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Research Article

Generalized Fractional Integral Operators on Generalized Local Morrey Spaces

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We study the continuity properties of the generalized fractional integral operator I_ρ on the generalized local Morrey spaces $LM_{p,\varphi}^{\{x_0\}}$ and generalized Morrey spaces $M_{p,\varphi}$. We find conditions on the triple $(\varphi_1, \varphi_2, \rho)$ which ensure the Spanne-type boundedness of I_ρ from one generalized local Morrey space $LM_{p,\varphi_1}^{\{x_0\}}$ to another $LM_{q,\varphi_2}^{\{x_0\}}$, $1 < p < q < \infty$, and from $LM_{1,\varphi_1}^{\{x_0\}}$ to the weak space $WLM_{q,\varphi_2}^{\{x_0\}}$, $1 < q < \infty$. We also find conditions on the pair (φ, ρ) which ensure the Adams-type boundedness of I_ρ from $M_{p,\varphi^{1/p}}$ to $M_{q,\varphi^{1/q}}$ for $1 < p < q < \infty$ and from $M_{1,\varphi}$ to $WM_{q,\varphi^{1/q}}$ for $1 < q < \infty$. In all cases the conditions for the boundedness of I_ρ are given in terms of Zygmund-type integral inequalities on $(\varphi_1, \varphi_2, \rho)$ and (φ, ρ), which do not assume any assumption on monotonicity of $\varphi_1(x, r)$, $\varphi_2(x, r)$, and $\varphi(x, r)$ in r.

1. Introduction

The theory of boundedness of classical operators of the real analysis, such as the maximal operator, Riesz potential, and the singular integral operators, from one weighted Lebesgue space to another one is well studied by now. Along with weighted Lebesgue spaces, Morrey-type spaces also play an important role in the theory of partial differential equations. Morrey spaces were first introduced by Morrey [1] in 1938 to study local behavior properties of the solutions of second-order elliptic partial differential equations. Furthermore, there are important applications for the theory of partial differential equations related to obtaining sharp a priori estimates and studying regularity properties of solutions in Morrey spaces. Recently, they proved to be useful also for the Navier-Stokes equations [2, 3]. However no attempt has been made to extend these results by using more generalized Morrey-type spaces. For example, sharp regularity properties of strong solutions to elliptic and parabolic equations with VMO coefficients in terms of general Morrey-type spaces are a good place to start the investigation.

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x, r)$ the open ball centered at x of radius r and by ${}^{\complement}B(x, r)$ denote its complement. Let $|B(x, r)|$ be the Lebesgue measure of the ball $B(x, r)$.

Let $f \in L_1^{\text{loc}}(\mathbb{R}^n)$. The fractional maximal operator M_α and the Riesz potential I_α are defined by

$$M_\alpha f(x) = \sup_{t>0} |B(x, t)|^{-1+\alpha/n} \int_{B(x,t)} |f(y)|\, dy, \quad 0 \le \alpha < n,$$
$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)\, dy}{|x - y|^{n-\alpha}}, \quad 0 < \alpha < n. \tag{1}$$

If $\alpha = 0$, then $M \equiv M_0$ is the Hardy-Littlewood maximal operator.

For a measurable function $\rho : (0, \infty) \to (0, \infty)$ the generalized Riesz potential I_ρ is defined by

$$I_\rho f(x) = \int_{\mathbb{R}^n} \frac{\rho(|x - y|)}{|x - y|^n} f(y)\, dy \tag{2}$$

for any suitable function f on $\mathbb{R}^n$. If $\rho(t) \equiv t^\alpha$, $0 < \alpha < n$, then we get the Riesz potential operator I_α.

The generalized fractional integral operator I_ρ was initially investigated in [4–6]. Nowadays many authors have been culminating important observations about I_ρ especially in connection with Morrey spaces. Nakai [6] proved the boundedness of I_ρ from the generalized Morrey spaces M_{1,φ_1} to the spaces M_{1,φ_2} for suitable functions φ_1, φ_2 satisfying the doubling condition. The boundedness of I_ρ from the generalized Morrey spaces M_{p,φ_1} to the spaces M_{q,φ_2} is studied by Eridani [7], Gunawan [8], Eridani et al. [9], Nakai [10], and Eridani et al. [11]. Guliyev [12] proved the Spanne- and Adams-type boundedness of I_α in the spaces $M_{p,\varphi}(\mathbb{R}^n)$ without any assumption on monotonicity of φ.

In this study, by using the method given by Guliyev in [13] (see also [12, 14]), we prove the Spanne-type boundedness of the operator I_ρ from one generalized local Morrey space $LM^{\{x_0\}}_{p,\varphi_1}$ to another one $LM^{\{x_0\}}_{q,\varphi_2}$, $1 < p < q < \infty$, and from $LM^{\{x_0\}}_{1,\varphi_1}$ to the weak space $WLM^{\{x_0\}}_{q,\varphi_2}$, $1 < q < \infty$. We also prove the Adams-type boundedness of the operator I_ρ from generalized Morrey space $M_{p,\varphi^{1/p}}$ to another one $M_{q,\varphi^{1/q}}$ for $1 < p < q < \infty$ and from $M_{1,\varphi}$ to $WM_{q,\varphi^{1/q}}$ for $1 < q < \infty$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Generalized Local Morrey Spaces $LM^{\{x_0\}}_{p,\varphi}$

We find it convenient to define the generalized Morrey spaces in the form as follows.

