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A certain generalized Pochhammer symbol and its applications to hypergeometric functions

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ABSTRACT

In this article, we first introduce an interesting new generalization of the familiar Pochhammer symbol by means of a certain one-parameter family of generalized gamma functions. With the help of this new generalized Pochhammer symbol, we then introduce an extension of the generalized hypergeometric function ${}_rF_s$ with r numerator and s denominator parameters. Finally, we present a systematic study of the various fundamental properties of the class of the generalized hypergeometric functions introduced here.

1. Introduction and preliminaries

In many areas of applied mathematics, various types of special functions become essential tools for scientists and engineers. The continuous development of mathematical physics, probability theory and other areas has led to new classes of special functions and their extensions and generalizations (see, for details, [22] and the references cited therein; see also [20,21,23,24]).

The (Euler's) gamma function $\Gamma(z)$ is one of the most fundamental special functions, because of its important rôle in various fields in the mathematical, physical, engineering and statistical sciences. Various generalizations of the gamma function can be found in the literature [6,9,11,12,26]. In particular, Chaudhry and Zubair [6] (see also [5]) introduced an interesting generalization of the gamma function $\Gamma(z)$ as follows:

$$\Gamma_p(z) := \begin{cases} \int_0^\infty t^{z-1}\, \exp\left(-t-\frac{p}{t}\right)dt & \left(\Re(p)>0;\ z\in\mathbb{C}\right) \\ \Gamma(z) & \left(p=0;\ \Re(z)>0\right). \end{cases} \tag{1}$$

The classical Pochhammer symbol $(\lambda)_\nu$ $(\lambda,\nu\in\mathbb{C})$ is defined, in terms of the gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu=0;\ \lambda\in\mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu=n\in\mathbb{N};\ \lambda\in\mathbb{C}), \end{cases} \tag{2}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists (see, for details, [25, p. 21 *et seq.*]), $\mathbb{N}$ being (as usual) the set of positive integers (see also [10,19]).

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Recently, in terms of the *incomplete gamma functions* $\gamma(s,x)$ and $\Gamma(s,x)$ defined, respectively, by

$$\gamma(\kappa,x) := \int_0^x t^{\kappa-1} e^{-t}\, dt \quad \left(\Re(\kappa) > 0;\ x \geqq 0\right) \tag{3}$$

and

$$\Gamma(\kappa,x) := \int_x^\infty t^{\kappa-1} e^{-t}\, dt \quad \left(x \geqq 0;\ \Re(\kappa) > 0 \quad \text{when} \quad x = 0\right), \tag{4}$$

which are known to satisfy the following decomposition formula:

$$\gamma(\kappa,x) + \Gamma(\kappa,x) = \Gamma(\kappa) \quad \left(\Re(\kappa) > 0\right), \tag{5}$$

the so-called *incomplete* Pochhammer symbols

$$(\lambda;x)_\nu \quad \text{and} \quad [\lambda;x]_\nu \quad \left(\lambda, \nu \in \mathbb{C};\ x \geqq 0\right)$$

were defined by Srivastava et al. [22] as follows:

$$(\lambda;x)_\nu := \frac{\gamma(\lambda+\nu,x)}{\Gamma(\lambda)} \quad \left(\lambda, \nu \in \mathbb{C};\ x \geqq 0\right) \tag{6}$$

and

$$[\lambda;x]_\nu := \frac{\Gamma(\lambda+\nu,x)}{\Gamma(\lambda)} \quad \left(\lambda, \nu \in \mathbb{C};\ x \geqq 0\right). \tag{7}$$

As a matter of fact, Srivastava et al. [22] and other authors (see, for example, [3,27–29]) made use of the incomplete Pochhammer symbols defined by (6) and (7) with a view to investigating the various properties of the corresponding incomplete hypergeometric functions and incomplete hypergeometric polynomials (see also several recent works on the subject including, for example, [4,15–17]).

Motivated essentially by these aforecited investigations, here we first introduce a generalization of the Pochhammer symbol in (2) by using the generalized gamma function $\Gamma_p(z)$ defined by (1). We then derive its useful properties and make use of it to define and investigate the corresponding extension of the generalized hypergeometric function ${}_rF_s$ with r numerator and s denominator parameters. Various other related functions and their integral representations are also considered. Moreover, for *further* reading by the interested reader, we have chosen to include a number of useful references which would provide the needed details about such topics as (for example) the gamma and related functions, the Bessel and the modified Bessel functions, the Gauss and Kummer hypergeometric functions, and the generalized hypergeometric functions and polynomials, each of which is dealt with in our investigation here.

