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CHARACTERIZATIONS OF SPACELIKE SLANT HELICES IN MINKOWSKI 3-SPACE

BY

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Abstract. In this paper, we investigate tangent indicatrix, principal normal indicatrix and binormal indicatrix of a spacelike curve with spacelike, timelike and null principal normal vector in Minkowski 3-space $\mathbb{E}_1^3$ and we construct their Frenet equations and curvature functions. Moreover, we obtain some differential equations which characterize for a spacelike curve to be a slant helix by using the Frenet apparatus of spherical indicatrix of the curve. Also related examples and their illustrations are given.

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Key words: Minkowski 3-space, spacelike curve, slant helix, genaral helix, spherical helix, tangent indicatrix, principal normal indicatrix and binormal indicatrix.

1. Introduction

In classical differential geometry; a general helix in the Euclidean 3-space, $\mathbb{E}^3$, is a curve with constant slope which means that it makes a constant angle with a fixed direction (the axis of the helix). A classical result stated by *M. A. Lancret* in 1802 and first proved by *B. de Saint Venant* in 1845 (see for details [14, 18]) is: A necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant. In particular circular helices where both curvatures and torsion are constant so as plane curve where the torsion vanish identically provide two subclasses of general helices.

The Lancret theorem was revisited and solved by Barros [4] in three dimensional real space forms by using the notion of Killing vector fields along curves. Characterizations for helices and Cornu spirals in those backgrounds were also obtained by ARROYO, BARROS and GARAY in [1].

For general helices in semi-Riemannian settings, including Lorentzian ones, we refer the reader to [5, 6, 7, 8, 9].

Recently, IZUMIYA and TAKEUCHI in [10], have introduced the concept of slant helix in Euclidean 3-space. A slant helix in Euclidean space $\mathbb{E}^3$ was defined by the property that the principal normal makes a constant angle with a fixed direction. Moreover, Izumiya and Takeuchi showed that γ is a slant helix if and only if the geodesic curvature of the principal normal of a space curve γ is a constant function.

In [12], KULA and YAYLI studied the spherical images under both tangent and binormal indicatrices of slant helices and obtained that the spherical images of a slant helix are spherical helix. In [13], the authors characterize slant helices by certain differential equations verified for each one of obtained spherical indicatrix in Euclidean 3-space. Recently, ALI and LOPEZ in [2], have studied slant helix in Minkowski 3-space. They showed that the spherical indicatrix of a slant helix in $\mathbb{E}^3_1$ are helices. Also in [3], ALI and TURGUT, studied the position vector of a timelike slant helix in $\mathbb{E}^3_1$.

In this paper, we consider a spacelike curve in Minkowski 3-space and we obtained its spherical indicatrix and their Frenet apparatus. Finally, we obtain some certain differential equations for a space like curve to be a slant helix by the help of spherical indicatrix of the curve and well known results obtained by ALI and LOPEZ in [2].

2. Preliminaries

The Minkowski $3-$space $\mathbb{E}^3_1$ is the Euclidean 3-space $\mathbb{E}^3$ equipped with indefinite flat metric given by $g = -dx_1^2 + dx_2^2 + dx_3^2$, where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}^3_1$. Recall that a vector $v \in \mathbb{E}^3_1$ can be *spacelike* if $g(v, v) > 0$ or $v = 0$, *timelike* if $g(v, v) < 0$ and *null* (*lightlike*) if $g(v, v) = 0$ and $v \neq 0$. The norm of a vector v is given by $||v|| = \sqrt{|g(v, v)|}$ and two vectors v and w are said to be orthogonal if $g(v, w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}^3_1$ can locally be *spacelike*, *timelike* or *null* (*lightlike*), if all its velocity vectors $\alpha'(s)$ are spacelike, timelike or null, respectively. Spacelike or a timelike curve α has unit speed, if $g(\alpha'(s), \alpha'(s)) = \pm 1$. A null curve α is parameterized by pseudo-arc s, if $g(\alpha''(s), \alpha''(s)) = 1$ (see [15]). For a non-null unit speed space curve $\alpha(s)$

in the space $\mathbb{E}_1^3$ with non-null normals, the following Frenet formulae are given in [6, 9]

$$
\begin{array}{rl}
T'(s) = & \varkappa(s) N(s), \\
N'(s) = & -\varepsilon_0\varepsilon_1\varkappa(s) T(s) + \tau(s) B(s), \\
B'(s) = & -\varepsilon_1\varepsilon_2\tau(s) N(s),
\end{array}
\tag{2.1}
$$

where $g(T(s),T(s)) = \varepsilon_0 = \pm 1$, $g(N(s),N(s)) = \varepsilon_1 = \pm 1$ and $g(B(s), B(s)) = \varepsilon_2 = \pm 1$.

If α is a pseudo null curve, i.e. α is a spacelike curve with a null principal normal vector N, then the following Frenet formulas hold (see [16])

$$
\begin{array}{rl}
T'(s) = & \varkappa(s) N(s), \\
N'(s) = & \tau(s) N(s), \\
B'(s) = & -\varkappa(s) T(s) - \tau(s) B(s),
\end{array}
\tag{2.2}
$$

where $T(s) = \alpha'(s), N(s) = \alpha''(s)$ and $g(T,T) = 1, g(N,N) = g(B,B) = 0, g(T,N) = g(T,\ B) = 0, g(N,B) = 1$.

For a pseudo null curve, $\varkappa$ can take only two values: $\varkappa = 0$ when α is a straight line, or $\varkappa = 1$ in all other cases.

If α is a null curve then the following Frenet formulas hold (see [16, 19])

$$
\begin{array}{rl}
T'(s) = & \varkappa(s) N(s), \\
N'(s) = & \tau(s) T(s) - \varkappa(s) B(s), \\
B'(s) = & -\tau(s) N(s),
\end{array}
\tag{2.3}
$$

where $g(T,T) = 0$, $g(N,N) = 1$, $g(B,B) = 0$, $g(T,N) = g(N,B) = 0$, $g(T,B) = 1$.

For a null curve, $\varkappa = 0$ can take only two values: $\varkappa = 0$ when α is a straight line, or $\varkappa = 1$ in all other cases (see [16, 19]).

It is well known that, the pseudo - Riemannian sphere with radius $r = 1$ and centered at origin is defined by $S_1^2 = \left\{p \in \mathbb{E}_1^3 : g(p,p) = 1\right\}$, the pseudohyperbolic space with radius $r = 1$ and centered at origin is defined by $H_0^2 = \left\{p \in \mathbb{E}_1^3 : g(p,p) = -1\right\}$ are the hyperquadrics with dimension 2 and index 1 and with dimension 2 and index 0, respectively (see [15]).

3. Spherical indicatrix of a spacelike curve in Minkowski 3-space

In Euclidean geometry, the spherical indicatrix of a space curve defined as follows:

Let α be a unit speed regular curve in Euclidean 3-space with Frenet vectors t, n and b. The unit tangent vectors along the curve α generate a curve (t) on the sphere of radius 1 about the origin. The curve (t) is called the spherical indicatrix of t or more commonly, (t) is called tangent indicatrix of the curve α. If $\alpha = \alpha(s)$ is a natural representation of α, then $(t) = t(s)$ will be a representation of (t). Similarly one considers the principal normal indicatrix $(n) = n(s)$ and binormal indicatrix $(b) = b(s)$. It is clear that, this definition is related with spherical curve (see [18]).