Definition 1. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L^{\text{loc}}_p(\mathbb{R}^n)$ with finite quasinorm:

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-1/p} \|f\|_{L_p(B(x,r))}. \quad (3)$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL^{\text{loc}}_p(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-1/p} \|f\|_{WL_p(B(x,r))} < \infty. \quad (4)$$

According to this definition, we recover the Morrey space $M_{p,\lambda}$ and weak Morrey space $WM_{p,\lambda}$ under the choice $\varphi(x,r) = r^{(\lambda-n)/p}$:

$$M_{p,\lambda} = M_{p,\varphi}\big|_{\varphi(x,r)=r^{(\lambda-n)/p}}, \qquad WM_{p,\lambda} = WM_{p,\varphi}\big|_{\varphi(x,r)=r^{(\lambda-n)/p}}. \quad (5)$$

Definition 2. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $LM_{p,\varphi} \equiv LM_{p,\varphi}(\mathbb{R}^n)$ the generalized local (central) Morrey space, the space of all functions $f \in L^{\text{loc}}_p(\mathbb{R}^n)$ with finite quasinorm:

$$\|f\|_{LM_{p,\varphi}} = \sup_{r>0} \varphi(0,r)^{-1} |B(0,r)|^{-1/p} \|f\|_{L_p(B(0,r))}. \quad (6)$$

Also by $WLM_{p,\varphi} \equiv WLM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized local (central) Morrey space of all functions $f \in WL^{\text{loc}}_p(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\varphi}} = \sup_{r>0} \varphi(0,r)^{-1} |B(0,r)|^{-1/p} \|f\|_{WL_p(B(0,r))} < \infty. \quad (7)$$

Definition 3. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $LM^{\{x_0\}}_{p,\varphi} \equiv LM^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n)$ the generalized local Morrey space, the space of all functions $f \in L^{\text{loc}}_p(\mathbb{R}^n)$ with finite quasinorm:

$$\|f\|_{LM^{\{x_0\}}_{p,\varphi}} = \|f(x_0+\cdot)\|_{LM_{p,\varphi}}. \quad (8)$$

Also by $WLM^{\{x_0\}}_{p,\varphi} \equiv WLM^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized local Morrey space of all functions $f \in WL^{\text{loc}}_p(\mathbb{R}^n)$ for which

$$\|f\|_{WLM^{\{x_0\}}_{p,\varphi}} = \|f(x_0+\cdot)\|_{WLM_{p,\varphi}} < \infty. \quad (9)$$

According to this definition, we recover the local Morrey space $LM^{\{x_0\}}_{p,\lambda}$ and weak local Morrey space $WLM^{\{x_0\}}_{p,\lambda}$ under the choice $\varphi(x_0,r) = r^{(\lambda-n)/p}$:

$$LM^{\{x_0\}}_{p,\lambda} = LM^{\{x_0\}}_{p,\varphi}\big|_{\varphi(x_0,r)=r^{(\lambda-n)/p}}, \qquad WLM^{\{x_0\}}_{p,\lambda} = WLM^{\{x_0\}}_{p,\varphi}\big|_{\varphi(x_0,r)=r^{(\lambda-n)/p}}. \quad (10)$$

Furthermore, we have the following embeddings:

$$M_{p,\varphi} \subset LM^{\{x_0\}}_{p,\varphi}, \quad \|f\|_{LM^{\{x_0\}}_{p,\varphi}} \le \|f\|_{M_{p,\varphi}}, \qquad WM_{p,\varphi} \subset WLM^{\{x_0\}}_{p,\varphi}, \quad \|f\|_{WLM^{\{x_0\}}_{p,\varphi}} \le \|f\|_{WM_{p,\varphi}}. \quad (11)$$

Wiener [15, 16] looked for a way to describe the behavior of a function at infinity. The conditions he considered are related to appropriate weighted L_q spaces. Beurling [17] extended this idea and defined a pair of dual Banach spaces A_q and $B_{q'}$, where $1/q + 1/q' = 1$. To be precise, A_q is a Banach algebra with respect to the convolution, expressed as a union of certain weighted L_q spaces; the space $B_{q'}$ is expressed as the intersection of the corresponding weighted $L_{q'}$ spaces. Feichtinger [18] observed that the space B_q can be described by

$$\|f\|_{B_q} = \sup_{k\ge 0} 2^{-kn/q} \|f\chi_k\|_{L_q(\mathbb{R}^n)}, \quad (12)$$

where χ_0 is the characteristic function of the unit ball $\{x \in \mathbb{R}^n : |x| \le 1\}$, χ_k is the characteristic function of the annulus

$\{x \in \mathbb{R}^n : 2^{k-1} < |x| \le 2^k\}$, $k = 1, 2, \ldots$. By duality, the space $A_q(\mathbb{R}^n)$, called Beurling algebra now, can be described by

$$\|f\|_{A_q} = \sum_{k=0}^{\infty} 2^{-kn/q'} \|f\chi_k\|_{L_q(\mathbb{R}^n)}. \tag{13}$$

Let $\dot{B}_q(\mathbb{R}^n)$ and $\dot{A}_q(\mathbb{R}^n)$ be the homogeneous versions of $B_q(\mathbb{R}^n)$ and $A_q(\mathbb{R}^n)$ by taking $k \in \mathbb{Z}$ in (42) and (13) instead of $k \ge 0$ there.