2. A generalized Pochhammer symbol

Our proposed generalization of the Pochhammer symbol $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is defined by

$$(\lambda;p)_\nu := \begin{cases} \dfrac{\Gamma_p(\lambda+\nu)}{\Gamma(\lambda)} & \left(\Re(p) > 0;\ \lambda, \nu \in \mathbb{C}\right) \\ (\lambda)_\nu & (p = 0;\ \lambda, \nu \in \mathbb{C}), \end{cases} \tag{8}$$

which leads us easily to the following integral representation for the generalized Pochhammer symbol $(\lambda;p)_\nu$:

$$(\lambda;p)_\nu = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda+\nu-1} \exp\left(-t - \frac{p}{t}\right) dt \tag{9}$$

$$\left(\Re(p) > 0;\ \Re(\lambda+\nu) > 0 \quad \text{when} \quad p = 0\right).$$

Since the generalized gamma function $\Gamma_p(z)$ is related to the modified Bessel function of the third kind (or the Macdonald function) $K_\mu(z)$ by (see [6]; see also [1,2], [7, pp. 265 *et seq.*], [13,14,30] for details)

$$\Gamma_p(\lambda) = 2p^{\frac{\lambda}{2}} K_\lambda(2\sqrt{p}) \quad \left(\Re(p) > 0\right),$$

the generalized Pochhammer symbol $(\lambda;p)_\nu$ can also be written as follows:

$$(\lambda;p)_\nu = \frac{2p^{\frac{\lambda+\nu}{2}}}{\Gamma(\lambda)} K_{\lambda+\nu}(2\sqrt{p}) \quad \left(\Re(p) > 0\right). \tag{10}$$

Each of the following closed-form representations can be deduced from the representation (10):

$$\left(\frac{1}{2};p\right)_n = p^{\frac{n}{2}} e^{-2\sqrt{p}} \sum_{m=0}^{n} \frac{(4\sqrt{p})^{-m}}{m!} \frac{(n+m)!}{(n-m)!} \quad \left(\Re(p) > 0;\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\right) \tag{11}$$

and

$$(1;p)_\nu = 2p^{\frac{\nu+1}{2}} K_{\nu+1}(2\sqrt{p}) \quad \left(\Re(p) > 0\right). \tag{12}$$

In the particular case when $n = 0$ and $\nu = 0$, we find from (11) and (12) that

$$\left(\frac{1}{2};p\right)_0 = e^{-2\sqrt{p}} \quad \text{and} \quad (1;p)_0 = 2\sqrt{p}\, K_1(2\sqrt{p}),$$

respectively (see, for details, [1,2], [7, pp. 265 *et seq.*] and [13,14,30]).

Theorem 1. *Let $\lambda, \mu, \nu \in \mathbb{C}$. Then*

$$(\lambda;p)_{\nu+\mu} = (\lambda)_\nu (\lambda+\nu;p)_\mu. \tag{13}$$

Proof. From the definitions (2) and (8) of the classical and the generalized Pochhammer symbols, we find that

$$(\lambda;p)_{\nu+\mu} = \frac{\Gamma_p(\lambda+\nu+\mu)}{\Gamma(\lambda)} = \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda+\nu)} \frac{\Gamma_p(\lambda+\nu+\mu)}{\Gamma(\lambda)} = (\lambda)_\nu (\lambda+\nu;p)_\mu. \qquad \square$$

Remark 1. It readily follows from the assertion (13) of Theorem 1 that

$$(\lambda;p)_{n+m} = (\lambda)_n (\lambda+n;p)_m \quad (\lambda \in \mathbb{C};\ m,n \in \mathbb{N}_0). \tag{14}$$

Thus, by applying the well-known properties of the classical Pochhammer symbol $(\lambda)_n$ in (14) (see, for example, [18,19]), it is fairly straightforward to derive the corresponding properties of the generalized Pochhammer symbol $(\lambda;p)_n$ as follows.

