)} \le p' \|f\|_{pq}$$

that is, the quasi-norms $\|f\|_{pq}$ and $\|f\|_{(pq)}$ are equivalent.

Lemma 2.7 [7] *Let $f_1, f_2, \ldots, f_k \in L_0(\mathbb{R}^n)$, $k \ge 2$. Then, for all $x \in \mathbb{R}^n$ and nonzero real numbers $\theta_1, \ldots, \theta_k$,*

$$\int_{\mathbb{R}^n} \left| f_1(x - \theta_1 y) f_2(x - \theta_2 y) \cdots f_k(x - \theta_k y) \right| dy \le C_\theta \int_0^\infty f_1^*(t) f_2^*(t) \cdots f_k^*(t)\, dt, \tag{2.6}$$

where $C_\theta = |\theta_1 \ldots \theta_k|^{-n}$.

Let $\mathbf{f} = (f_1, f_2, \ldots, f_k)$ and define

$$\mathbf{f}^*(t) = f_1^*(t) \cdots f_k^*(t), \qquad \mathbf{f}^{**}(t) = \frac{1}{t} \int_0^t f_1^*(\tau) \cdots f_k^*(\tau)\, d\tau, \quad t > 0.$$

In the following, we give the O'Neil inequality for rearrangements of the multilinear convolution operator $\mathbf{f} \otimes g$ proved in [7].

Lemma 2.8 [7] *Let $f_1, f_2, \ldots, f_k, g \in L_0(\mathbb{R}^n)$. Then, for all $0 < t < \infty$, the following inequality holds*:

$$(\mathbf{f} \otimes g)^{**}(t) \le C_\theta \left(t \mathbf{f}^{**}(t) g^{**}(t) + \int_t^\infty \mathbf{f}^*(s) g^*(s)\, ds \right). \tag{2.7}$$

Corollary 2.9 [7] *Let $f_1, f_2, \ldots, f_k \in L_0(\mathbb{R}^n)$ and $g \in WL_m(\mathbb{R}^n)$, $1 < m < \infty$. Then*

$$\begin{aligned} (\mathbf{f} \otimes g)^*(t) &\le (\mathbf{f} \otimes g)^{**}(t) \\ &\le C_\theta \|g\|_{WL_m} \left(m' t^{-1/m} \int_0^t \mathbf{f}^*(\tau)\, d\tau + \int_t^\infty \tau^{-1/m} \mathbf{f}^*(\tau)\, d\tau \right). \end{aligned} \tag{2.8}$$

Lemma 2.10 [7] *Let $f_1, f_2, \dots, f_k, g \in L_0(\mathbb{R}^n)$. Then for any $t > 0$*

$$(\mathbf{f} \otimes g)^{**}(t) \le C_\theta \int_t^\infty \mathbf{f}^{**}(t) g^{**}(t)\, dt. \tag{2.9}$$

Corollary 2.11 *Let $f_1, f_2, \dots, f_k \in L_0(\mathbb{R}^n)$ and $g \in WL_m(\mathbb{R}^n)$, $1 < m < \infty$. Then*

$$(\mathbf{f} \otimes g)^{*}(t) \le (\mathbf{f} \otimes g)^{**}(t) \le m' C_\theta \|g\|_{WL_m} \int_t^\infty \tau^{-1/m} \mathbf{f}^{**}(\tau)\, d\tau. \tag{2.10}$$

3 O'Neil inequality for the multilinear convolutions in the Lorentz spaces

In this section, we prove the O'Neil inequality for the multilinear convolutions in the Lorentz spaces. It is said that p is the harmonic mean of $p_1, p_2, \dots, p_k > 1$ if $1/p = 1/p_1 + 1/p_2 + \cdots + 1/p_k$. If $f_j \in L_{p_j r_j}(\mathbb{R}^n)$, $j = 1, 2, \dots, k$, then we say that $\mathbf{f} \in L_{p_1 r_1} \times L_{p_2 r_2} \times \cdots \times L_{p_k r_k}(\mathbb{R}^n)$.

Theorem 3.1 (O'Neil inequality for k-linear convolution in the Lorentz spaces) *Suppose that $1 < m < \infty$, $g \in WL_m(\mathbb{R}^n)$, p and r are the harmonic means of $p_1, p_2, \dots, p_k > 1$ and $r_1, r_2, \dots, r_k > 0$, respectively. If $1 < p < m'$, $1 < r \le s < \infty$ or $m'/(1+m') \le p \le 1$, $0 < r \le 1$, $r \le s < \infty$ or $p = m'$, $1 < r < \infty$, $s = \infty$ or $p = m'$, $0 < r \le 1$, $s = \infty$ $\mathbf{f} \in L_{p_1 r_1} \times L_{p_2 r_2} \times \cdots \times L_{p_k r_k}(\mathbb{R}^n)$ and $1/p - 1/q = 1/m'$, then $\mathbf{f} \otimes g \in L_{qs}(\mathbb{R}^n)$ and*

$$\|\mathbf{f} \otimes g\|_{qs} \lesssim C_\theta K(p,q,r,s,m) \prod_{j=1}^k \|f_j\|_{p_j r_j} \|g\|_{WL_m},$$

where $K(p,q,r,s,m) = \kappa$ and

$$\kappa \approx \begin{cases} m'\mathcal{A}_1 + m'\mathcal{A}_2 + \mathcal{A}_4 + \mathcal{A}_5, & \text{if } 1 < p < m', 1 < r \le s < \infty, \\ m'\mathcal{A}_1 + m'\mathcal{A}_3 + \mathcal{A}_4, & \text{if } \frac{m'}{1+m'} \le p \le 1, 0 < r \le 1, r \le s < \infty \\ m'\mathcal{A}_6 + m'\mathcal{A}_8, & \text{if } p = m', 1 < r < \infty, s = \infty, \\ m'\mathcal{A}_7 + m'\mathcal{A}_9, & \text{if } p = m', 0 < r \le 1, s = \infty \end{cases}$$