If $\lambda < 0$ or $\lambda > n$, then $LM^{\{x_0\}}_{p,\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$. Note that $LM_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ and $LM_{p,n}(\mathbb{R}^n) = \dot{B}_p(\mathbb{R}^n)$:

$$\dot{B}_{p,\mu} = LM_{p,\varphi}\big|_{\varphi(0,r)=r^{n+np\mu}}, \quad W\dot{B}_{p,\mu} = WLM_{p,\varphi}\big|_{\varphi(0,r)=r^{n+np\mu}}, \quad \mu \in \left[-\frac{1}{p}, 0\right]. \tag{14}$$

In order to study the relationship between central BMO spaces and Morrey spaces, Alvárez et al. [19] introduced λ-central bounded mean oscillation spaces and central Morrey spaces $\dot{B}_{p,\mu}(\mathbb{R}^n) \equiv LM_{p,r^{n+np\mu}}(\mathbb{R}^n)$, $\mu \in [-1/p, 0]$. If $\mu < -1/p$ or $\mu > 0$, then $\dot{B}_{p,\mu}(\mathbb{R}^n) = \Theta$. Note that $\dot{B}_{p,-1/p}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ and $\dot{B}_{p,0}(\mathbb{R}^n) = \dot{B}_p(\mathbb{R}^n)$. Also define the weak central Morrey spaces $W\dot{B}_{p,\mu}(\mathbb{R}^n) \equiv WLM_{p,n+np\mu}(\mathbb{R}^n)$.

The classical result by Hardy-Littlewood-Sobolev states that if $1 < p < q < \infty$, then the operator I_α is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if and only if $\alpha = n(1/p - 1/q)$ and for $p = 1 < q < \infty$, the operator I_α is bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ if and only if $\alpha = n(1 - 1/q)$. Spanne and Adams studied boundedness of the Riesz potential in Morrey spaces. Their results can be summarized as follows.

Theorem 4 (Spanne, but published by Peetre [20]). *Let $0 < \alpha < n$, $1 < p < n/\alpha$, and $0 < \lambda < n - \alpha p$. Moreover, let $1/p - 1/q = \alpha/n$ and $\lambda/p = \mu/q$. Then, for $p > 1$, the operator I_α is bounded from $M_{p,\lambda}$ to $M_{q,\mu}$ and, for $p = 1$, I_α is bounded from $M_{1,\lambda}$ to $WM_{q,\mu}$.*

Theorem 5 (Adams [21]). *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $0 < \lambda < n - \alpha p$, and $1/p - 1/q = \alpha/(n - \lambda)$. Then, for $p > 1$, the operator I_α is bounded from $M_{p,\lambda}$ to $M_{q,\lambda}$ and, for $p = 1$, I_α is bounded from $M_{1,\lambda}$ to $WM_{q,\lambda}$.*

Some authors [8, 12, 22–25] generalized Theorems 4 and 5 to generalized Morrey spaces and called them Spanne-type and Adams-type results for I_α.

In [23] the following condition was imposed on $\varphi(x, r)$:

$$c^{-1}\varphi(x, r) \le \varphi(x, t) \le c\varphi(x, r) \tag{15}$$

whenever $r \le t \le 2r$, where $c \ge 1$ does not depend on t, r and $x \in \mathbb{R}^n$, jointly with the condition

$$\int_r^\infty t^{\alpha p} \varphi(x, t)^p \frac{dt}{t} \le C r^{\alpha p} \varphi(x, r)^p, \tag{16}$$

where $C > 0$ does not depend on r and $x \in \mathbb{R}^n$.

In [23] the following Spanne-type result was proved for I_α on $M_{p,\varphi}$.

Theorem 6. *Let $1 < p < \infty$, $0 < \alpha < n/p$, $1/q = 1/p - \alpha/n$, and $\varphi(x, r)$ satisfy the conditions (15) and (16). Then the operator I_α is bounded from $M_{p,\varphi}$ to $M_{q,\varphi}$.*

The following Spanne-type result for I_α on $LM^{\{x_0\}}_{p,\varphi}$, containing results obtained in [23, 26], was proved in [12, 13] (see also [14]).

Theorem 7. *Let $x_0 \in \mathbb{R}^n$, $1 \le p < \infty$, $0 < \alpha < n/p$, $1/q = 1/p - \alpha/n$, and (φ_1, φ_2) satisfy the condition*

$$\int_t^\infty r^\alpha \varphi_1(x_0, r) \frac{dr}{r} \le C\varphi_2(x_0, t), \tag{17}$$

where C does not depend on x_0 and t. Then the operator I_α is bounded from $LM^{\{x_0\}}_{p,\varphi_1}$ to $LM^{\{x_0\}}_{q,\varphi_2}$ for $p > 1$ and from $LM^{\{x_0\}}_{1,\varphi_1}$ to $WLM^{\{x_0\}}_{q,\varphi_2}$ for $p = 1$.

From Theorem 7 we get the following Spanne-type result for I_α on $M_{p,\varphi}$.

Corollary 8. *Let $1 \le p < \infty$, $0 < \alpha < n/p$, $1/q = 1/p - \alpha/n$, and (φ_1, φ_2) satisfy the condition*

$$\int_t^\infty r^\alpha \varphi_1(x, r) \frac{dr}{r} \le C\varphi_2(x, t), \tag{18}$$

where C does not depend on x and t. Then the operator I_α is bounded from M_{p,φ_1} to M_{q,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{q,φ_2} for $p = 1$.

The following Spanne-type result for I_α on $M_{p,\varphi}$, containing results obtained in [12], was proved in [27].

Theorem 9. *Let $1 \le p < \infty$, $0 < \alpha < n/p$, $1/q = 1/p - \alpha/n$, and (φ_1, φ_2) satisfy the condition*

$$\int_t^\infty \frac{\operatorname{ess\,inf}_{r<s<\infty} \varphi_1(x, s)\, s^{n/q}}{r^{n/q+1}} dr \le C\varphi_2(x, t), \tag{19}$$

where C does not depend on x and t. Then the operator I_α is bounded from M_{p,φ_1} to M_{q,φ_2} for $1 < p < q < \infty$ and from M_{1,φ_1} to WM_{q,φ_2} for $1 < q < \infty$.