Corollary 1. *Let $k, \ell, m, n \in \mathbb{N}_0$ and $N \in \mathbb{N}$. Then*

$$(\lambda;p)_{n+m+\ell} = (\lambda)_n (\lambda+n)_m (\lambda+n+m;p)_\ell,$$

$$(\lambda;p)_{n-m+\ell} = \frac{(-1)^m (\lambda)_n}{(1-\lambda-n)_m} (\lambda+n-m;p)_\ell,$$

$$(\lambda;p)_{2m+\ell} = 2^{2m} \left(\frac{\lambda}{2}\right)_m \left(\frac{\lambda+1}{2}\right)_m (\lambda+2m;p)_\ell,$$

$$(\lambda;p)_{Nm+\ell} = N^{Nm} \left(\frac{\lambda}{N}\right)_m \left(\frac{\lambda+1}{N}\right)_m \cdots \left(\frac{\lambda+N-1}{N}\right)_m (\lambda+Nm;p)_\ell,$$

$$(\lambda+n;p)_{n+\ell} = (\lambda+n)_n (\lambda+2n;p)_\ell = \frac{(\lambda)_{2n}}{(\lambda)_n} (\lambda+2n;p)_\ell,$$

$$(\lambda+m;p)_{n+\ell} = \frac{(\lambda)_n (\lambda+n)_m}{(\lambda)_m} (\lambda+m+n;p)_\ell,$$

$$(\lambda+km;p)_{kn+\ell} = \frac{(\lambda)_{km+kn}}{(\lambda)_{km}} (\lambda+km+kn;p)_\ell,$$

$$(\lambda-n;p)_{n+\ell} = (-1)^n (1-\lambda)_n (\lambda;p)_\ell,$$

$$(\lambda-m;p)_{n+\ell} = \frac{(1-\lambda)_m (\lambda)_n}{(1-\lambda-n)_m} (\lambda+n-m;p)_\ell,$$

$$(\lambda-km;p)_{kn+\ell} = (-1)^{km} (\lambda)_{kn-km} (1-\lambda)_{km} (\lambda+kn-km;p)_\ell,$$

$$(\lambda+m;p)_{n-m+\ell}=\frac{(\lambda)_n}{(\lambda)_m}\,(\lambda+n;p)_\ell,$$

$$(\lambda-m;p)_{n-m+\ell}=\frac{(-1)^m(\lambda)_n(1-\lambda)_m}{(1-\lambda-n)_{2m}}\,(\lambda+n-2m;p)_\ell$$

and

$$(-\lambda;p)_{n+\ell}=(-1)^n(\lambda-n+1)_n(-\lambda+n;p)_\ell.$$

3. Extension and generalization of the hypergeometric function

In terms of the generalized Pochhammer symbol $(\lambda;p)_n$ $(n\in\mathbb{N}_0)$, an extension of the generalized hypergeometric function ${}_rF_s$ of r numerator parameters $a_1,\ldots,a_r$ and s denominator parameters $b_1,\ldots,b_s$ can now be given as follows:

$${}_rF_s\left[\begin{matrix}(a_1,p), & a_2,\ldots,a_r; & \\ & & z\\ & b_1,\ldots,b_s; & \end{matrix}\right]:=\sum_{n=0}^{\infty}\frac{(a_1;p)_n(a_2)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n}\,\frac{z^n}{n!},\tag{15}$$

provided that the series on the right-hand side converges, it being understood (as usual) that

$$a_j\in\mathbb{C}\quad(j=1,\ldots,r)\quad\text{and}\quad b_j\in\mathbb{C}\setminus\mathbb{Z}_0^-$$

$$(j=1,\ldots,s;\ \mathbb{Z}_0^-:=\{0,-1,-2,\ldots\}).$$

In particular, the corresponding extensions of the confluent hypergeometric function ${}_1F_1$ and the Gauss hypergeometric function ${}_2F_1$ are given by

$${}_1F_1[(a,p);c;z]:=\sum_{n=0}^{\infty}\frac{(a;p)_n}{(c)_n}\,\frac{z^n}{n!}\tag{16}$$

and

$${}_2F_1[(a,p),b;c;z]:=\sum_{n=0}^{\infty}\frac{(a;p)_n(b)_n}{(c)_n}\,\frac{z^n}{n!},\tag{17}$$

respectively.