and

$$\mathcal{A}_1 = \left(\frac{m'q}{s(m'+q)}\right)^{1/s} \left(\frac{r}{p}\right)^{1/r}, \qquad \mathcal{A}_2 = \frac{r}{p}\left(\frac{mq}{s(q-m)}\right)^{1/s} \left(\frac{p'}{r'}\right)^{1/r'},$$

$$\mathcal{A}_3 = \left(\frac{r}{p}\right)^{1/r} \left(\frac{mq}{s(q-m)}\right)^{1/s}, \qquad \mathcal{A}_4 = \left(m'\right)^{1+1/s} \left(\frac{r}{p}\right)^{1/r} \left(B\left(s+1, sm'/q\right)\right)^{1/s},$$

$$\mathcal{A}_5 = \left(m'\right)^{1+1/r'} \frac{r}{p}\left(\frac{q}{s}\right)^{1/s} \left(B\left(r'+1, r'm'/p - r'\right)\right)^{1/r'}, \qquad \mathcal{A}_6 = m'\left(\frac{r}{p}\right)^{1/r},$$

$$\mathcal{A}_7 = \left(m'\right)^{1+1/r'} \left(B\left(r'+1, r'm'/p - r'\right)\right)^{1/r'},$$

$$\mathcal{A}_8 = \left(\frac{r}{p}\right)^{1/r} \left(\frac{p}{p-r}\right)^{1+1/r'} \left(B\left(r'+1, \frac{r}{p-r}\right)\right)^{1/r'}, \qquad \mathcal{A}_9 = \left(\frac{r}{p}\right)^{1/r}.$$

Here $B(s,r) = \int_0^1 (1-\tau)^{s-1} \tau^{r-1}\, d\tau$ is the beta function.

Proof Let $1 < m < \infty$, $m'/(1+m') \le p < m'$, $1/p - 1/q = 1/m'$, p be the harmonic mean of $p_1, p_2, \dots, p_k > 1$, r be the harmonic mean of $r_1, r_2, \dots, r_k > 0$, $0 < r \le s \le \infty$ and $\mathbf{f} \in$

$L_{p_1r_1} \times L_{p_2r_2} \times \cdots \times L_{p_kr_k}(\mathbb{R}^n)$. By using inequality (2.8), we have

$$\begin{aligned}\|\mathbf{f} \otimes g\|_{qs} &= \left\|(\mathbf{f} \otimes g)^*(t)t^{1/q-1/s}\right\|_{L_s(0,\infty)} \\ &\le C_\theta\left(\int_0^\infty \left(m't^{-1/m}\int_0^t \mathbf{f}^*(\tau)\,d\tau + \int_t^\infty \tau^{-1/m}\mathbf{f}^*(\tau)\,d\tau\right)^s t^{s/q-1}\,dt\right)^{1/s} \\ &\le C_\theta m'\left(\int_0^\infty \left(\int_0^t \mathbf{f}^*(\tau)\,d\tau\right)^s t^{-s/m+s/q-1}\,dt\right)^{1/s} \\ &\quad + C_\theta\left(\int_0^\infty \left(\int_t^\infty \tau^{-1/m}\mathbf{f}^*(\tau)\,d\tau\right)^s t^{s/q-1}\,dt\right)^{1/s}.\end{aligned}$$

Case I. Suppose that $1 < p < m'$ (equivalently $m < q < \infty$), $1 < r \le s < \infty$. From Lemma 2.2, for the validity of the inequality for $1 < r \le s < \infty$

$$\left(\int_0^\infty \left(\frac{1}{t}\int_0^t \mathbf{f}^*(\tau)\,d\tau\right)^s t^{s-s/m+s/q-1}\,dt\right)^{1/s} \le C_1\left(\int_0^\infty \left(t^{1/p}\mathbf{f}^*(t)\right)^r \frac{dt}{t}\right)^{1/r}, \tag{3.1}$$

the necessary and sufficient condition is

$$\begin{aligned}\mathcal{A}_1 &= \sup_{t>0} W^{1/s}(t)V^{-1/r}(t) = \left(\frac{m'q}{s(m'+q)}\right)^{1/s}\left(\frac{r}{p}\right)^{1/r}\sup_{t>0} t^{1/m'+1/q-1/p} < \infty \\ &\Leftrightarrow \quad 1/p - 1/q = 1/m' \text{ and } \mathcal{A}_1 = \left(\frac{m'q}{s(m'+q)}\right)^{1/s}\left(\frac{r}{p}\right)^{1/r}\end{aligned}$$

and

$$\begin{aligned}\mathcal{A}_2 &= \sup_{t>0}\left(\int_t^\infty \frac{w(\tau)}{\tau^s}\,d\tau\right)^{1/s}\left(\int_0^t \frac{v(\tau)\tau^{r'}}{V^{p'}(\tau)}\,d\tau\right)^{1/r'} \\ &= \frac{r}{p}\sup_{t>0}\left(\int_t^\infty \tau^{-s/m+s/q-1}\,d\tau\right)^{1/s}\left(\int_0^t \tau^{r/p-1+r'-rr'/p}\,d\tau\right)^{1/r'} \\ &= \frac{r}{p}\left(\frac{mq}{s(q-m)}\right)^{1/s}\left(\frac{p'}{r'}\right)^{1/r'}\sup_{t>0} t^{-1/m+1/q-1/p'} < \infty \\ &\Leftrightarrow \quad 1/p - 1/q = 1/m' \text{ and } \mathcal{A}_2 = \frac{r}{p}\left(\frac{mq}{s(q-m)}\right)^{1/s}\left(\frac{p'}{r'}\right)^{1/r'}.\end{aligned}$$