3. Some Weighted Inequalities

Let v be a nonnegative function on $(0, \infty)$. We denote by $L_{\infty,v}(0, \infty)$ the space of all functions $g(t)$, $t > 0$, with finite norm

$$\|g\|_{L_{\infty,v}(0,\infty)} = \sup_{t>0} v(t) |g(t)| \tag{20}$$

and $L_\infty(0, \infty) \equiv L_{\infty,1}(0, \infty)$. Let $\mathfrak{M}(0, \infty)$ be the set of all Lebesgue-measurable functions on $(0, \infty)$ and $\mathfrak{M}^+(0, \infty)$ its subset consisting of all nonnegative functions on $(0, \infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are nondecreasing on $(0,\infty)$ and

$$\mathbb{A}=\left\{\varphi\in\mathfrak{M}^+(0,\infty;\uparrow):\lim_{t\to 0+}\varphi(t)=0\right\}. \quad (21)$$

The following theorem is valid.

Theorem 10. *Let v_1, v_2 be nonnegative measurable functions satisfying $0<\|v_j\|_{L_\infty(t,\infty)}<\infty$, $j=1,2$, for any $t>0$.*

Then the identity operator I is bounded from $L_{\infty,v_1}(0,\infty)$ to $L_{\infty,v_2}(0,\infty)$ on the cone $\mathbb{A}$ if and only if

$$\left\|v_2\|v_1\|^{-1}_{L_\infty(\cdot,\infty)}\right\|_{L_\infty(0,\infty)}<\infty. \quad (22)$$

Proof. If F, G are nonnegative functions on $(0,\infty)$ and F is nondecreasing, then

$$\operatorname*{ess\,sup}_{t\in(0,\infty)}F(t)G(t)=\operatorname*{ess\,sup}_{t\in(0,\infty)}F(t)\operatorname*{ess\,sup}_{s\in(t,\infty)}G(s),\quad t\in(0,\infty). \quad (23)$$

Also if F, G are nonnegative functions on $(0,\infty)$ and F is nonincreasing, then

$$\operatorname*{ess\,sup}_{t\in(0,\infty)}F(t)G(t)=\operatorname*{ess\,sup}_{t\in(0,\infty)}F(t)\operatorname*{ess\,sup}_{s\in(0,t)}G(s),\quad t\in(0,\infty). \quad (24)$$

Therefore for all $\varphi\in\mathbb{A}$

$$\operatorname*{ess\,sup}_{t>0}v(t)\varphi(t)=\operatorname*{ess\,sup}_{t>0}\varphi(t)\bar{S}v(t), \quad (25)$$

where

$$\left(\bar{S}g\right)(t):=\|g\|_{L_\infty(t,\infty)},\quad t\in(0,\infty). \quad (26)$$

First we prove sufficiency. Assume that condition (22) holds. Then for all $\varphi\in\mathbb{A}$

$$\begin{aligned}\|I\varphi\|_{L_{\infty,v_2}(0,\infty)}&=\left\|(\bar{S}v_1)^{-1}\bar{S}v_1\varphi\right\|_{L_{\infty,v_2}(0,\infty)}\\&\le\operatorname*{ess\,sup}_{t>0}\bar{S}v_1(t)\varphi(t)\cdot\left\|(\bar{S}v_1)^{-1}\right\|_{L_{\infty,v_2}(0,\infty)}\\&=\left\|v_2\|v_1\|^{-1}_{L_\infty(\cdot,\infty)}\right\|_{L_\infty(0,\infty)}\operatorname*{ess\,sup}_{t>0}v_1(t)\varphi(t)\end{aligned} \quad (27)$$

by (25).

To prove necessity assume that I is bounded from $L_{\infty,v_1}(0,\infty)$ to $L_{\infty,v_2}(0,\infty)$ on the cone $\mathbb{A}$; that is,

$$\|I\varphi\|_{L_{\infty,v_2}(0,\infty)}\le c\|\varphi\|_{L_{\infty,v_1}(0,\infty)},\quad\varphi\in\mathbb{A}, \quad (28)$$

where $c>0$ is independent of φ.

We note that $(\bar{S}v_1)^{-1}\chi_{(1/n,\infty)}\in\mathbb{A}$ for all $n\in\mathbb{N}$ and take $\varphi=(\bar{S}v_1)^{-1}\chi_{(1/n,\infty)}$. Observe that

$$\begin{aligned}\left\|(\bar{S}v_1)^{-1}\chi_{(1/n,\infty)}\right\|_{L_{\infty,v_1}(0,\infty)}&\le\left\|(\bar{S}v_1)^{-1}\right\|_{L_{\infty,v_1}(0,\infty)}\\&=\operatorname*{ess\,sup}_{t>0}v_1(t)\left(\operatorname*{ess\,sup}_{t\le\tau<\infty}v_1(\tau)\right)^{-1}\\&\le 1.\end{aligned} \quad (29)$$

Also for all $\tau>0$ for sufficiently large n

$$\begin{aligned}&\left\|(\bar{S}v_1)^{-1}\chi_{(1/n,\infty)}\right\|_{L_{\infty,v_2}(0,\infty)}\\&\quad\ge\left\|\operatorname*{ess\,sup}_{t\le s<\infty}\chi_{(1/n,\infty)}(s)(\bar{S}v_1)^{-1}(s)\right\|_{L_{\infty,v_2}(\tau,\infty)}\\&\quad=\left\|(\bar{S}_1)^{-1}v_2\right\|_{L_\infty(\tau,\infty)};\end{aligned} \quad (30)$$

hence

$$\left\|\left(\bar{S}_1\right)^{-1}v_2\right\|_{L_\infty(\tau,\infty)}\le c \quad (31)$$

for all $\tau>0$ and condition (22) follows. □

We will use the following statement on the boundedness of the weighted Hardy operator:

$$H_wg(t):=\int_t^\infty g(s)w(s)\,ds,\quad 0<t<\infty, \quad (32)$$

where w is a weight.