In Minkowski 3-space $\mathbb{E}_1^3$, the definition of spherical indicatrix of a space curve is similar with the Euclidean case but richer than Euclidean case. For example, if position vector of a spacelike curve is a spacelike then the curve lies on the pseudo-Riemannian sphere S_1^2, if its position vector is a timelike then the curve lies on pseudohyperbolic space H_0^2. In Minkowski space, for the characterizations of spherical curves, we refer the papers of Petrović-Torgašev and Šućurović [16, 17] and Inoguchi and Lee [11].

In this section, we investigate the Frenet apparatus of the tangent indicatrix, principal normal indicatrix and binormal indicatrix of a spacelike curve with spacelike, timelike and null principal normal vectors in Minkowski 3-space. We will give only some theorems with their proofs. Because the others can easily prove that, using the similar method. Here, by D we denote the covariant differentiation of $\mathbb{E}_1^3$.

We will give to "Lemma 3.1" and "Lemma 3.2" as unproved, since we will use in next sections.

Lemma 3.1. *Let α be a unit speed spacelike curve with space principal normal vector in $\mathbb{E}_1^3$. Geodesic curvature of the spherical image of spacelike principal normal indicatrix (N) of α is $\sigma_1 = \frac{\varkappa^2}{(\varkappa^2-\tau^2)^{3/2}}\left(\frac{\tau}{\varkappa}\right)'$ and geodesic curvature of the spherical image of timelike principal normal indicatrix (N) of α is $\sigma_2 = \frac{\varkappa^2}{(\tau^2-\varkappa^2)^{3/2}}\left(\frac{\tau}{\varkappa}\right)'$, where $\tau^2 - \varkappa^2$ does not vanish.*

Lemma 3.2. *Let α be a unit speed spacelike curve with space principal normal vector in $\mathbb{E}_1^3$. Geodesic curvature of the spherical image of timelike principal normal indicatrix (N) of α is $\sigma_3 = \frac{\varkappa^2}{(\tau^2+\varkappa^2)^{3/2}}\left(\frac{\tau}{\varkappa}\right)'$, where $\tau^2 + \varkappa^2$ does not vanish.*

In the next three theorems, we obtain Frenet formulae of *tangent indicatrix β, principal normal indicatrix γ* and *binormal indicatrix δ of the spacelike curve α* with spacelike principal normal vector in $\mathbb{E}_1^3$.

Proposition 3.1. *Let α be a unit speed spacelike curve with spacelike principal normal vector in $\mathbb{E}_1^3$ with Frenet vectors T, N, B and curvatures $\varkappa$, τ. If the Frenet frame of the tangent indicatrix β of the space curve α is $\{T_\beta, N_\beta, B_\beta\}$, then we have the Frenet-Serret formulae:*

$$D_{T_\beta}T_\beta = \varkappa_\beta N_\beta, D_{T_\beta}N_\beta = -\varkappa_\beta T_\beta + \tau_\beta B_\beta, D_{T_\beta}B_\beta = \tau_\beta N_\beta, \tag{3.1}$$

where

$$\begin{aligned} T_\beta &= N, \\ N_\beta &= \frac{1}{\sqrt{\varepsilon\left(\varkappa^2-\tau^2\right)}}\left(-\varkappa T+\tau B\right), \\ B_\beta &= \frac{1}{\sqrt{\varepsilon\left(\varkappa^2-\tau^2\right)}}\left(-\tau T+\varkappa B\right) \end{aligned} \tag{3.2}$$

and $\varkappa_\beta = \frac{\sqrt{\varepsilon(\varkappa^2-\tau^2)}}{\varkappa}$ *is the curvature of β,* $\tau_\beta = -\frac{\varkappa\left(\frac{\tau}{\varkappa}\right)'}{(\varkappa^2-\tau^2)}$ *is the torsion of β.*

Theorem 3.1. *Let α be a unit speed spacelike curve with spacelike principal normal vektor in $\mathbb{E}_1^3$ with Frenet vectors T, N, B and curvatures $\varkappa$, τ. If the Frenet frame of the principal normal indicatrix γ of the space curve α is $\{T_\gamma, N_\gamma, B_\gamma\}$, then we have the Frenet-Serret formulae for three cases:*

Case I. *If $\varkappa^2 > \tau^2$, γ is a spacelike curve.*

a. *If $-1 < \sigma_1 < 1$, Frenet frame of γ is $\{T_\gamma, N_\gamma, B_\gamma\}$, then we have the Frenet-Serret formulae:*

$$D_{T_\gamma}T_\gamma = \varkappa_\gamma N_\gamma,\ D_{T_\gamma}N_\gamma = -\varkappa_\gamma T_\gamma + \tau_\gamma B_\gamma,\ D_{T_\gamma}B_\gamma = \tau_\gamma N_\gamma, \tag{3.3a}$$

where

$$\begin{aligned} T_\gamma &= \frac{1}{\sqrt{\varkappa^2-\tau^2}}\left(-\varkappa T+\tau B\right), \\ N_\gamma &= \frac{1}{\sqrt{1-\sigma_1^2}}\left(-\frac{\sigma_1\tau}{\sqrt{\varkappa^2-\tau^2}}T - N + \frac{\sigma_1\varkappa}{\sqrt{\varkappa^2-\tau^2}}B\right), \\ B_\gamma &= \frac{1}{\sqrt{(1-\sigma_1^2)(\varkappa^2-\tau^2)}}\left(-\tau T - \sigma_1\sqrt{\varkappa^2-\tau^2}N + \varkappa B\right). \end{aligned} \tag{3.4a}$$

Moreover, the curvature of γ is $\varkappa_\gamma = \sqrt{1-\sigma_1^2}$ and the torsion of γ is

$$\tau_\gamma = \frac{1}{\sqrt{\varkappa^2-\tau^2}(1-\sigma_1^2)}\sigma_1'.$$

b. *If $\sigma < -1$ or $1 < \sigma_1$, Frenet frame of γ is $\{T_\gamma, N_\gamma, B_\gamma\}$, then we have the Frenet-Serret formulae:*

$$D_{T_\gamma} T_\gamma = \varkappa_\gamma N_\gamma,\ D_{T_\gamma} N_\gamma = \varkappa_\gamma T_\gamma + \tau_\gamma B_\gamma,\ D_{T_\gamma} B_\gamma = \tau_\gamma N_\gamma, \tag{3.3b}$$

where

$$\begin{aligned} T_\gamma &= \frac{1}{\sqrt{\varkappa^2 - \tau^2}} \left(-\varkappa T + \tau B\right), \\ N_\gamma &= \frac{1}{\sqrt{\sigma_1^2 - 1}} \left(-\frac{\sigma_1 \tau}{\sqrt{\varkappa^2 - \tau^2}} T - N + \frac{\sigma_1 \varkappa}{\sqrt{\varkappa^2 - \tau^2}} B\right), \\ B_\gamma &= \frac{1}{\sqrt{(\sigma_1^2 - 1)(\varkappa^2 - \tau^2)}} \left(-\tau T - \sigma_1 \sqrt{\varkappa^2 - \tau^2} N + \varkappa B\right). \end{aligned} \tag{3.4b}$$

Moreover, the curvature of γ is $\varkappa_\gamma = \sqrt{\sigma_1^2 - 1}$ and the torsion of γ is $\tau_\gamma = \frac{1}{\sqrt{\varkappa^2 - \tau^2}(\sigma_1^2 - 1)} \sigma_1'$.