Theorem 2. *The following integral representation holds true:*

$${}_rF_s\left[\begin{matrix}(a_1,p), & a_2,\ldots,a_r; & \\ & & z\\ & b_1,\ldots,b_s; & \end{matrix}\right]=\frac{1}{\Gamma(a_1)}\int_0^\infty t^{a_1-1}\,\exp\left(-t-\frac{p}{t}\right)\cdot{}_{r-1}F_s\left[\begin{matrix}a_2,\ldots,a_r; & \\ & zt\\ b_1,\ldots,b_s; & \end{matrix}\right]dt\tag{18}$$

$$\Big(\Re(p)>0;\ \Re(a_1)>0\quad\text{when}\quad p=0\Big).$$

Proof. Replacing the generalized Pochhammer symbol $(a_1,p)_n$ in the definition (15) by its integral representation given by (9), we get the desired result (18). □

Theorem 3. *The following Beta-type integral representation holds true:*

$${}_rF_s\left[\begin{matrix}(a_1,p),\ a_2,\ldots,a_{r-1},b; & \\ & z\\ b_1,\ldots,b_{s-1},c; & \end{matrix}\right]=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\cdot{}_{r-1}F_{s-1}\left[\begin{matrix}(a_1,p),\ a_2,\ldots,a_{r-1}; & \\ & zt\\ b_1,\ldots,b_{s-1}; & \end{matrix}\right]dt$$
$$\Big(\Re(c)>\Re(b)>0;\ \Re(p)\geqq 0\Big).\tag{19}$$

Proof. The integral representation (19) involves the classical Beta function $B(\alpha,\beta)$ defined by

$$B(\alpha,\beta):=\begin{cases}\int_0^1 t^{\alpha-1}\,(1-t)^{\beta-1}\,dt & \Big(\min\{\Re(\alpha),\Re(\beta)\}>0\Big)\\[4pt] \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\beta\in\mathbb{C}\setminus\mathbb{Z}_0^-).\end{cases}\tag{20}$$

It can indeed be easily obtained by using the following elementary identity:

$$\frac{(b)_n}{(c)_n}=\frac{B(b+n,c-b)}{B(b,c-b)}=\frac{1}{B(b,c-b)}\int_0^1 t^{b+n-1}(1-t)^{c-b-1}\,dt \quad \big(\Re(c)>\Re(b)>0;\ n\in\mathbb{N}_0\big). \qquad \square$$

Theorem 4. *The following derivative formula holds true:*

$$\frac{d^n}{dz^n}\left\{{}_rF_s\left[\begin{matrix}(a_1,p), & a_2,\ldots,a_r; \\ & b_1,\ldots,b_s;\end{matrix}\ z\right]\right\}=\frac{(a_1)_n(a_2)_n\cdots(a_r)_n}{(b_1)_n(b_2)_n\cdots(b_s)_n}\cdot {}_rF_s\left[\begin{matrix}(a_1+n,p), & a_2+n,\ldots,a_r+n; \\ & b_1+n,\ldots,b_s+n;\end{matrix}\ z\right]$$
$$\big(n\in\mathbb{N}_0;\ \Re(p)\geqq 0\big). \tag{21}$$

Proof. The result (21) is obviously valid in the trivial case when $n=0$. For $n=1$, by using the series representation (15) of ${}_rF_s$, we find from (21) that

$$\frac{d}{dz}\left\{\sum_{n=0}^{\infty}\frac{(a_1;p)_n(a_2)_n\cdots(a_r)_n}{(b_1)_n(b_2)_n\cdots(b_s)_n}\frac{z^n}{n!}\right\}=\sum_{n=1}^{\infty}\frac{(a_1;p)_n(a_2)_n\cdots(a_r)_n}{(b_1)_n(b_2)_n\cdots(b_s)_n}\frac{z^{n-1}}{(n-1)!}=\sum_{n=0}^{\infty}\frac{(a_1;p)_{n+1}(a_2)_{n+1}\cdots(a_r)_{n+1}}{(b_1)_{n+1}(b_2)_{n+1}\cdots(b_s)_{n+1}}\frac{z^n}{n!},$$

which, in view of (13), yields

$$\frac{d}{dz}\left\{{}_rF_s\left[\begin{matrix}(a_1,p), & a_2,\ldots,a_r; \\ & b_1,\ldots,b_s;\end{matrix}\ z\right]\right\}=\frac{a_1a_2\cdots a_r}{b_1b_2\cdots b_s}\cdot {}_rF_s\left[\begin{matrix}(a_1+1,p), & a_2+1,\ldots,a_r+1; \\ & b_1+1,\ldots,b_s+1;\end{matrix}\ z\right].$$