Note that the best constant C_1 in (3.1) satisfies $C_1 \approx \mathcal{A}_1 + \mathcal{A}_2$. Furthermore, from Lemma 2.3 for the validity of the inequality for $1 < r \le s < \infty$

$$\left(\int_0^\infty \left(\int_t^\infty \tau^{-1/m}\mathbf{f}^*(\tau)\,d\tau\right)^s t^{s/q-1}\,dt\right)^{1/s} \le C_2\left(\int_0^\infty \left(t^{1/p}\mathbf{f}^*(t)\right)^r \frac{dt}{t}\right)^{1/r}, \tag{3.2}$$

the necessary and sufficient condition is

$$\begin{aligned}\mathcal{A}_4 &= m'\sup_{t>0}\left(\int_0^t \left(t^{1/m'} - \tau^{1/m'}\right)^s \tau^{s/q-1}\,d\tau\right)^{1/s}\left(\int_0^t \tau^{r/p-1}\,d\tau\right)^{-1/r} \\ &= m'\left(\frac{r}{p}\right)^{1/r}\sup_{t>0}\left(\int_0^t \left(t^{1/m'} - \tau^{1/m'}\right)^s \tau^{s/q-1}\,d\tau\right)^{1/s} t^{-1/p}\end{aligned}$$

$$
\begin{aligned}
&= (m')^{1+1/s}\left(\frac{r}{p}\right)^{1/r}\bigl(B(s+1, sm'/q)\bigr)^{1/s}\sup_{t>0} t^{-1/m'+1/q-1/p} < \infty \\
&\Leftrightarrow \quad 1/p - 1/q = 1/m' \text{ and } \mathcal{A}_4 = (m')^{1+1/s}\left(\frac{r}{p}\right)^{1/r}\bigl(B(s+1, sm'/q)\bigr)^{1/s}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{A}_5 &= \sup_{t>0} W^{1/s}(t)\left(\int_t^\infty U^{r'}(\tau,t)V^{-r'}(\tau)v(\tau)\,d\tau\right)^{1/r'} \\
&= \frac{m'r}{p}\left(\frac{q}{s}\right)^{1/s}\sup_{t>0} t^{1/q}\left(\int_t^\infty \bigl(\tau^{1/m'} - t^{1/m'}\bigr)^{r'}\tau^{-rr'/p+r/p-1}\,d\tau\right)^{1/r'} \\
&= \frac{m'r}{p}\left(\frac{q}{s}\right)^{1/s}\left(\int_1^\infty \bigl(\lambda^{1/m'} - 1\bigr)^{r'}\lambda^{-r'/p-1}\,d\lambda\right)^{1/r'}\sup_{t>0} t^{1/q+1/m'-1/p} \\
&= (m')^{1+1/r'}\frac{r}{p}\left(\frac{q}{s}\right)^{1/s}\left(\int_0^1 \bigl(1 - \lambda^{1/m'}\bigr)^{r'}\lambda^{-r'/m'+r'/p-1}\,d\lambda\right)^{1/r'}\sup_{t>0} t^{1/q+1/m'-1/p} \\
&= (m')^{1+1/r'}\frac{r}{p}\left(\frac{q}{s}\right)^{1/s}\left(\int_0^1 (1-\tau)^{r'}\tau^{-r'+r'm'/p-1}\,d\tau\right)^{1/r'}\sup_{t>0} t^{1/q+1/m'-1/p} \\
&= (m')^{1+1/r'}\frac{r}{p}\left(\frac{q}{s}\right)^{1/s}\bigl(B(r'+1, r'm'/p - r')\bigr)^{1/r'}\sup_{t>0} t^{1/q+1/m'-1/p} < \infty \\
&\Leftrightarrow \quad 1/p - 1/q = 1/m' \text{ and } \mathcal{A}_5 = (m')^{1+1/r'}\frac{r}{p}\left(\frac{q}{s}\right)^{1/s}\bigl(B(r'+1, r'm'/p - r')\bigr)^{1/r'}.
\end{aligned}
$$

Note that the best constant C_2 in (3.2) satisfies $C_2 \approx \mathcal{A}_4 + \mathcal{A}_5$.

Case II. Let $m'/(1+m') \le p \le 1$, $0 < r \le 1$ and $r \le s < \infty$. From Lemma 2.3, for the validity of inequality (3.1), the necessary and sufficient condition is $\mathcal{A}_1 < \infty$ and

$$
\begin{aligned}
\mathcal{A}_3 &= \sup_{t>0} t\left(\int_t^\infty \frac{w(\tau)}{\tau^s}\,d\tau\right)^{1/s} V^{-1/r}(t) \\
&= \left(\frac{r}{p}\right)^{1/r}\sup_{t>0} t\left(\int_t^\infty \tau^{-s/m+s/q-1}\,d\tau\right)^{1/s} t^{-1/p} \\
&= \left(\frac{r}{p}\right)^{1/r}\left(\frac{mq}{s(q-m)}\right)^{1/s}\sup_{t>0} t^{1-1/m+1/q-1/p} \\
&\Leftrightarrow \quad 1/p - 1/q = 1/m' \text{ and } \mathcal{A}_3 = \left(\frac{r}{p}\right)^{1/r}\left(\frac{mq}{s(q-m)}\right)^{1/s}.
\end{aligned}
$$