The following theorem was proved in [28] (see, also [29]).

Theorem 11. *Let v_1, v_2, and w be weights on $(0,\infty)$ and let $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\operatorname*{ess\,sup}_{t>0}v_2(t)H_wg(t)\le C\operatorname*{ess\,sup}_{t>0}v_1(t)g(t) \quad (33)$$

holds for some $C>0$ for all nonnegative and nondecreasing g on $(0,\infty)$ if and only if

$$B:=\sup_{t>0}v_2(t)\int_t^\infty\frac{w(s)\,ds}{\operatorname{ess\,sup}_{s<\tau<\infty}v_1(\tau)}<\infty. \quad (34)$$

Moreover, the value $C=B$ is the best constant for (33).

Remark 12. In (33) and (34) it is assumed that $1/\infty=0$ and $0\cdot\infty=0$.

4. Spanne-Type Result for the Operator I_ρ in $LM^{\{x_0\}}_{p,\varphi}$

We assume that

$$\int_1^\infty\frac{\rho(t)}{t^n}\frac{dt}{t}<\infty, \quad (35)$$

so that the fractional integrals $I_\rho f$ are well defined, at least for characteristic functions $1/|x|^{2n}$ of complementary balls:

$$f(x)=\frac{\chi_{\mathbb{R}^n\backslash B(0,1)}(x)}{|x|^{2n}}. \quad (36)$$

In addition, we will also assume that ρ satisfies the growth condition: there exist constants $C_1>0$ and $0<2k_1<k_2<\infty$ such that

$$\sup_{r/2<s\le 3r/2}\frac{\rho(s)}{s^n}\le C_1\int_{k_1r}^{k_2r}\frac{\rho(t)}{t^n}\frac{dt}{t},\quad r>0. \quad (37)$$

This condition is weaker than the usual doubling condition for the function $\rho(t)/t^n$: there exists a constant $C_2 > 0$ such that

$$\frac{1}{C_2}\frac{\rho(t)}{t^n} \leq \frac{\rho(r)}{r^n} \leq C_2\frac{\rho(t)}{t^n}, \tag{38}$$

whenever r and t satisfy $r, t > 0$ and $1/2 \leq r/t \leq 2$.

In the sequel for the generalized fractional integral operator I_ρ we always assume that ρ satisfies the conditions (37) and then denote the set of all such functions by $\widetilde{G}_0$. We will write, when $\rho \in \widetilde{G}_0$,

$$\widetilde{\rho}(r) := Cr^n \int_r^\infty \frac{\rho(t)}{t^n}\frac{dt}{t}. \tag{39}$$

Remark 13. Typical examples of $\rho(t)$ that we envisage are, for $0 < \alpha < n$,

$$\rho(t) \equiv \begin{cases} t^\alpha \log\left(\dfrac{e}{t}\right), & 0 < t \leq 1, \\ \dfrac{t^\alpha}{\log(et)}, & 1 \leq t < \infty, \end{cases} \tag{40}$$

and, for $c > 0$,

$$\rho(t) \equiv \begin{cases} t^\alpha, & 0 < t \leq 1, \\ e^c e^{-ct^2}, & 1 \leq t < \infty. \end{cases} \tag{41}$$

The second one is used to control the Bessel potential (see also [30]).

The following theorem was proved in [11].

Theorem 14. *(1) Let $1 < p < q < \infty$. Then the operator I_ρ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if and only if there exists $C > 0$ such that for all $r > 0$*

$$\rho(r) \leq Cr^{n/p-n/q}. \tag{42}$$

(2) Let $1 < q < \infty$. Then the operator I_ρ is bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ if and only if there exists $C > 0$ such that for all $r > 0$

$$\rho(r) \leq Cr^{n-n/q}. \tag{43}$$

The following lemma is valid.

Lemma 15. *Let $1 \leq p < \infty$ and $\rho(t)$ satisfy the conditions (35) and (37). If the condition (42) is fulfilled, then for $p > 1$ the inequality*

$$\begin{aligned}\left\|I_\rho f\right\|_{L_q(B(x_0,r))} &\lesssim \|f\|_{L_p(B(x_0,2r))} \\ &\quad + r^{n/q}\int_{2r}^\infty \|f\|_{L_p(B(x_0,t))}\frac{\rho(t)}{t^{n/p}}\frac{dt}{t}\end{aligned} \tag{44}$$

holds for any ball $B(x_0, r)$ and for all $f \in L_p^{\text{loc}}(\mathbb{R}^n)$.

If the condition (43) is fulfilled, then for $p = 1$ the inequality

$$\begin{aligned}\left\|I_\rho f\right\|_{WL_q(B(x_0,r))} &\lesssim \|f\|_{L_1(B(x_0,2r))} \\ &\quad + r^{n/q}\int_{2r}^\infty \|f\|_{L_1(B(x_0,t))}\frac{\rho(t)}{t^{n}}\frac{dt}{t}\end{aligned} \tag{45}$$

holds for any ball $B(x_0, r)$ and for all $f \in L_1^{\text{loc}}(\mathbb{R}^n)$.