Case II. *If $\varkappa^2 < \tau^2$, γ is a timelike curve Frenet frame of γ is $\{T_\gamma, N_\gamma, B_\gamma\}$, then we have the Frenet-Serret formulae:*

$$D_{T_\gamma} T_\gamma = \varkappa_\gamma N_\gamma,\ D_{T_\gamma} N_\gamma = \varkappa_\gamma T_\gamma + \tau_\gamma B_\gamma,\ D_{T_\gamma} B_\gamma = -\tau_\gamma N_\gamma, \tag{3.5}$$

where

$$\begin{aligned} T_\gamma &= \frac{1}{\sqrt{\tau^2 - \varkappa^2}} \left(-\varkappa T + \tau B\right), \\ N_\gamma &= \frac{1}{\sqrt{1 + \sigma_2^2}} \left(\frac{\tau \sigma_2}{\sqrt{\tau^2 - \varkappa^2}} T + N - \frac{\varkappa \sigma_2}{\sqrt{\tau^2 - \varkappa^2}} B\right), \\ B_\gamma &= \frac{1}{\sqrt{(1 + \sigma_2^2)(\tau^2 - \varkappa^2)}} \left(\tau T - \sigma_2 \sqrt{\tau^2 - \varkappa^2} N - \varkappa B\right). \end{aligned} \tag{3.6}$$

Moreover, the curvature of γ is $\varkappa_\gamma = \sqrt{1 + \sigma_2^2}$ and the torsion of γ is

$$\tau_\gamma = -\frac{1}{\sqrt{\tau^2 - \varkappa^2}(1 + \sigma_2^2)} \sigma_2'.$$

Case III. *If $\varkappa^2 = \tau^2$, γ is a null curve and $\varkappa_\gamma = 0$ so that γ is a null straight line.*

Proof. Case 1: If $\varkappa^2 > \tau^2$, γ is a spacelike curve.

a. For $-1 < \sigma_1 < 1$. Let s be arc-parameter of α and s_γ be the arc-parameter of γ

$$\gamma\left(s_\gamma\right) = N\left(s\right). \tag{3.7}$$

Differentiating (3.7) with respect to s and by using Frenet formulas given in (3.1), we get $\frac{d\gamma}{ds_\gamma} \cdot \frac{ds_\gamma}{ds} = \frac{dN(s)}{ds}$,

$$T_\gamma\left(s_\gamma\right) \cdot \frac{ds_\gamma}{ds} = -\varkappa\left(s\right) T\left(s\right) + \tau\left(s\right) B\left(s\right) \tag{3.8}$$

and we have

$$g\left(T_\gamma\left(s_\gamma\right), T_\gamma\left(s_\gamma\right)\right) = \left(\frac{ds}{ds_\gamma}\right)^2 \cdot \left(\varkappa^2\left(s\right) - \tau^2\left(s\right)\right). \tag{3.9}$$

Using the Eq.(3.9) we can easily show that $g\left(T_\gamma\left(s_\gamma\right), T_\gamma\left(s_\gamma\right)\right) = 1$ and

$$\frac{ds_\gamma}{ds} = \sqrt{\varkappa^2\left(s\right) - \tau^2\left(s\right)}.$$

So, we can rewrite the Eq.(3.8)

$$T_\gamma\left(s_\gamma\right) = \frac{1}{\sqrt{\varkappa^2\left(s\right) - \tau^2\left(s\right)}}\left(-\varkappa\left(s\right) T\left(s\right) + \tau(s) B\left(s\right)\right). \tag{3.10}$$

Differentiating (3.10) with respect to s

$$\frac{dT_\gamma\left(s_\gamma\right)}{ds_\gamma} = \frac{-\tau\left(\frac{\tau}{\varkappa}\right)'\varkappa^2}{\left(\varkappa^2\left(s\right) - \tau^2\left(s\right)\right)^2} T\left(s\right) - N\left(s\right) + \frac{\varkappa\left(\frac{\tau}{\varkappa}\right)'\varkappa^2}{\left(\varkappa^2\left(s\right) - \tau^2\left(s\right)\right)^2} B\left(s\right). \tag{3.11}$$

From the norm of $\frac{dT_\gamma(s_\gamma)}{ds_\gamma}$

$$\left\|\frac{dT_\gamma\left(s_\gamma\right)}{ds_\gamma}\right\| = \sqrt{1 - \sigma_1^2}.$$

If we consider $N_\gamma\left(s_\gamma\right) = \frac{\frac{dT_\gamma(s_\gamma)}{ds_\gamma}}{\left\|\frac{dT_\gamma(s_\gamma)}{ds_\gamma}\right\|}$, we can write

$$N_\gamma\left(s_\gamma\right) = \frac{1}{\sqrt{1 - \sigma_1^2}}\left(-\frac{\sigma_1 \tau}{\sqrt{\varkappa^2 - \tau^2}} T(s) - N(s) + \frac{\sigma_1 \varkappa}{\sqrt{\varkappa^2 - \tau^2}} B(s)\right). \tag{3.12}$$

Now we know that $B_\gamma(s_\gamma) = T_\gamma(s_\gamma) \times N_\gamma(s_\gamma)$ and using the equations (3.10), (3.12) we show that

$$B_\gamma(s_\gamma) = \frac{1}{\sqrt{(1-\sigma_1^2)(\varkappa^2-\tau^2)}}\left(-\tau T(s) - \sigma_1\sqrt{\varkappa^2-\tau^2}N(s) + \varkappa B(s)\right),$$

where we can easily see that $g(N_\gamma(s_\gamma), N_\gamma(s_\gamma)) = 1$ and $g(B_\gamma(s_\gamma), B_\gamma(s_\gamma)) = -1$, that is, N_γ is spacelike vector and B_γ is timelike vector. So, γ is a spacelike curve with spacelike principal normal and timelike binormal. Moreover, the Frenet formulas of γ is given by $D_{T_\gamma}T_\gamma = \varkappa_\gamma N_\gamma$, $D_{T_\gamma}N_\gamma = -\varkappa_\gamma T_\gamma + \tau_\gamma B_\gamma$, $D_{T_\gamma}B_\gamma = \tau_\gamma N_\gamma$. By using the equality of $D_{T_\gamma}T_\gamma = \frac{dT_\gamma(s_\gamma)}{ds_\gamma} = \varkappa_\gamma N_\gamma$ with the Eq. (3.11) and (3.12), we get $\varkappa_\gamma = \sqrt{1-\sigma_1^2}$. Similarly, from the equation $D_{T_\gamma}N_\gamma = -\varkappa_\gamma T_\gamma + \tau_\gamma B_\gamma$ we can easily see that $\tau_\gamma = -\frac{1}{\sqrt{\varkappa^2-\tau^2}(1-\sigma_1^2)}\sigma_1'$.

b. The proof of (**b**) is obvious.