Note that the best constant C_1 in (3.1) satisfies $C_1 \approx \mathcal{A}_1 + \mathcal{A}_3$. From Lemma 2.3, for the validity of inequality (3.2), the necessary and sufficient condition is $\mathcal{A}_4 < \infty$. Consequently, using inequalities (3.1), (3.2) and applying the Hölder inequality, we obtain

$$
\begin{aligned}
\|\mathbf{f} \otimes g\|_{qs} &\le C_\theta\bigl(m'C_1 + C_2\bigr)\left(\int_0^\infty \bigl(t^{1/p}\mathbf{f}^*(t)\bigr)^r\frac{dt}{t}\right)^{1/r}\|g\|_{WL_m} \\
&= C_\theta K(p,q,r,s,m)\left(\int_0^\infty \prod_{j=1}^k \bigl(f_j^*(t)t^{1/p_j}\bigr)^r\frac{dt}{t}\right)^{1/r}\|g\|_{WL_m}
\end{aligned}
$$

$$\leq C_\theta K(p,q,r,s,m)\prod_{j=1}^{k}\left(\int_0^\infty \left(f_j^*(t)t^{1/p_j}\right)^{r_j}\frac{dt}{t}\right)^{1/r_j}\|g\|_{WL_m}$$
$$= C_\theta K(p,q,r,s,m)\prod_{j=1}^{k}\|f_j\|_{p_jr_j}\|g\|_{WL_m}.$$

Case III. Let $p = m'$, $q = s = \infty$, $1 < r < \infty$ or $p = m'$, $q = s = \infty$, $0 < r \leq 1$ and $\mathbf{f} \in L_{p_1r_1} \times L_{p_2r_2} \times \cdots \times L_{p_kr_k}(\mathbb{R}^n)$. By using inequality (2.8), we have

$$\begin{aligned}\|\mathbf{f}\otimes g\|_\infty &= \sup_{t>0}(\mathbf{f}\otimes g)^*(t)\\ &\leq C_\theta \sup_{t>0}\left(m' t^{-1/m}\int_0^t \mathbf{f}^*(\tau)\,d\tau + \int_t^\infty \tau^{-1/m}\mathbf{f}^*(\tau)\,d\tau\right)\|g\|_{WL_m}\\ &\leq C_\theta m' \sup_{t>0}\left(t^{-1/m}\int_0^t \mathbf{f}^*(\tau)\,d\tau\right) + \sup_{t>0}\left(\int_t^\infty \tau^{-1/m}\mathbf{f}^*(\tau)\,d\tau\right)\|g\|_{WL_m}\\ &\leq C_\theta m'\left(\int_0^\infty \left(t^{1/p}\mathbf{f}^*(t)\right)^r\frac{dt}{t}\right)\|g\|_{WL_m}.\end{aligned}$$

From Lemma 2.5, for the validity of the inequality for $1 < r < \infty$

$$\sup_{t>0}\left(t^{-1/m}\int_0^t \mathbf{f}^*(\tau)\,d\tau\right) \leq C_3\left(\int_0^\infty \left(t^{1/p}\mathbf{f}^*(t)\right)^r\frac{dt}{t}\right)^{1/r}, \tag{3.3}$$

the necessary and sufficient condition is

$$\begin{aligned}\mathcal{A}_8 &= \left(\frac{r}{p}\right)^{1/r}\sup_{t>0} t^{-1/m}\left(\int_0^t\left(\int_s^t \tau^{-r/p}\,d\tau\right)^{r'} s^{r/p-1}\,ds\right)^{1/r'}\\ &= \left(\frac{r}{p}\right)^{1/r}\frac{r}{p-r}\sup_{t>0} t^{-1/m}\left(\int_0^t \left(t^{1-r/p}-\tau^{1-r/p}\right)^{r'}\tau^{r/p-1}\,d\tau\right)^{1/r'}\\ &= \left(\frac{r}{p}\right)^{1/r}\left(\frac{p}{p-r}\right)^{1+1/r'}\left(\int_0^1 \left(1-\tau^{1-r/p}\right)^s\tau^{r/p-1}\,d\tau\right)^{1/r'}\sup_{t>0} t^{-1/m-1/p+1}\\ &= \left(\frac{r}{p}\right)^{1/r}\left(\frac{p}{p-r}\right)^{1+1/r'}\left(B\left(r'+1,\frac{r}{p-r}\right)\right)^{1/r'}\sup_{t>0} t^{-1/m-1/p+1} < \infty\\ \Leftrightarrow\quad & p = m' \text{ and } \mathcal{A}_8 = \left(\frac{r}{p}\right)^{1/r}\left(\frac{p}{p-r}\right)^{1+1/r'}\left(B\left(r'+1,\frac{r}{p-r}\right)\right)^{1/r'}.\end{aligned}$$

From Lemma 2.5, for the validity of the inequality for $0 < r \leq 1$

$$\sup_{t>0}\left(t^{-1/m}\int_0^t \mathbf{f}^*(\tau)\,d\tau\right) \leq C_3\left(\int_0^\infty \left(t^{1/p}\mathbf{f}^*(t)\right)^r\frac{dt}{t}\right)^{1/r}, \tag{3.4}$$

the necessary and sufficient condition is

$$\begin{aligned}\mathcal{A}_9 &= \sup_{t>0}\sup_{\tau>0} K\bigl(t,\min(\tau,t)\bigr)w(\tau)V^{-1/r}(t) = \left(\frac{r}{p}\right)^{1/r}\sup_{t>0} t^{1/m'-1/p} < \infty\\ \Leftrightarrow\quad & p = m' \text{ and } \mathcal{A}_9 = \left(\frac{r}{p}\right)^{1/r}.\end{aligned}$$