Proof. Let $1 < p < \infty$, $0 < \alpha < n/p$, and $1/q = 1/p - \alpha/n$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement(2B)}$. Hence

$$\left\|I_\rho f\right\|_{L_q(B)} \leq \left\|I_\rho f_1\right\|_{L_q(B)} + \left\|I_\rho f_2\right\|_{L_q(B)}. \tag{46}$$

Since $f_1 \in L_p(\mathbb{R}^n)$, $I_\rho f_1 \in L_q(\mathbb{R}^n)$ and from condition (42) we get the boundedness of I_ρ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ (see Theorem 14) and it follows that

$$\left\|I_\rho f_1\right\|_{L_q(B)} \leq \left\|I_\rho f_1\right\|_{L_q(\mathbb{R}^n)} \leq C\left\|f_1\right\|_{L_p(\mathbb{R}^n)} = C\|f\|_{L_p(2B)}, \tag{47}$$

where constant $C > 0$ is independent of f.

It is clear that $x \in B$, $y \in {}^\complement(2B)$ implies $(1/2)|x_0 - y| \leq |x - y| \leq (3/2)|x_0 - y|$. Then from conditions (35), (37) and by Fubini's theorem we have

$$\begin{aligned}&\int_{\complement(2B)} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\,dy \\ &\lesssim \int_{\complement(2B)} |f(y)|\left(\int_{k_1|x_0-y|}^{k_2|x_0-y|}\frac{\rho(t)}{t^{n+1}}dt\right)dy \\ &\approx \int_{2k_1 r}^\infty \left(\int_{2k_1 r<|x_0-y|<t}|f(y)|\,dy\right)\frac{\rho(t)}{t^{n+1}}dt \\ &\lesssim \int_{2r}^\infty\left(\int_{B(x_0,t)}|f(y)|\,dy\right)\frac{\rho(t)}{t^{n+1}}dt.\end{aligned} \tag{48}$$

Applying Hölder's inequality, we get

$$\int_{\complement(2B)}\frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\,dy \lesssim \int_{2r}^\infty \|f\|_{L_p(B(x_0,t))}\frac{\rho(t)}{t^{n/p+1}}dt. \tag{49}$$

Moreover, for all $p \in [1, \infty)$, the inequality

$$\left\|I_\rho f_2\right\|_{L_q(B)} \lesssim r^{n/q}\int_{2r}^\infty \|f\|_{L_p(B(x_0,t))}\frac{\rho(t)}{t^{n/p+1}}dt \tag{50}$$

is valid. Thus

$$\left\|I_\rho f\right\|_{L_q(B)} \lesssim \|f\|_{L_p(2B)} + r^{n/q}\int_{2r}^\infty \|f\|_{L_p(B(x_0,t))}\frac{\rho(t)}{t^{n/p+1}}dt. \tag{51}$$

Let $p = 1$. From the weak $(1, q)$ boundedness of I_ρ and (43) it follows that

$$\left\|I_\rho f_1\right\|_{WL_q(B)} \leq \left\|I_\rho f_1\right\|_{WL_q(\mathbb{R}^n)} \lesssim \left\|f_1\right\|_{L_1(\mathbb{R}^n)} = \|f\|_{L_1(2B)}. \tag{52}$$

Then from (50) and (52) we get the inequality (45). □

The following theorem is one of the main results of this paper.

Theorem 16. *Let $x_0 \in \mathbb{R}^n$, $1 \le p < \infty$, and the function ρ satisfy the conditions (35), (37), and (42). Let also (φ_1, φ_2) satisfy the conditions*

$$\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0, s)\, s^{n/p} \le C\varphi_2\left(x_0, \frac{t}{2}\right) t^{n/q},$$
$$\int_r^\infty \left(\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0, s)\, s^{n/p}\right) \frac{\rho(t)}{t^{n/p+1}} dt \le C\varphi_2(x_0, r), \quad (53)$$

where C does not depend on x_0 and r. Then the operator I_ρ is bounded from $LM_{p,\varphi_1}^{\{x_0\}}$ to $LM_{q,\varphi_2}^{\{x_0\}}$ for $p > 1$ and from $LM_{1,\varphi_1}^{\{x_0\}}$ to $WLM_{q,\varphi_2}^{\{x_0\}}$ for $p = 1$. Moreover, for $p > 1$,

$$\left\| I_\rho f \right\|_{LM_{q,\varphi_2}^{\{x_0\}}} \lesssim \| f \|_{LM_{p,\varphi_1}^{\{x_0\}}}, \quad (54)$$

and, for $p = 1$,

$$\left\| I_\rho f \right\|_{WLM_{q,\varphi_2}^{\{x_0\}}} \lesssim \| f \|_{LM_{1,\varphi_1}^{\{x_0\}}}. \quad (55)$$