Case 2: If $\varkappa^2 < \tau^2$, γ is a timelike curve. By using the method in the Case (1) the proof of Case (2) is obvious.

Case 3: If $\varkappa^2(s) = \tau^2(s)$, γ is a null curve, using the Eq.(3.9) we can easily show that $g(T_\gamma(s_\gamma), T_\gamma(s_\gamma)) = 0$.

Differentiating (3.8) with respect to s and using Frenet formulas of γ null curve $\varkappa_\gamma N_\gamma(s_\gamma)\frac{ds_\gamma}{ds}\cdot\frac{ds_\gamma}{ds} + T_\gamma(s_\gamma)\frac{d^2s_\gamma}{ds^2} = -\varkappa'(s)T(s) - \varkappa^2(s)N(s) + \tau'(s)B(s) + \tau^2(s)N(s)$ or

$$\varkappa_\gamma N_\gamma(s_\gamma)\left(\frac{ds_\gamma}{ds}\right)^2 + T_\gamma(s_\gamma)\frac{d^2s_\gamma}{ds^2} = -\varkappa'(s)T(s) + \tau'(s)B(s). \tag{3.13}$$

From the last equation we have

$$g\left(\varkappa_\gamma N_\gamma(s_\gamma)\left(\frac{ds_\gamma}{ds}\right)^2 + T_\gamma(s_\gamma)\frac{d^2s_\gamma}{ds^2},\right.$$
$$\left.\varkappa_\gamma N_\gamma(s_\gamma)\left(\frac{ds_\gamma}{ds}\right)^2 + T_\gamma(s_\gamma)\frac{d^2s_\gamma}{ds^2}\right) = (\varkappa')^2 - (\tau')^2,\ \ \varkappa_\gamma^2\left(\frac{ds_\gamma}{ds}\right)^4 = 0,$$

where $\frac{ds_\gamma}{ds} \neq 0$. So, we get $\varkappa_\gamma = 0$, that is, γ is a null straight line. These complete the proof. □

Corollary 3.1. *If α is a spacelike general helix with spacelike principal normal vector and non-zero curvatures $\varkappa$, τ in $\mathbb{E}_1^3$ then the principal normal indicatrix of α is null-geodesic lying in pseudo sphere $\mathbb{S}_1^2$.*

Proposition 3.2. *Let α be a unit speed spacelike curve with spacelike principal normal vector in $\mathbb{E}_1^3$ with Frenet vectors T, N, B and curvatures $\varkappa$, τ. If Frenet frame of the binormal indicatrix δ of the space curve α is $\{T_\delta, N_\delta, B_\delta\}$, then we have the Frenet-Serret formulae:*

$$D_{T_\delta} T_\delta = \varkappa_\delta N_\delta,\ D_{T_\delta} N_\delta = -\varkappa_\delta T_\delta + \tau_\delta B_\delta,\ D_{T_\delta} B_\delta = \tau_\delta N_\delta, \tag{3.14}$$

where

$$\begin{aligned} T_\delta &= N, \\ N_\delta &= \frac{1}{\sqrt{\varepsilon\left(\varkappa^2 - \tau^2\right)}}\left(-\varkappa T + \tau B\right), \\ B_\delta &= \frac{1}{\sqrt{\varepsilon\left(\varkappa^2 - \tau^2\right)}}\left(-\tau T + \varkappa B\right) \end{aligned} \tag{3.15}$$

and $\varkappa_\delta = \frac{\sqrt{\varepsilon(\varkappa^2-\tau^2)}}{\tau}$ is the curvature of δ, $\tau_\delta = \frac{\varkappa^2\left(\frac{\tau}{\varkappa}\right)'}{\tau(\varkappa^2-\tau^2)}$ is the torsion of δ.

In the next three theorems, we obtain Frenet formulae of *tangent indicatrix β, principal normal indicatrix γ* and *binormal indicatrix δ of the spacelike curve α* with timelike principal normal vector in $\mathbb{E}_1^3$.

Proposition 3.3. *Let α be a unit speed spacelike curve with timelike principal normal vector in $\mathbb{E}_1^3$ with Frenet vectors T, N, B and curvatures $\varkappa$, τ. If the Frenet frame of the tangent indicatrix β of the space curve α is $\left\{\mathbf{T}_\beta, \mathbf{N}_\beta, \mathbf{B}_\beta\right\}$, then we have the Frenet-Serret formulae:*

$$D_{\mathbf{T}_\beta} \mathbf{T}_\beta = \varkappa_\beta \mathbf{N}_\beta,\ D_{\mathbf{T}_\beta} \mathbf{N}_\beta = \varkappa_\beta \mathbf{T}_\beta + \tau_\beta \mathbf{B}_\beta,\ D_{\mathbf{T}_\beta} \mathbf{B}_\beta = -\tau_\beta \mathbf{N}_\beta, \tag{3.16}$$

where

$$\begin{aligned} \mathbf{T}_\beta &= N, \\ \mathbf{N}_\beta &= \tfrac{1}{\sqrt{(\varkappa^2+\tau^2)}}\left(\varkappa T + \tau B\right), \\ \mathbf{B}_\beta &= \tfrac{1}{\sqrt{(\varkappa^2-\tau^2)}}\left(\tau T + \varkappa B\right) \end{aligned} \tag{3.17}$$

and $\varkappa_\beta = \frac{\sqrt{(\varkappa^2+\tau^2)}}{\varkappa}$ is the curvature of β, $\tau_\beta = -\frac{\varkappa\left(\frac{\tau}{\varkappa}\right)'}{(\varkappa^2+\tau^2)}$ is the torsion of β.

Theorem 3.2. *Let α be a unit speed spacelike curve with timelike principal normal vector in $\mathbb{E}_1^3$ with Frenet vectors T, N, B and curvatures $\varkappa, \tau$.*

a. *If $-1<\sigma_3<1$ and the Frenet frame of the principal normal indicatrix γ of the space curve α is $\left\{\mathbf{T}_\gamma, \mathbf{N}_\gamma, \mathbf{B}_\gamma\right\}$, then we have the Frenet-Serret formulae:*

$$D_{\mathbf{T}_\gamma}\mathbf{T}_\gamma = \varkappa_\gamma \mathbf{N}_\gamma,\ D_{\mathbf{T}_\gamma}\mathbf{N}_\gamma = \varkappa_\gamma \mathbf{T}_\gamma + \tau_\gamma \mathbf{B}_\gamma,\ D_{\mathbf{T}_\gamma}\mathbf{B}_\gamma = \tau_\gamma \mathbf{N}_\gamma, \tag{3.18a}$$

where

$$\begin{aligned}
\mathbf{T}_\gamma &= \frac{1}{\sqrt{\varkappa^2+\tau^2}}\left(\varkappa T+\tau B\right),\\
\mathbf{N}_\gamma &= \frac{1}{\sqrt{1-\sigma_3^2}}\left(\frac{\tau\sigma_3}{\varkappa^2+\tau^2}T+N-\frac{\varkappa\sigma_3}{\varkappa^2+\tau^2}B\right),\\
\mathbf{B}_\gamma &= \frac{1}{\sqrt{(1-\sigma_3^2)(\varkappa^2+\tau^2)}}\left(-\tau T-\sigma_3\sqrt{\varkappa^2+\tau^2}N-\varkappa B\right).
\end{aligned} \tag{3.19a}$$