The general result (21) can now be easily derived by using the principle of mathematical induction on $n\in\mathbb{N}_0$. $\square$

We state the following results without proof. Each of these results would follow readily from the corresponding known result involving the generalized hypergeometric functions, which are asserted by Theorems 2–4.

Corollary 2. *Each of the following integral representations holds true:*

$${}_1F_1[(a,p);b;z]=\frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}\ \exp\left(-t-\frac{p}{t}\right){}_0F_1(\text{—};b;zt)\,dt, \tag{22}$$

$${}_2F_1[(a,p),b;c;z]=\frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}\ \exp\left(-t-\frac{p}{t}\right){}_1F_1(b;c;zt)\,dt \tag{23}$$

and

$${}_2F_1[(a,p),b;c;z]=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\,{}_1F_0[(a,p);\text{—};zt]\,dt, \tag{24}$$

provided that the integrals involved are convergent.

Corollary 3. *The following derivative formulas hold true:*

$$\frac{d^n}{dz^n}\{{}_1F_1[(a,p);c;z]\}=\frac{(a)_n}{(c)_n}\,{}_1F_1[(a+n,p);c+n;z] \tag{25}$$

and

$$\frac{d^n}{dz^n}\{{}_2F_1[(a,p),b;c;z]\}=\frac{(a)_n(b)_n}{(c)_n}\,{}_2F_1[(a+n,p),b+n;c+n;z]. \tag{26}$$

Remark 2. The Bessel function $J_\nu(z)$ and the *modified* Bessel function $I_\nu(z)$ are expressible as hypergeometric functions as follows (see, for example, [19]; see also [1,2], [7, pp. 265 *et seq.*] and [13,14,30]):

$$J_\nu(z)=\frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\ {}_0F_1\left(\text{—};\nu+1;-\frac{1}{4}z^2\right)\quad \left(\nu\in\mathbb{C}\setminus\mathbb{Z}^-\ \ (\mathbb{Z}^-:=\{-1,-2,-3,\cdots\})\right) \tag{27}$$

and

$$I_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)}\ {}_0F_1\left(\text{—};\nu+1;\frac{1}{4}z^2\right) \quad (\nu \in \mathbb{C} \setminus \mathbb{Z}^-). \tag{28}$$

Moreover, for the *incomplete gamma function* $\gamma(s,x)$ defined by (3), we know that

$${}_1F_1(\kappa;\kappa+1;-x) = \kappa x^{-\kappa}\gamma(\kappa,x). \tag{29}$$

Thus, by applying the relationships (27) to (29) in Corollary 2, we can deduce Corollaries 4 and 5 below.

Corollary 4. *Each of the following integral representations holds true:*

$${}_1F_1[(a,p);b+1;-z] = \frac{\Gamma(b+1)}{\Gamma(a)}\ z^{-\frac{b}{2}} \cdot \int_0^\infty t^{a-\frac{b}{2}-1}\ \exp\left(-t-\frac{p}{t}\right)J_b\left(2\sqrt{zt}\right)\,dt \tag{30}$$

and

$${}_1F_1[(a,p);b+1;z] = \frac{\Gamma(b+1)}{\Gamma(a)}\ z^{-\frac{b}{2}} \cdot \int_0^\infty t^{a-\frac{b}{2}-1}\ \exp\left(-t-\frac{p}{t}\right)I_b\left(2\sqrt{zt}\right)\,dt, \tag{31}$$

provided that the integrals involved are convergent.

Corollary 5. *The following integral representations holds true:*

$${}_2F_1[(a,p),b;b+1;-z] = \frac{bz^{-b}}{\Gamma(a)}\int_0^\infty t^{a-b-1}\ \exp\left(-t-\frac{p}{t}\right)\gamma(b,zt)\,dt, \tag{32}$$

provided that the integral involved is convergent.