From Lemma 2.4, for the validity of the inequality for $1 < r < \infty$

$$\sup_{t>0}\left(\int_t^\infty \tau^{-1/m}\mathbf{f}^*(\tau)\,d\tau\right) \le C_3\left(\int_0^\infty \left(t^{1/p}\mathbf{f}^*(t)\right)^r \frac{dt}{t}\right)^{1/r}, \tag{3.5}$$

the necessary and sufficient condition is $\mathcal{A}_6$

$$\begin{aligned}
\mathcal{A}_6 &= \sup_{t>0}\left(\int_t^\infty U^{r'}(\tau,t)V^{-r'}(\tau)v(\tau)\,d\tau\right)^{1/r'}\\
&= \left(\int_t^\infty \left(\tau^{1/m'}-t^{1/m'}\right)^{r'}\tau^{-rr'/p+r/p-1}\,d\tau\right)^{1/r'}\\
&= \left(\int_1^\infty \left(\lambda^{1/m'}-1\right)^{r'}\lambda^{-r'/p-1}\,d\lambda\right)^{1/r'}\sup_{t>0} t^{1/m'-1/p}\\
&= \frac{m'r}{p}\left(\int_0^1 \left(1-\lambda^{1/m'}\right)^{r'}\lambda^{-r'/m'+r'/p-1}\,d\lambda\right)^{1/r'}\sup_{t>0} t^{1/m'-1/p}\\
&= \frac{m'r}{p}\left(\int_0^1 (1-\tau)^{r'}\tau^{-r'+r'm'/p-1}\,d\tau\right)^{1/r'}\sup_{t>0} t^{1/m'-1/p}\\
&= \frac{m'r}{p}\left(B\left(r'+1,r'm'/p-r'\right)\right)^{1/r'}\sup_{t>0} t^{1/m'-1/p} < \infty\\
&\Leftrightarrow \quad p = m' \text{ and } \mathcal{A}_6 = \frac{m'r}{p}\left(B\left(r'+1,r'm'/p-r'\right)\right)^{1/r'}.
\end{aligned}$$

Furthermore, from Lemma 2.4, for the validity of the inequality for $0 < r \le 1$

$$\sup_{t>0}\left(\int_t^\infty \tau^{-1/m}\mathbf{f}^*(\tau)\,d\tau\right) \le C_3\left(\int_0^\infty \left(t^{1/p}\mathbf{f}^*(t)\right)^r \frac{dt}{t}\right)^{1/r}, \tag{3.6}$$

the necessary and sufficient condition is

$$\begin{aligned}
\mathcal{A}_7 &= \sup_{t>0}\sup_{0<\tau<t} U(\tau,t)w(\tau)V^{-1/r}(t)\\
&= m'\sup_{t>0}\sup_{0<\tau<t}\left(t^{1/m'}-\tau^{1/m'}\right)\left(\int_0^t \tau^{r/p-1}\,d\tau\right)^{-1/r}\\
&= m'\left(\frac{r}{p}\right)^{1/r}\sup_{t>0}\sup_{0<\tau<t}\left(t^{1/m'}-\tau^{1/m'}\right)t^{-1/p}\\
&= m'\left(\frac{r}{p}\right)^{1/r}\sup_{t>0} t^{1/m'-1/p} < \infty\\
&\Leftrightarrow \quad p = m' \text{ and } \mathcal{A}_7 = m'\left(\frac{r}{p}\right)^{1/r}.
\end{aligned}$$

Thus the proof of Theorem 3.1 is completed. □

Corollary 3.2 [8] *Suppose that* $1 < m < \infty$, $g \in WL_m(\mathbb{R}^n)$ *and* p *is the harmonic mean of* $p_1, p_2, \ldots, p_k > 1$. *If* $m'/(1+m') \le p < m'$, $\mathbf{f} \in L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$ *and* q *satisfy*

$1/p - 1/q = 1/m'$, *then* $\mathbf{f} \otimes g \in L_q(\mathbb{R}^n)$ *and*

$$\|\mathbf{f} \otimes g\|_q \le C_\theta K(p,q,m) \prod_{j=1}^k \|f_j\|_{p_j} \|g\|_{WL_m},$$

where in the case $1 < p = r < m'$, $q = s$

$$K(p,q,m) = m'\left(\frac{m'}{m'+q}\right)^{1/q} + m'\left(\frac{m}{q-m}\right)^{1/q} + (m')^{1+1/q}\big(B(q+1,m')\big)^{1/q} + (m')^{1+1/p'}\big(B(p'+1,p'm'/p-p')\big)^{1/p'},$$

and in the case $m'/(1+m') \le p = r \le 1$, $m < q = s$

$$K(p,q,m) = m'\left(\frac{m'}{m'+q}\right)^{1/q} + (m'+1)\left(\frac{m}{q-m}\right)^{1/q} + (m')^{1+1/q}\big(B(q+1,m')\big)^{1/q}.$$

4 The $L_{p_1r_1} \times L_{p_2r_2} \times \cdots \times L_{p_kr_k}$ boundedness of rough multilinear fractional integral operators

In this section, we prove the Sobolev type theorem for the rough multilinear fractional integral $I_{\Omega,\alpha}\mathbf{f}$.