Moreover, the curvature of γ is $\varkappa_\gamma = \sqrt{1-\sigma_3^2}$ and the torsion of γ is $\tau_\gamma = \frac{1}{\sqrt{(1-\sigma_3^2)(\varkappa^2+\tau^2)}}\sigma_3'$.

b. *If $\sigma_3<-1$ or $1<\sigma_3$ and the Frenet frame of the principal normal indicatrix γ of a space curve α is $\left\{\mathbf{T}_\gamma, \mathbf{N}_\gamma, \mathbf{B}_\gamma\right\}$, then we have the Frenet-Serret formulae:*

$$D_{\mathbf{T}_\gamma}\mathbf{T}_\gamma = \varkappa_\gamma \mathbf{N}_\gamma,\ D_{\mathbf{T}_\gamma}\mathbf{N}_\gamma = -\varkappa_\gamma \mathbf{T}_\gamma + \tau_\gamma \mathbf{B}_\gamma,\ D_{\mathbf{T}_\gamma}\mathbf{B}_\gamma = \tau_\gamma \mathbf{N}_\gamma, \tag{3.18b}$$

where

$$\begin{aligned}
\mathbf{T}_\gamma &= \frac{1}{\sqrt{\varkappa^2+\tau^2}}\left(\varkappa T+\tau B\right),\\
\mathbf{N}_\gamma &= \frac{1}{\sqrt{\sigma_3^2-1}}\left(\frac{\tau\sigma_3}{\varkappa^2+\tau^2}T+N-\frac{\varkappa\sigma_3}{\varkappa^2+\tau^2}B\right),\\
\mathbf{B}_\gamma &= \frac{1}{\sqrt{(\sigma_3^2-1)(\varkappa^2+\tau^2)}}\left(-\tau T-\sigma_3\sqrt{\varkappa^2+\tau^2}N-\varkappa B\right).
\end{aligned} \tag{3.19b}$$

Moreover, the curvature of γ is $\varkappa_\gamma = \sqrt{\sigma_3^2-1}$ and the torsion of γ is

$$\tau_\gamma = \frac{1}{\sqrt{(\sigma_3^2-1)(\varkappa^2+\tau^2)}}\sigma_3'.$$

Proposition 3.4. *Let α be a unit speed spacelike curve with timelike principal normal vector in $\mathbb{E}_1^3$ with Frenet vectors T, N, B and curvatures*

$\varkappa$, τ. If the Frenet frame of the binormal indicatrix δ of the space curve α is $\{\mathbf{T}_\delta, \mathbf{N}_\delta, \mathbf{B}_\delta\}$, then we have the Frenet-Serret formulae:

$$D_{\mathbf{T}_\delta}\mathbf{T}_\delta = \varkappa_\delta \mathbf{N}_\delta,\ D_{\mathbf{T}_\delta}\mathbf{N}_\delta = \varkappa_\delta \mathbf{T}_\delta + \tau_\delta \mathbf{B}_\delta,\ D_{\mathbf{T}_\delta}\mathbf{B}_\delta = -\tau_\delta \mathbf{N}_\delta, \tag{3.20}$$

where

$$\begin{aligned} \mathbf{T}_\delta &= N, \\ \mathbf{N}_\delta &= \frac{1}{\sqrt{(\varkappa^2+\tau^2)}}\left(\varkappa T + \tau B\right), \\ \mathbf{B}_\delta &= \frac{1}{\sqrt{(\varkappa^2+\tau^2)}}\left(\tau T - \varkappa B\right) \end{aligned} \tag{3.21}$$

and $\varkappa_\delta = \frac{\sqrt{(\varkappa^2+\tau^2)}}{\tau}$ is the curvature of δ, $\tau_\delta = \frac{\varkappa^2\left(\frac{\tau}{\varkappa}\right)'}{\tau(\varkappa^2+\tau^2)}$ is the torsion of δ.

In the next three theorems, we obtain Frenet formulae of *tangent indicatrix β, principal normal indicatrix γ* and *binormal indicatrix δ of the spacelike curve α* with null principal normal vector in $\mathbb{E}_1^3$.

Theorem 3.3. *Let α be a unit speed spacelike curve with null principal normal vector in $\mathbb{E}_1^3$ with Frenet vectors T, N, B and curvatures $\varkappa$, τ. If the Frenet frame of the tangent indicatrix β of the space curve α is $\left\{\boldsymbol{T}_\beta, \boldsymbol{N}_\beta, \boldsymbol{B}_\beta\right\}$, then the tangent indicatrix β of α is a null straight line.*

Theorem 3.4. *Let α be a unit speed spacelike curve with null principal normal vector in $\mathbb{E}_1^3$ with Frenet vectors T, N, B and curvatures $\varkappa$, τ. If the Frenet frame of the principal normal indicatrix γ of the space curve α is $\left\{\boldsymbol{T}_\gamma, \boldsymbol{N}_\gamma, \boldsymbol{B}_\gamma\right\}$, then the principal normal indicatrix γ of the space curve α is a null straight line.*

Proof. Let α be a spacelike curve with null principal normal vector and the Frenet frame of the tangent indicatrix γ of a space curve α is $\{\boldsymbol{T}_\gamma,\boldsymbol{N}_\gamma,\boldsymbol{B}_\gamma\}$. In this case, we can easily show that

$$g\left(\boldsymbol{T}_\gamma\left(s_\gamma\right), \boldsymbol{T}_v\left(s_\gamma\right)\right) = 0. \tag{3.23}$$

Differentiating this equation$\boldsymbol{T}_\gamma\left(s_\gamma\right)\frac{ds_\gamma}{ds} = -\varkappa\left(s\right)T\left(s\right)+\tau\left(s\right)B\left(s\right)$ with respect to s and using Frenet formulas of γ null curve we found $\varkappa_\gamma \boldsymbol{N}_\gamma\left(s_\gamma\right)\frac{ds_\gamma}{ds}\cdot\frac{ds_\gamma}{ds} + \boldsymbol{T}_\gamma\left(s_\gamma\right)\frac{d^2 s_\gamma}{ds^2} = -\varkappa'\left(s\right)T\left(s\right) - \varkappa^2\left(s\right)N\left(s\right) + \tau'\left(s\right)B\left(s\right) + \tau^2\left(s\right)N\left(s\right)$ or

$$\varkappa_\gamma \boldsymbol{N}_\gamma\left(s_\gamma\right)\left(\frac{ds_\gamma}{ds}\right)^2 + \boldsymbol{T}_\gamma\left(s_\gamma\right)\frac{d^2 s_\gamma}{ds^2} = \tau^2\left(s\right)N\left(s\right). \tag{3.24}$$