4. Families of generalized hypergeometric generating functions

In order to derive several families of generalized hypergeometric generating functions, we find it to be convenient to abbreviate by $\Delta(N,\lambda)$ the following array of N parameters:

$$\frac{\lambda}{N},\frac{\lambda+1}{N},\ldots,\frac{\lambda+N-1}{N} \quad (\lambda\in\mathbb{C};\ N\in\mathbb{N}),$$

the array $\Delta(N;\lambda)$ being assumed to be empty when $N=0$ (see, for details, [2], [7, Chapter 4], [14] and [25, Chapter 2]).

Theorem 5. *The following generating function holds true:*

$$\sum_{n=0}^\infty \frac{(\lambda)_n}{n!}\ {}_{r+N}F_s\left[\begin{matrix}\Delta(N;\lambda+n), & (a_1,p), & a_2,\cdots,a_r; \\ & & b_1,\ldots,b_s;\end{matrix}\ z\right]t^n = (1-t)^{-\lambda}\ {}_{r+N}F_s\left[\begin{matrix}\Delta(N;\lambda), & (a_1,p), & a_2,\cdots,a_r; \\ & & b_1,\ldots,b_s;\end{matrix}\ \frac{z}{(1-t)^N}\right]$$
$$(|t|<1;\ \lambda\in\mathbb{C};\ N\in\mathbb{N}), \tag{33}$$

provided that each member of (33) *exists.*

Proof. Our derivation of the generating function (33) is based upon the definition (15) and the following elementary identity:

$$\sum_{n=0}^\infty \frac{(\lambda)_n}{n!}\ z^n = (1-z)^{-\lambda} \quad (|z|<1;\ \lambda\in\mathbb{C}). \tag{34}$$

The details involved are being omitted here. □

Remark 3. Whenever any of the numerator parameters $a_2,\ldots,a_r$ is a *nonpositive* integer, the series in the definition (15) would terminate and define a generalized hypergeometric polynomial. Theorem 6 below provides a general family of generating functions for such classes of hypergeometric polynomials.

Theorem 6. *Each of the following generating functions holds true:*

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\ {}_{r+N}F_s\left[\begin{matrix}\Delta(N;-n),(a_1,p), & a_2,\ldots,a_r; \\ & b_1,\ldots,b_s;\end{matrix}\ z\right]t^n=(1-t)^{-\lambda}\ {}_{r+N}F_s\left[\begin{matrix}\Delta(N;\lambda),(a_1,p), & a_2,\ldots,a_r; \\ & b_1,\ldots,b_s;\end{matrix}\ z\left(-\frac{t}{1-t}\right)^N\right]$$
$$(|t|<1;\ \lambda\in\mathbb{C};\ N\in\mathbb{N}), \tag{35}$$

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\ {}_{r+2N}F_s\left[\begin{matrix}\Delta(N;-n),\Delta(N;\lambda+n),(a_1,p), & a_2,\ldots,a_r; \\ & b_1,\ldots,b_s;\end{matrix}\ z\right]t^n$$
$$=(1-t)^{-\lambda}\ {}_{r+2N}F_s\left[\begin{matrix}\Delta(2N;\lambda),(a_1,p), & a_2,\ldots,a_r; \\ & b_1,\ldots,b_s;\end{matrix}\ z\left(-\frac{4t}{(1-t)^2}\right)^N\right]\quad(|t|<1;\ \lambda\in\mathbb{C};\ N\in\mathbb{N}) \tag{36}$$

and

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\ {}_{r+N}F_{s+N}\left[\begin{matrix}\Delta(N;-n),(a_1,p), & a_2,\ldots,a_r; \\ \Delta(N;1-\lambda-n), & b_1,\ldots,b_s;\end{matrix}\ z\right]t^n=(1-t)^{-\lambda}\ {}_rF_s\left[\begin{matrix}(a_1,p), & a_2,\ldots,a_r; \\ & b_1,\ldots,b_s;\end{matrix}\ zt^N\right]$$
$$(|t|<1;\ \lambda\in\mathbb{C};\ N\in\mathbb{N}), \tag{37}$$

provided that each member of the assertions (35) to (37) exists.

Proof. The proof of Theorem 6 is much akin to that of Theorem 5. □

Finally, we choose to state the *simplest* consequences of the generating functions (33) and 35, 36, 37 when $N=1$ as Corollary 6 below.