Lemma 4.1 *Let* $0 < \alpha < n$, Ω *be homogeneous of degree zero on* $\mathbb{R}^n$, $\Omega \in L_{n/(n-\alpha)}(S^{n-1})$ *and*

$$g(x) = \frac{\Omega(x)}{|x|^{n-\alpha}}.$$

Then $g \in WL_{n/(n-\alpha)}(\mathbb{R}^n)$ *and*

$$\|g\|_{WL_{n/(n-\alpha)}} = n^{\alpha/n-1}\|\Omega\|_{L_{n/(n-\alpha)}}, \tag{4.1}$$

where

$$\|\Omega\|_{L_{n/(n-\alpha)}} = \left(\int_{S^{n-1}} \big|\Omega(x')\big|^{n/(n-\alpha)}\, d\sigma(x')\right)^{(n-\alpha)/n}.$$

Proof Note that

$$g^*(t) = (nt)^{\alpha/n-1}\|\Omega\|_{L_{n/(n-\alpha)}}, \qquad g^{**}(t) = \frac{n}{\alpha} g^*(t),$$

therefore $g \in WL_{n/(n-\alpha)}(\mathbb{R}^n)$ and equality (4.1) is valid. □

Lemma 4.2 *Suppose that* $0 < \alpha < n$, $\Omega \in L_s(S^{n-1})$ *and* $s \ge 1$. *Then*

$$M_{\Omega,\alpha}\mathbf{f}(x) \le I_{|\Omega|,\alpha}\big(|\mathbf{f}|\big)(x), \tag{4.2}$$

where $|\mathbf{f}| = (|f_1|, \dots, |f_k|)$.

Proof Indeed, for all $r > 0$, we have

$$\begin{aligned} I_{|\Omega|,\alpha}\big(|f|\big)(x) &\geq \int_{E(0,r)} \frac{|\Omega(y)|}{|y|^{n-\alpha}} \big|f_1(x-\theta_1 y)\dots f_k(x-\theta_k y)\big|\,dy \\ &\geq \frac{1}{r^{n-\alpha}} \int_{E(0,r)} \big|\Omega(y)\big| \big|f_1(x-\theta_1 y)\dots f_k(x-\theta_k y)\big|\,dy, \end{aligned}$$

where $E(0,r)$ is the open ball centered at the origin of radius r. Taking supremum over all $r > 0$, we get (4.2). □

By Lemmas 2.8 and 4.2, we obtain a pointwise rearrangement estimate of the rough k-sublinear fractional maximal integral $M_{\Omega,\alpha}\mathbf{f}$ and k-linear fractional integral $I_{\Omega,\alpha}\mathbf{f}$.

Lemma 4.3 [7] *Suppose that Ω is homogeneous of degree zero on $\mathbb{R}^n$ and $\Omega \in L_{n/(n-\alpha)}(S^{n-1})$, $0 < \alpha < n$. Then the following inequalities hold*:

$$\begin{aligned} (I_{\Omega,\alpha}\mathbf{f})^*(t) &\leq (I_{\Omega,\alpha}\mathbf{f})^{**}(t) \\ &\leq C_\theta n^{\alpha/n-1}\|\Omega\|_{L_{n/(n-\alpha)}} \left(\frac{n}{\alpha} t^{\alpha/n-1} \int_0^t \mathbf{f}^*(\tau)\,d\tau + \int_t^\infty \tau^{\alpha/n-1}\mathbf{f}^*(\tau)\,d\tau\right), \\ (M_{\Omega,\alpha}\mathbf{f})^*(t) &\leq (M_{\Omega,\alpha}\mathbf{f})^{**}(t) \\ &\leq C_\theta n^{\alpha/n-1}\|\Omega\|_{L_{n/(n-\alpha)}} \left(\frac{n}{\alpha} t^{\alpha/n-1} \int_0^t \mathbf{f}^*(\tau)\,d\tau + \int_t^\infty \tau^{\alpha/n-1}\mathbf{f}^*(\tau)\,d\tau\right). \end{aligned}$$

From Theorem 3.1 and Lemma 4.3, we get the following.

Theorem 4.4 *Let Ω be homogeneous of degree zero on $\mathbb{R}^n$, $\Omega \in L_{n/(n-\alpha)}(S^{n-1})$, $0 < \alpha < n$, p and r be the harmonic means of $p_1, p_2, \dots, p_k > 1$ and $r_1, r_2, \dots, r_k > 0$, respectively, and $0 < r \leq s \leq \infty$, q satisfy $1/q = 1/p - \alpha/n$. If $1 < p < n/\alpha$, $1 < r \leq s < \infty$ or $n/(n+\alpha) \leq p \leq 1$, $0 < r \leq s < \infty$ or $p = n/\alpha$, $r = 1$, then $I_{\Omega,\alpha}$ is a bounded operator from $L_{p_1 r_1} \times L_{p_2 r_2} \times \cdots \times L_{p_k r_k}(\mathbb{R}^n)$ to $L_{qs}(\mathbb{R}^n)$ and*

$$\|I_{\Omega,\alpha}\mathbf{f}\|_{qs} \leq C_\theta n^{\alpha/n-1} K\big(p,q,r,s,n/(n-\alpha)\big) \|\Omega\|_{L_{n/(n-\alpha)}} \prod_{j=1}^k \|f_j\|_{p_j r_j}.$$

Corollary 4.5 [8] *Let Ω be homogeneous of degree zero on $\mathbb{R}^n$, $\Omega \in L_{n/(n-\alpha)}(S^{n-1})$, $0 < \alpha < n$, p be the harmonic mean of $p_1, p_2, \dots, p_k > 1$, and q satisfy $1/q = 1/p - \alpha/n$. Then $I_{\Omega,\alpha}$ is a bounded operator from $L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $n/(n+\alpha) \leq p < n/\alpha$ (equivalently $1 \leq q < \infty$) and*

$$\|I_{\Omega,\alpha}\mathbf{f}\|_q \leq C_\theta n^{\alpha/n-1} K\big(p,q,n/(n-\alpha)\big) \|\Omega\|_{L_{n/(n-\alpha)}} \prod_{j=1}^k \|f_j\|_{p_j}.$$