From the last equation we have

$$g\left(\varkappa_\gamma \boldsymbol{N}_\gamma\left(s_\gamma\right)\left(\frac{ds_\gamma}{ds}\right)^2+\boldsymbol{T}_\gamma\left(s_\gamma\right)\frac{d^2s_\gamma}{ds^2},\right.$$
$$\left.\varkappa_\gamma \boldsymbol{N}_\gamma\left(s_\gamma\right)\left(\frac{ds_\gamma}{ds}\right)^2+\boldsymbol{T}_\gamma\left(s_\gamma\right)\frac{d^2s_\gamma}{ds^2}\right)=0,\ \varkappa_\gamma^2\left(\frac{ds_\gamma}{ds}\right)^4=0,$$

where $\frac{ds_\gamma}{ds}\neq 0$. So, we get $\varkappa_\gamma=0$, that is, γ is a null straight line. These complete the proof. □

Proposition 3.5. *Let α be a unit speed spacelike curve with null principal normal vector in $\mathbb{E}_1^3$ with Frenet vectors T,N,B and curvatures $\varkappa$, τ. If the Frenet frame of the binormal indicatrix δ of the space curve α is $\{\boldsymbol{T}_\delta,\boldsymbol{N}_\delta,\boldsymbol{B}_\delta\}$, then we have the Frenet-Serret formulae:*

$$D_{\boldsymbol{T}_\delta}\boldsymbol{T}_\delta=\varkappa_\delta\boldsymbol{N}_\delta,\ D_{\boldsymbol{T}_\delta}\boldsymbol{N}_\delta=-\varkappa_\delta\boldsymbol{T}_\delta+\tau_\delta\boldsymbol{B}_\delta,\ D_{\boldsymbol{T}_\delta}\boldsymbol{B}_\delta=\tau_\delta\boldsymbol{N}_\delta, \tag{3.25}$$

where

$$\begin{aligned}\boldsymbol{T}_\delta&=-T-\tau B,\\ \boldsymbol{N}_\delta&=-T-\frac{1}{\tau}N-\left(1+\frac{\tau'}{\tau}\right)B,\\ \boldsymbol{B}_\delta&=T+\left(1-\tau+\frac{\tau'}{\tau}\right)N+\frac{1}{\tau}B.\end{aligned}\tag{3.26}$$

Moreover, the curvature of δ is $\varkappa_\delta=\tau$ and the torsion of δ

$$\tau_\delta=-\left\{\begin{array}{c}\frac{\varkappa\tau+\varkappa\tau'}{\tau}+\left(1-\tau+\frac{\tau'}{\tau}\right)\left(\frac{\tau^3+\tau^2\tau'-\tau''\tau+(\tau')^2}{\tau^2}\right)\\ -\frac{1}{\tau}\left(1+\varkappa-\frac{\tau'}{\tau^2}\right)\end{array}\right\}.$$

4. Characterizations of spacelike slant helices in Minkowski 3-space

In Minkowski 3-space, slant helix and its properties was studied by Ali and Lopez in [2]. They proved the following theorems:

Theorem 4.1. *Let α be a spacelike slant helix with spacelike principal normal vector. In this case, the spherical image on pseudosphere S_1^2 of the tangent indicatrix β of α and the binormal indicatrix δ of α are a pseudospherical helices* (*see* [2]).

Theorem 4.2. *Let α be a spacelike curve with spacelike principal normal vector in $\mathbb{E}_1^3$. Then α is a slant helix if and only if function of geodesic curvature of the spherical image on pseudosphere S_1^2 of the principal normal indicatrix (N) of α is*

$$\sigma_N(s) = \left(\frac{\varkappa^2}{\left(\varepsilon\left(\varkappa^2-\tau^2\right)\right)^{3/2}}\left(\frac{\tau}{\varkappa}\right)'\right)(s) \tag{4.1}$$

constant. Where $\varepsilon = \left\{\begin{array}{cc} 1 & ,\varkappa^2-\tau^2\rangle 0, \\ -1 & ,\varkappa^2-\tau^2\langle 0 \end{array}\right\}$ *and everywhere $\varkappa^2-\tau^2$ does not vanish* (*see* [2]).

In this section, by using the above results, we obtain certain differential equations for a *spacelike curve* α with spacelike principal normal vector to be a slant helix in $\mathbb{E}_1^3$ according to the *tangent vector field* T_β, *principal normal vector field* N_β and *binormal vector field* B_β *of the curve* β which is the *tangent indicatrix* of the curve α.

Theorem 4.3. *Let α be a unit speed spacelike curve with spacelike principal normal vector and Frenet vectors T, N, B and non-zero curvatures $\varkappa$ and τ in $\mathbb{E}_1^3$. The curve α is a spacelike slant helix with spacelike principal normal vector if and only if tangent vector field T_β of the curve β satisfies one of the following equations:*

$$D_{T_\beta}^3 T_\beta - 3\frac{\varkappa_\beta'}{\varkappa_\beta}D_{T_\beta}^2 T_\beta - \left\{\frac{\varkappa_\beta''}{\varkappa_\beta} - 3\left(\frac{\varkappa_\beta'}{\varkappa_\beta}\right)^2 - \lambda_1\varkappa_\beta^2\right\}D_{T_\beta}T_\beta = 0 \quad \text{or} \tag{4.2}$$

$$D_{T_\beta}^3 T_\beta - 3\frac{\tau_\beta'}{\tau_\beta}D_{T_\beta}^2 T_\beta - \left\{\frac{\tau_\beta''}{\tau_\beta} - 3\left(\frac{\tau_\beta'}{\tau_\beta}\right)^2 + \lambda_2\tau_\beta^2\right\}D_{T_\beta}T_\beta = 0, \tag{4.3}$$

where $\lambda_1 \in \mathbb{R}$ $(\lambda_1 = 1-\mu^2)$, $\lambda_2 \in \mathbb{R}$ $(\lambda_2 = 1-\frac{1}{\mu^2})$, $\mu \in \mathbb{R}_0$ and $\varkappa_\beta$, τ_β are curvature and torsion of the curve β, respectively.

Proof. Suppose that α is a *spacelike slant helix with spacelike principal normal vector*. Thus the tangent indicatrix β of α is a spherical helix then we have $\frac{\varkappa_\beta}{\tau_\beta} = \mu$, $\mu \in \mathbb{R}_0$, where $\varkappa_\beta$ and τ_β are curvature functions of β. From (3.1), we have $D_{T_\beta}T_\beta = \varkappa_\beta N_\beta$. By differentiating two times of $D_{T_\beta}T_\beta = \varkappa_\beta N_\beta$ with respect to s_β, we get $D_{T_\beta}^3 T_\beta = -2\varkappa_\beta\varkappa_\beta' T_\beta - \varkappa_\beta^2 D_{T_\beta}T_\beta +$

$\varkappa''_\beta N_\beta + \varkappa'_\beta D_{T_\beta} N_\beta + 2\varkappa'_\beta \tau_\beta B_\beta + \varkappa_\beta \tau_\beta D_{T_\beta} B_\beta$. By using the Frenet equations in (3.1), we get (4.2) easily. Also using the relation $\varkappa_\beta = \mu\tau_\beta, \mu \in \mathbb{R}_0$ we get the equation (4.3).