Proof. By Lemma 15 and Theorems 10 and 11 we have, for $p > 1$,

$$\begin{aligned}\left\| I_\rho f \right\|_{LM_{q,\varphi_2}^{\{x_0\}}} &\lesssim \sup_{r>0} \varphi_2(x_0, r)^{-1} r^{-n/q} \| f \|_{L_p(B(x_0,2r))} \\ &\quad + \sup_{r>0} \varphi_2(x_0, r)^{-1} \int_r^\infty \| f \|_{L_p(B(x_0,t))} \frac{\rho(t)}{t^{n/p+1}} dt \\ &\approx \sup_{r>0} \varphi_1(x_0, r)^{-1} r^{-n/p} \| f \|_{L_p(B(x_0,r))} \\ &= \| f \|_{LM_{p,\varphi_1}^{\{x_0\}}},\end{aligned} \quad (56)$$

and, for $p = 1$,

$$\begin{aligned}\left\| I_\rho f \right\|_{WLM_{q,\varphi_2}^{\{x_0\}}} &\lesssim \sup_{r>0} \varphi_2(x_0, r)^{-1} r^{-n/q} \| f \|_{L_1(B(x_0,2r))} \\ &\quad + \sup_{r>0} \varphi_2(x_0, r)^{-1} \int_r^\infty \| f \|_{L_1(B(x_0,t))} \frac{\rho(t)}{t^{n+1}} dt \\ &\approx \sup_{r>0} \varphi_1(x_0, r)^{-1} r^{-n} \| f \|_{L_1(B(x_0,r))} \\ &= \| f \|_{LM_{1,\varphi_1}^{\{x_0\}}}.\end{aligned} \quad (57)$$

□

Corollary 17. *Let $1 \le p < \infty$, the function ρ satisfies the conditions (35), (37), and (42). Let also (φ_1, φ_2) satisfy the conditions*

$$\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x, s)\, s^{n/p} \le C\varphi_2\left(x, \frac{t}{2}\right) t^{n/q},$$
$$\int_r^\infty \left(\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x, s)\, s^{n/p}\right) \frac{\rho(t)}{t^{n/p+1}} dt \le C\varphi_2(x, r), \quad (58)$$

where C does not depend on x and r. Then the operator I_ρ is bounded from M_{p,φ_1} to M_{q,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{q,φ_2} for $p = 1$.

In the case $\rho(t) = t^\alpha$ from Theorem 16 we get new Spanne-type result on generalized local Morrey spaces.

Corollary 18. *Let $x_0 \in \mathbb{R}^n$, $0 < \alpha < n$, $1 \le p < q < \infty$, and $1/p - 1/q = \alpha/n$. Let also (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \left(\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0, s)\, s^{n/p}\right) \frac{dt}{t^{n/q+1}} dt \le C\varphi_2(x_0, r), \quad (59)$$

where C does not depend on r. Then the operator I_α is bounded from $LM_{p,\varphi_1}^{\{x_0\}}$ to $LM_{q,\varphi_2}^{\{x_0\}}$ for $p > 1$ and from $LM_{1,\varphi_1}^{\{x_0\}}$ to $WLM_{q,\varphi_2}^{\{x_0\}}$ for $p = 1$.

Also in the cases $\rho(t) = t^\alpha$ and $\varphi(x,t) = t^{(\lambda-n)/p}$, $0 < \lambda < n$, from Theorem 16 we get local Morrey space variant of Theorem 4.

Corollary 19. *Let $x_0 \in \mathbb{R}^n$, $0 < \alpha < n$, $1 < p < n/\alpha$, and $0 < \lambda < n - \alpha p$. Moreover, let $1/p - 1/q = \alpha/n$ and $\lambda/p = \mu/q$. Then, for $p > 1$, the operator I_α is bounded from $LM_{p,\lambda}^{\{x_0\}}$ to $LM_{q,\lambda}^{\{x_0\}}$ and, for $p = 1$, I_α is bounded from $LM_{1,\lambda}^{\{x_0\}}$ to $WLM_{q,\lambda}^{\{x_0\}}$.*

5. Adams-Type Result for the Operator I_ρ in $M_{p,\varphi}$

The following Adams-type result was proved in [31] (see also [12]).

Theorem 20. *Let $1 \le p < \infty$, $0 < \alpha < n/p$, and $q > p$ and let $\varphi(x,t)$ satisfy the conditions*

$$\sup_{r<t<\infty} \varphi(x, t) \le C\varphi(x, r), \quad (60)$$

$$\int_r^\infty t^\alpha \varphi(x, t)^{1/p} \frac{dt}{t} \le Cr^{-\alpha p/(q-p)}, \quad (61)$$

where C does not depend on $x \in \mathbb{R}^n$ and $r > 0$.

Then the operator I_α is bounded from $M_{p,\varphi^{1/p}}$ to $M_{q,\varphi^{1/q}}$ for $p > 1$ and from $M_{1,\varphi}$ to $WM_{q,\varphi^{1/q}}$ for $p = 1$.

The following Theorem was proved in [32].

Theorem 21. *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfies the condition*

$$\sup_{r<t<\infty} \varphi_1(x, t) \le C\varphi_2(x, r), \quad (62)$$

where C does not depend on x and r. Then, for $p > 1$, the Hardy-Littlewood maximal operator M is bounded from M_{p,φ_1} to M_{p,φ_2} and, for $p = 1$, M is bounded from M_{1,φ_1} to WM_{1,φ_2}.

The following theorem is a main result of this paper on Adams-type estimate for generalized fractional integral operator I_ρ. In the case $\rho(t) = t^\alpha$ we get Theorem 20 from this theorem.

Theorem 22. *Let $1 \le p < \infty$, $q > p$, and $\rho(t)$ satisfy the conditions (37) and (42). Let also $\varphi(x,t)$ satisfy the condition (60) and*

$$\int_r^\infty \varphi(x,t)^{1/p} \frac{\rho(t)}{t} dt \le C\rho(r)^{-p/(q-p)}, \quad (63)$$

where C does not depend on $x \in \mathbb{R}^n$ and $r > 0$.

Then the operator I_ρ is bounded from $M_{p,\varphi^{1/p}}$ to $M_{q,\varphi^{1/q}}$ for $p > 1$ and from $M_{1,\varphi}$ to $WM_{q,\varphi^{1/q}}$ for $p = 1$.