Corollary 4.6 [8] *Let Ω be homogeneous of degree zero on $\mathbb{R}^n$, $\Omega \in L_{n/(n-\alpha)}(S^{n-1})$, $0 < \alpha < n$, p be the harmonic mean of $p_1, p_2, \dots, p_k > 1$, and q satisfy $1/q = 1/p - \alpha/n$. Then $M_{\Omega,\alpha}$ is a bounded operator from $L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $n/(n+\alpha) \leq p \leq n/\alpha$ (equiv-*

alently $1 \le q \le \infty$) *and*

$$\|M_{\Omega,\alpha}\mathbf{f}\|_q \le C_\theta n^{\alpha/n-1} K\big(p,q,n/(n-\alpha)\big)\|\Omega\|_{L_{n/(n-\alpha)}} \prod_{j=1}^{k} \|f_j\|_{p_j},$$

when $n/(n+\alpha) \le p < n/\alpha$, *and*

$$\|M_{\Omega,\alpha}\mathbf{f}\|_\infty \le C_\theta \|\Omega\|_{L_{n/(n-\alpha)}} \prod_{j=1}^{k} \|f_j\|_{p_j}, \quad p = n/\alpha.$$

Finally, in the following theorem we obtain the necessary and sufficient conditions for the rough k-linear fractional integral operator $I_{\Omega,\alpha}$ to be bounded from the Lorentz spaces $L_{p_1r_1} \times L_{p_2r_2} \times \cdots \times L_{p_kr_k}(\mathbb{R}^n)$ to $L_{qs}(\mathbb{R}^n)$, $n/(n+\alpha) \le p < q < \infty$, $0 < r \le s < \infty$.

Theorem 4.7 *Let* $0 < \alpha < n$, Ω *be homogeneous of degree zero on* $\mathbb{R}^n$, $\Omega \in L_{n/(n-\alpha)}(S^{n-1})$, p *and* r *be the harmonic means of* $p_1, p_2, \ldots, p_k > 1$ *and* $r_1, r_2, \ldots, r_k > 0$, *respectively. If* $1 < p < n/\alpha$, $1 < r \le s < \infty$ *or* $n/(n+\alpha) \le p \le 1$, $0 < r \le s < \infty$, *then the condition* $1/p - 1/q = \alpha/n$ *is necessary and sufficient for the boundedness of* $I_{\Omega,\alpha}$ *from* $L_{p_1r_1} \times L_{p_2r_2} \times \cdots \times L_{p_kr_k}(\mathbb{R}^n)$ *to* $L_{qs}(\mathbb{R}^n)$.

Proof Sufficiency of the theorem follows from Theorem 4.4.

Necessity. Suppose that the operator $I_{\Omega,\alpha}$ is bounded from $L_{p_1r_1} \times L_{p_2r_2} \times \cdots \times L_{p_kr_k}(\mathbb{R}^n)$ to $L_{qs}(\mathbb{R}^n)$, and $n/(n+\alpha) \le p < n/\alpha$ (equivalently $1 \le q < \infty$). Define $\mathbf{f}_t(x) =: \mathbf{f}(tx)$ for $t > 0$ and $\|\mathbf{f}\|_{pr} = \prod_{j=1}^{k} \|f_j\|_{p_jr_j}$. Then it can be easily shown that

$$\|\mathbf{f}_t\|_{pr} = \prod_{j=1}^{k} \big\|(f_j)_t\big\|_{p_jr_j} = \prod_{j=1}^{k} t^{-n/p_j}\|f_j\|_{p_jr_j} = t^{-n/p}\|\mathbf{f}\|_{pr}$$

and

$$I_{\Omega,\alpha}\mathbf{f}_t(x) = t^{-\alpha} I_{\Omega,\alpha}\mathbf{f}(tx), \qquad \|I_{\Omega,\alpha}\mathbf{f}_t\|_{qs} = t^{-\alpha-n/q}\|I_{\Omega,\alpha}\mathbf{f}\|_{qs}.$$

Since the operator $I_{\Omega,\alpha}$ is bounded from $L_{p_1r_1} \times L_{p_2r_2} \times \cdots \times L_{p_kr_k}(\mathbb{R}^n)$ to $L_{qs}(\mathbb{R}^n)$, we have

$$\|I_{\Omega,\alpha}\mathbf{f}\|_{qs} \le C\|\mathbf{f}\|_{pr},$$

where C is independent of $\mathbf{f}$. Then we get

$$\|I_{\Omega,\alpha}\mathbf{f}\|_{qs} = t^{\alpha+n/q}\|I_{\Omega,\alpha}\mathbf{f}_t\|_{qs} \le Ct^{\alpha+n/q}\|\mathbf{f}_t\|_{pr} = Ct^{\alpha+n/q-n/p}\|f\|_{pr}.$$

If $1/p < 1/q + \alpha/n$, then for all $\mathbf{f} \in L_{p_1r_1} \times L_{p_2r_2} \times \cdots \times L_{p_kr_k}(\mathbb{R}^n)$ we have $\|I_{\Omega,\alpha}\mathbf{f}\|_{L_{q,s}} = 0$ as $t \to 0$. If $1/p > 1/q + \alpha/n$, then for all $\mathbf{f} \in L_{p_1r_1} \times L_{p_2r_2} \times \cdots \times L_{p_kr_k}(\mathbb{R}^n)$ we have $\|I_{\Omega,\alpha}\mathbf{f}\|_{qs} = 0$ as $t \to \infty$. Therefore we get $1/p = 1/q + \alpha/n$. □

Corollary 4.8 [8] *Let* $0 < \alpha < n$, p *be the harmonic mean of* $p_1, p_2, \ldots, p_k > 1$, Ω *be homogeneous of degree zero on* $\mathbb{R}^n$ *and* $\Omega \in L_{n/(n-\alpha)}(S^{n-1})$. *If* $n/(n+\alpha) \le p < n/\alpha$, *then the condition* $1/p - 1/q = \alpha/n$ *is necessary and sufficient for the boundedness of* $I_{\Omega,\alpha}$ *from* $L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$.