Conversely let us assume that (4.2) holds. From (3.1), we have

$$B_\beta = \frac{1}{\tau_\beta} D_{T_\beta} N_\beta + \frac{\varkappa_\beta}{\tau_\beta} T_\beta. \tag{4.4}$$

Differentiating the last equality with respect to s_β, we have

$$\begin{aligned} D_{T_\beta} B_\beta = \frac{1}{\varkappa_\beta \tau_\beta} \Bigg\{ & D^3_{T_\beta} T_\beta - 3\frac{\varkappa'_\beta}{\varkappa_\beta} D^2_{T_\beta} T_\beta \\ & - \left[\frac{\varkappa''_\beta}{\varkappa_\beta} - 3\left(\frac{\varkappa'_\beta}{\varkappa_\beta}\right)^2 - \varkappa^2_\beta + \tau^2_\beta \right] D_{T_\beta} T_\beta \Bigg\} \\ & + \frac{1}{\varkappa^2_\beta}\left(\frac{\varkappa_\beta}{\tau_\beta}\right)' D^2_{T_\beta} T_\beta - \left(\frac{\tau_\beta}{\varkappa_\beta} + \frac{\varkappa'_\beta}{\varkappa^3_\beta}\left(\frac{\varkappa_\beta}{\tau_\beta}\right)' \right) D_{T_\beta} T_\beta + \left(\frac{\varkappa_\beta}{\tau_\beta}\right)' T_\beta. \end{aligned} \tag{4.5}$$

Using equations (3.1) and (4.2) in (4.5) we get $(\frac{\varkappa_\beta}{\tau_\beta})' = 0$ and $\frac{\varkappa_\beta}{\tau_\beta} = \sqrt{\frac{1}{1-\lambda_2}}$ (non-zero constant). Thus, from (3.1) and (3.2), we obtain $\sigma_n = \frac{\tau_\beta}{\varkappa_\beta} =$ constant. According to the Theorem 4.2., α is a *spacelike slant helix in* $\mathbb{E}^3_1$. The similar proof can be done by using the equation (4.3).

□

For the next two theorems, we omit their proofs since they can be done easily with similar way with above proof.

Theorem 4.4. *Let α be a unit speed spacelike curve with spacelike principal normal vector and Frenet vectors T, N, B and non-zero curvatures $\varkappa$ and τ in $\mathbb{E}^3_1$. The curve α is a spacelike slant helix with spacelike principal normal vector if and only if the principal normal vector field N_β of the curve β satisfies one of the following equations:*

$$D^2_{T_\beta} N_\beta - \frac{\varkappa'_\beta}{\varkappa_\beta} D_{T_\beta} N_\beta + \lambda_2 \varkappa^2_\beta N_\beta = 0 \ \textit{or} \tag{4.6}$$

$$D^2_{T_\beta} N_\beta - \frac{\tau'_\beta}{\tau_\beta} D_{T_\beta} N_\beta - \lambda_1 \tau^2_\beta N_\beta = 0, \tag{4.7}$$

where $\lambda_1 \in \mathbb{R}$ $(\lambda_1 = 1 - \mu^2)$, $\lambda_2 \in \mathbb{R}$ $(\lambda_2 = 1 - \frac{1}{\mu^2})$, $\mu \in \mathbb{R}_0$ and $\varkappa_\beta$, τ_β are curvature and torsion of the curve β, respectively.

Theorem 4.5. *Let α be a unit speed spacelike curve with spacelike principal normal vector and Frenet vectors T, N, B and non-zero curvatures $\varkappa$ and τ in $\mathbb{E}_1^3$. The curve α is a spacelike slant helix with spacelike principal normal vector if and only if the binormal vector field B_β of the tangent indicatrix β of the curve α satisfies one of the following equations:*

$$D^3_{T_\beta}B_\beta - 3\frac{\varkappa'_\beta}{\varkappa_\beta}D^2_{T_\beta}B_\beta - \left\{\frac{\varkappa''_\beta}{\varkappa_\beta} - 3\left(\frac{\varkappa'_\beta}{\varkappa_\beta}\right)^2 - \lambda_1\varkappa_\beta^2\right\}D_{T_\beta}B_\beta = 0 \quad or \tag{4.8}$$

$$D^3_{T_\beta}B_\beta - 3\frac{\tau'_\beta}{\tau_\beta}D^2_{T_\beta}B_\beta - \left\{\frac{\tau''_\beta}{\tau_\beta} - 3\left(\frac{\tau'_\beta}{\tau_\beta}\right)^2 + \lambda_2\tau_\beta^2\right\}D_{T_\beta}B_\beta = 0, \tag{4.9}$$

where $\lambda_1 \in \mathbb{R}$ $(\lambda_1 = 1-\mu^2)$, $\lambda_2 \in \mathbb{R}$ $(\lambda_2 = 1-\frac{1}{\mu^2})$, $\mu \in \mathbb{R}_0$ and $\varkappa_\beta$, τ_β are curvature and torsion of the curve β, respectively.

Remark 4.1. By similar way with above theorems, If we use eq. $(3.3a)$, $(3.3b)$, (3.5) and (3.14), than we can easily find that differential equations for principal normal indicatrix γ and principal binormal indicatrix δ of the curve α, respectively.

Also, Ali and Lopez in [2] proved the following theorems:

Theorem 4.6. *Let α be a spacelike curve with timelike principal normal vector. Then α is a slant helix if and only if the function of geodesic curvature of the spherical image on pseudohyperbolic space H_0^2 of the principal normal indicatrix (N) of α is*

$$\sigma_N(s) = \left(\frac{\varkappa^2}{(\varkappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\varkappa}\right)'\right)(s). \tag{4.19}$$

constant (*see* [2]).

Theorem 4.7. *Let α be a spacelike slant helix with timelike principal normal vector. In this case, the spherical image on pseudohyperbolic space H_0^2 of the tangent indicatrix β of α and the binormal indicatrix δ of α are a pseudohyperbolic helices* (*see* [2]).

In the next three theorems, we obtain the differential equations of a *spacelike curve α* with timelike principal normal vector to be a slant helix in $\mathbb{E}_1^3$ according to the *tangent vector field $\mathbf{T}_\beta$, principal normal vector field $\mathbf{N}_\beta$* and *binormal vector field $\mathbf{B}_\beta$ of the curve β* which is the *tangent indicatrix* of the curve α. We omit their proofs since they are similar with the proof of Theorem 4.3.

Theorem 4.8. *Let α be a unit speed spacelike curve with timelike principal normal vector curve with Frenet vectors T, N, B and with non-zero curvatures $\varkappa$ and τ in $\mathbb{E}_1^3$. The curve α is a spacelike slant helix with timelike principal normal vector if and only if tangent vector field $\mathbf{T}_\beta$ of the curve β satisfies one of the following equations:*

$$D^3_{\mathbf{T}_\beta}\mathbf{T}_\beta - 3\frac{\varkappa'_\beta}{\varkappa_\beta}D^2_{\mathbf{T}_\beta}\mathbf{T}_\beta - \left\{\frac{\varkappa''_\beta}{\varkappa_\beta} - 3\left(\frac{\varkappa'_\beta}{\varkappa_\beta}\right)^2 + \lambda_1\varkappa_\beta^2\right\}D_{\mathbf{T}_\beta}\mathbf{T}_\beta = 0 \text{ or} \tag{4.20}$$

$$D^3_{\mathbf{T}_\beta}\mathbf{T}_\beta - 3\frac{\tau'_\beta}{\tau_\beta}D^2_{\mathbf{T}_\beta}\mathbf{T}_\beta - \left\{\frac{\tau''_\beta}{\tau_\beta} - 3\left(\frac{\tau'_\beta}{\tau_\beta}\right)^2 - \lambda_2\tau_\beta^2\right\}D_{\mathbf{T}_\beta}\mathbf{T}_\beta = 0, \tag{4.21}$$

where λ_1 and $\lambda_2 \in \mathbb{R}$ ($\lambda_1 = 1-\mu^2$, $\lambda_2 = 1-\frac{1}{\mu^2}$, $\mu \in \mathbb{R}_0$) and $\varkappa_\beta$ and τ_β are curvature and torsion of the curve β.