Proof. Let $x_0 \in \mathbb{R}^n$, $1 < p < \infty$, $0 < \alpha < n/p$, $q > p$, and $f \in M_{p,\varphi^{1/p}}$. Write $f = f_1 + f_2$, where $B = B(x,r)$, $f_1 = f\chi_{2B}$, and $f_2 = f\chi_{\complement(2B)}$. Then we have

$$I_\rho f(x) = I_\rho f_1(x) + I_\rho f_2(x). \quad (64)$$

For $I_\rho f_1(y)$, $y \in B(x,r)$, following Hedberg's trick (see, e.g., [33, page 354]), we obtain

$$\begin{aligned}
\left|I_\rho f_1(y)\right| &\le \int_{B(x,2r)} \frac{\rho(|y-z|)}{|y-z|^n} |f(z)| dz \\
&\approx \sum_{k=-\infty}^{0} \int_{B(x,2^{k+1}r)\setminus B(x,2^k r)} \frac{\rho(|y-z|)}{|y-z|^n} |f(z)| dz \\
&\lesssim \sum_{k=-\infty}^{0} \int_{(2^{k+1}-1)k_1 r}^{(2^{k+1}-1)k_2 r} \frac{\rho(s)}{s^{n+1}} ds \int_{B(x,2^{k+1}r)} |f(z)| dz \\
&\approx Mf(x) \sum_{k=-\infty}^{0} \int_{(2^{k+1}-1)k_1 r}^{(2^{k+1}-1)k_2 r} \frac{\rho(s)}{s} ds \\
&= Mf(x) \int_0^{k_2 r} \frac{\rho(s)}{s} ds \\
&= Mf(x) \widetilde{\rho}(k_2 r) \\
&\lesssim Mf(x) \rho(r).
\end{aligned} \quad (65)$$

For $I_\rho f_2(y)$, $y \in B(x,r)$, from (49) we have

$$\begin{aligned}
\left|I_\rho f_2(y)\right| &\lesssim \int_{\complement B(x,2r)} \frac{\rho(|y-z|)}{|y-z|^n} |f(z)| dz \\
&\lesssim \int_{2r}^\infty \|f\|_{L_p(B(x,t))} \frac{\rho(t)}{t^{n/p+1}} dt.
\end{aligned} \quad (66)$$

Then from condition (63) and inequality (66) for all $y \in B(x,r)$ we get

$$\begin{aligned}
\left|I_\rho f(y)\right| &\lesssim \rho(r) Mf(x) + \int_r^\infty \|f\|_{L_p(B(x,t))} \frac{\rho(t)}{t^{n/p+1}} dt \\
&\le \rho(r) Mf(x) + \|f\|_{M_{p,\varphi^{1/p}}} \int_r^\infty \varphi(x,t)^{1/p} \frac{\rho(t)}{t} dt \\
&\lesssim \rho(r) Mf(x) + \rho(r)^{-p/(q-p)} \|f\|_{M_{p,\varphi^{1/p}}}.
\end{aligned} \quad (67)$$

Hence choosing $\rho(r) = (\|f\|_{M_{p,\varphi^{1/p}}}/Mf(x))^{(q-p)/q}$ for all $y \in B(x,r)$, we have

$$\left|I_\rho f(y)\right| \lesssim (Mf(x))^{p/q} \|f\|_{M_{p,\varphi^{1/p}}}^{1-p/q}. \quad (68)$$

Consequently the statement of the theorem follows in view of the boundedness of the maximal operator M in $M_{p,\varphi^{1/p}}$ provided by Theorem 21 in virtue of condition (60). If $1 < p < q < \infty$, then

$$\begin{aligned}
&\left\|I_\rho f\right\|_{M_{q,\varphi^{1/q}}} \\
&= \sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-1/q} t^{-n/q} \left\|I_\rho f\right\|_{L_q(B(x,t))} \\
&\lesssim \|f\|_{M_{p,\varphi^{1/p}}}^{1-p/q} \sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-1/q} t^{-n/q} \|Mf\|_{L_p(B(x,t))}^{p/q} \\
&= \|f\|_{LM_{p,\varphi^{1/p}}}^{1-p/q} \left(\sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-1/p} t^{-n/p} \|Mf\|_{L_p(B(x,t))} \right)^{p/q} \\
&= \|f\|_{M_{p,\varphi^{1/p}}}^{1-p/q} \|Mf\|_{M_{p,\varphi^{1/p}}}^{p/q} \\
&\lesssim \|f\|_{M_{p,\varphi^{1/p}}},
\end{aligned} \quad (69)$$

and if $1 < q < \infty$, then

$$\begin{aligned}
&\left\|I_\rho f\right\|_{WM_{q,\varphi^{1/q}}} \\
&= \sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-1/q} t^{-n/q} \left\|I_\rho f\right\|_{WL_q(B(x,t))} \\
&\lesssim \|f\|_{M_{1,\varphi}}^{1-1/q} \sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-1/q} t^{-n/q} \|Mf\|_{WL_1(B(x,t))}^{1/q} \\
&= \|f\|_{M_{1,\varphi}}^{1-1/q} \left(\sup_{x\in\mathbb{R}^n, t>0} \varphi(x,t)^{-1} t^{-n} \|Mf\|_{WL_1(B(x,t))} \right)^{1/q} \\
&= \|f\|_{M_{1,\varphi}}^{1-1/q} \|Mf\|_{WM_{1,\varphi}}^{1/q} \\
&\lesssim \|f\|_{M_{1,\varphi}}.
\end{aligned} \quad (70)$$

Thus the proof of the theorem is completed. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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