Remark 4.9 Note that the sufficiency part of Corollary 4.8 was proved in [7] and in the case $\Omega \equiv 1$ in [2], and in the case $\Omega \in L_s(S^{n-1})$, $s > n/(n-\alpha)$ in [6].

Theorem 4.10 *Let $0 < \alpha < n$, Ω be homogeneous of degree zero on $\mathbb{R}^n$, $\Omega \in L_{n/(n-\alpha)}(S^{n-1})$, p and r be the harmonic means of $p_1, p_2, \ldots, p_k > 1$ and $r_1, r_2, \ldots, r_k > 0$, respectively. If $1 < p < n/\alpha$, $1 < r \le s < \infty$ or $n/(n+\alpha) \le p \le 1$, $0 < r \le s < \infty$, then the condition $1/p - 1/q = \alpha/n$ is necessary and sufficient for the boundedness of $M_{\Omega,\alpha}$ from $L_{p_1 r_1} \times L_{p_2 r_2} \times \cdots \times L_{p_k r_k}(\mathbb{R}^n)$ to $L_{qs}(\mathbb{R}^n)$.*

Proof Sufficiency part of the theorem follows from Theorem 4.7 and Lemma 4.2.

Necessity. Suppose that the operator $M_{\Omega,\alpha}$ is bounded from $L_{p_1 r_1} \times L_{p_2 r_2} \times \cdots \times L_{p_k r_k}(\mathbb{R}^n)$ to $L_{qs}(\mathbb{R}^n)$, and $n/(n+\alpha) \le p < n/\alpha$, $0 < r \le s < \infty$. Then we have

$$M_{\Omega,\alpha} f_t(x) = t^{-\alpha} M_{\Omega,\alpha} f(tx)$$

and

$$\|M_{\Omega,\alpha} f_t\|_{qs} = t^{-\alpha - \frac{n}{q}} \|M_{\Omega,\alpha} f\|_{qs}.$$

By the same argument in Theorem 4.7, we obtain $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. □

Corollary 4.11 [8] *Let $0 < \alpha < n$, p be the harmonic mean of $p_1, p_2, \ldots, p_k > 1$, Ω be homogeneous of degree zero on $\mathbb{R}^n$ and $\Omega \in L_{n/(n-\alpha)}(S^{n-1})$. If $n/(n+\alpha) \le p \le n/\alpha$, then the condition $1/p - 1/q = \alpha/n$ is necessary and sufficient for the boundedness of $M_{\Omega,\alpha}$ from $L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$.*

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details

[1]Institute of Mathematics and Mechanics, Baku, Azerbaijan. [2]Department of Mathematics, Ahi Evran University, Kirsehir, Turkey. [3]Department of Mathematics, Dumlupınar University, Kütahya, Turkey. [4]Khazar University, Baku, Azerbaijan.

Acknowledgements

The research of V Guliyev was partially supported by the grant of Science Development Foundation under the President of the Republic of Azerbaijan, Grant EIF-2014-9(15)-46/10/1 and by the grant of Presidium of Azerbaijan National Academy of Science 2015.

Received: 5 November 2014 Accepted: 30 January 2015 Published online: 26 February 2015

References

1. Coifman, R, Grafakos, L: Hardy spaces estimates for multilinear operators. I. Rev. Mat. Iberoam. **8**, 45-68 (1992)
2. Grafakos, L: On multilinear fractional integrals. Stud. Math. **102**, 49-56 (1992)
3. Grafakos, L: Hardy spaces estimates for multilinear operators. II. Rev. Mat. Iberoam. **8**, 69-92 (1992)
4. Grafakos, L, Kalton, N: Some remarks on multilinear maps and interpolation. Math. Ann. **319**, 49-56 (2001)
5. Kenig, CE, Stein, EM: Multilinear estimates and fractional integration. Math. Res. Lett. **6**, 1-15 (1999)
6. Ding, Y, Lu, S: The $\mathbf{f} \in L_{p_1} \times L_{p_2} \times \cdots \times L_{p_k}$ boundedness for some rough operators. J. Math. Anal. Appl. **203**, 151-180 (1996)
7. Guliyev, VS, Nazirova, SA: A rearrangement estimate for the rough multilinear fractional integrals. Sib. Math. J. **48**(3), 1-12 (2007)
8. Guliyev, VS, Nazirova, SA: O'Neil inequality for multilinear convolutions and some applications. Integral Equ. Oper. Theory **60**(4), 485-497 (2008)
9. Ragusa, MA: Necessary and sufficient condition for a VMO function. Appl. Math. Comput. **128**, 11952-11958 (2012)

10. Maz'ya, VG: Sobolev Spaces. Springer, Berlin (1985)
11. Stepanov, VD: The weighted Hardy inequality for nonincreasing functions. Trans. Am. Math. Soc. **338**, 173-186 (1993)
12. Barza, S, Persson, LE, Soria, J: Sharp weighted multidimensional integral inequalities for monotone functions. Math. Nachr. **210**, 43-58 (2000)
13. Gogatishvili, A, Stepanov, VD: Reduction theorems for operators on the cones of monotone functions, Preprint No. 4, Institute of Mathematics, AS CR, Prague, pp. 1-25 (2012)
14. Kolyada, VI: Rearrangements of functions and embedding of anisotropic spaces of Sobolev type. East J. Approx. **4**(2), 111-119 (1999)
15. Bennett, C, Sharpley, R: Interpolation of Operators. Academic Press, Boston (1988)
16. Stein, EM, Weiss, G: Introduction to Fourier Analysis on Euclidean Spaces. Princeton University Press, Princeton (1971)