Theorem 4.9. *Let α be a unit speed spacelike curve with timelike principal normal vector curve with Frenet vectors T, N, B and with non-zero curvatures $\varkappa$ and τ in $\mathbb{E}_1^3$. The curve α is a spacelike slant helix with timelike principal normal vector if and only if the principal normal vector field $\mathbf{N}_\beta$ of the curve β satisfies one of the following equations:*

$$D^2_{\mathbf{T}_\beta}\mathbf{N}_\beta - \frac{\varkappa'_\beta}{\varkappa_\beta}D_{\mathbf{T}_\beta}\mathbf{N}_\beta - \lambda_2\varkappa_\beta^2\mathbf{N}_\beta = 0 \text{ or} \tag{4.22}$$

$$D^2_{\mathbf{T}_\beta}\mathbf{N}_\beta - \frac{\tau'_\beta}{\tau_\beta}D_{\mathbf{T}_\beta}\mathbf{N}_\beta + \lambda_1\tau_\beta^2\mathbf{N}_\beta = 0, \tag{4.23}$$

where λ_1 and $\lambda_2 \in \mathbb{R}$ ($\lambda_1 = 1-\mu^2$, $\lambda_2 = 1-\frac{1}{\mu^2}$, $\mu \in \mathbb{R}_0$) and $\varkappa_\beta$ and τ_β are curvature and torsion of the curve β.

Theorem 4.10. *Let α be a unit speed spacelike curve with timelike principal normal vector curve with Frenet vectors T, N, B and with non-zero curvatures $\varkappa$ and τ in $\mathbb{E}_1^3$. The curve α is a spacelike slant helix with timelike principal normal vector if and only if the binormal vector field $\mathbf{B}_\beta$ of the tangent indicatrix β of the curve α satisfies one of the following equations:*

$$D^3_{\mathbf{T}_\beta}\mathbf{B}_\beta - 3\frac{\varkappa'_\beta}{\varkappa_\beta}D^2_{\mathbf{T}_\beta}\mathbf{B}_\beta - \left\{\frac{\varkappa''_\beta}{\varkappa_\beta} - 3\left(\frac{\varkappa'_\beta}{\varkappa_\beta}\right)^2 + \lambda_1\varkappa_\beta^2\right\}D_{\mathbf{T}_\beta}\mathbf{B}_\beta = 0 \text{ or} \tag{4.24}$$

$$D^3_{\mathbf{T}_\beta}\mathbf{B}_\beta - 3\frac{\tau'_\beta}{\tau_\beta}D^2_{\mathbf{T}_\beta}\mathbf{B}_\beta - \left\{\frac{\tau''_\beta}{\tau_\beta} - 3\left(\frac{\tau'_\beta}{\tau_\beta}\right)^2 - \lambda_2\tau_\beta^2\right\}D_{\mathbf{T}_\beta}\mathbf{B}_\beta = 0, \tag{4.25}$$

where λ_1 *and* $\lambda_2 \in \mathbb{R}$ $(\lambda_1 = 1-\mu^2,\ \lambda_2 = 1-\frac{1}{\mu^2},\ \mu \in \mathbb{R}_0)$ *and* $\varkappa_\beta$ *and* τ_β *are curvature and torsion of the curve* β.

Remark 4.2. By similar way with above theorem, If we use eq. (3.18a), (3.18b) and (3.20), than we can easily find that differential equations for principal normal indicatrix γ and principal binormal indicatrix δ of the curve α, respectively.

5. Example

In this section we give an example of spacelike slant helix in Minkowski 3-space and draw its pictures and its tangent, binormal indicatrices by using **Mathematica**.

We consider a spacelike slant helix α is defined by $\alpha_1(s) = \frac{15}{136}\sin(17s)$, $\alpha_2(s) = \frac{9}{400}\sin(25s) + \frac{25}{144}\sin(9s)$, $\alpha_3(s) = -\frac{9}{400}\cos(25s) + \frac{25}{144}\cos(9s)$.

Since its position vector is a spacelike vector, the curve α lie on the circular hyperboloid of one sheet (pseudosphere) with the equation $-\frac{x^2}{\left(\frac{2}{15}\right)^2} + \frac{y^2}{\left(\frac{34}{225}\right)^2} + \frac{z^2}{\left(\frac{34}{225}\right)^2} = 1$. From [20], the curve α is closed. The picture of the curve α is rendered in Figure 1.

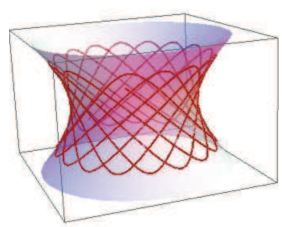

Figure 1: Spacelike slant helix α.

The parametrization of the tangent indicatrix $\beta = (\beta_1, \beta_2, \beta_3)$ of the spacelike slant helix α is $\beta_1(s) = \frac{30}{16}\cos(17s)$, $\beta_2(s) = \frac{9}{16}\cos(25s)+\frac{25}{16}\cos(9s)$, $\beta_3(s) = \frac{9}{16}\sin(25s) - \frac{25}{16}\sin(9s)$. The picture of the helix β is rendered in Figure 2.

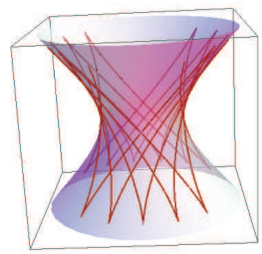

Figure 2: Tangent indicatrix $\beta = (\beta_1, \beta_2, \beta_3)$ of the spacelike slant helix α.

The parametrization of the binormal indicatrix $\delta = (\delta_1, \delta_2, \delta_3)$ of the spacelike slant helix α is $\delta_1(s) = \frac{30}{16}\sin(17s)$, $\delta_2(s) = \frac{9}{16}\sin(25s) + \frac{25}{16}\sin(9s)$, $\delta_3(s) = -\frac{9}{16}\cos(25s) + \frac{25}{16}\cos(9s)$.

The picture of the helix δ is rendered in Figure 3.

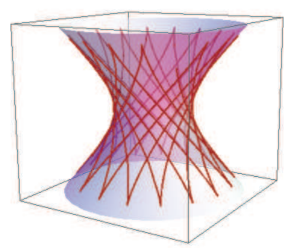

Figure 3: Binormal indicatrix $\delta = (\delta_1, \delta_2, \delta_3)$ of the spacelike slant helix α.

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