Corollary 6. *Each of the following generating functions holds true:*

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\ {}_{r+1}F_s\left[\begin{matrix}\lambda+n,(a_1,p), & a_2,\ldots,a_r; \\ & b_1,\ldots,b_s;\end{matrix}\ z\right]t^n=(1-t)^{-\lambda}\ {}_{r+1}F_s\left[\begin{matrix}\lambda,(a_1,p), & a_2,\ldots,a_r; \\ & b_1,\ldots,b_s;\end{matrix}\ \frac{z}{1-t}\right]\quad(|t|<1;\ \lambda\in\mathbb{C}), \tag{38}$$

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\ {}_{r+1}F_s\left[\begin{matrix}-n,(a_1,p), & a_2,\ldots,a_r; \\ & b_1,\ldots,b_s;\end{matrix}\ z\right]t^n=(1-t)^{-\lambda}\ {}_{r+1}F_s\left[\begin{matrix}\lambda,(a_1,p), & a_2,\ldots,a_r; \\ & b_1,\ldots,b_s;\end{matrix}\ -\frac{zt}{1-t}\right]\quad(|t|<1;\ \lambda\in\mathbb{C}), \tag{39}$$

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\ {}_{r+2}F_s\left[\begin{matrix}-n,\lambda+n,(a_1,p), & a_2,\ldots,a_r; \\ & b_1,\ldots,b_s;\end{matrix}\ z\right]t^n=(1-t)^{-\lambda}\ {}_{r+2}F_s\left[\begin{matrix}\Delta(2;\lambda),(a_1,p), & a_2,\ldots,a_r; \\ & b_1,\ldots,b_s;\end{matrix}\ -\frac{4zt}{(1-t)^2}\right]$$
$$(|t|<1;\ \lambda\in\mathbb{C}) \tag{40}$$

and

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\ {}_{r+1}F_{s+1}\left[\begin{matrix}-n,(a_1,p), & a_2,\ldots,a_r; \\ 1-\lambda-n, & b_1,\ldots,b_s;\end{matrix}\ z\right]t^n=(1-t)^{-\lambda}\ {}_rF_s\left[\begin{matrix}(a_1,p), & a_2,\ldots,a_r; \\ & b_1,\ldots,b_s;\end{matrix}\ zt\right]\quad(|t|<1;\ \lambda\in\mathbb{C}), \tag{41}$$

provided that each member of the generating functions (38) to (41) exists.

Finally, since

$$\lim_{|\lambda|\to\infty}\left\{(\lambda)_n\left(\frac{z}{\lambda}\right)^n\right\}=z^n=\lim_{|\mu|\to\infty}\left\{\frac{(\mu z)^n}{(\mu)_n}\right\}\qquad(\lambda,\mu\in\mathbb{C};\ n\in\mathbb{N}_0),$$

a limit case of the generating function (39) when t is replaced by $\frac{t}{\lambda}$ and $|\lambda|\to\infty$ yields the following *exponential* generating function:

$$\sum_{n=0}^{\infty}\ {}_{r+1}F_s\left[\begin{matrix}-n,(a_1,p), & a_2,\ldots,a_r; \\ & b_1,\ldots,b_s;\end{matrix}\ z\right]\frac{t^n}{n!}=e^t\ {}_rF_s\left[\begin{matrix}(a_1,p), & a_2,\ldots,a_r; \\ & b_1,\ldots,b_s;\end{matrix}\ -zt\right]. \tag{42}$$

As a matter of fact, this last generating function (42) can be deduced also as a limit case of two of the other generating functions (40) and (41) asserted by Corollary 6. For example, if we replace z and t in the generating function (41) by $-\lambda z$ and $\frac{t}{\lambda}$, respectively, and then proceed to the limit as $|\lambda|\to\infty$, we are led easily to the *exponential* generating function (42).

5. Concluding remarks and observations

In our present investigation, we first introduced a generalization of the Pochhammer symbol by means of the one-parameter family of generalized gamma functions defined by (1). Then, with the help of this new Pochhammer symbol $(\lambda;p)_\nu$

defined by (8), we introduced an extension of the generalized hypergeometric function ${}_rF_s$ with r numerator and s denominator parameters. Finally, we presented a systematic study of the various fundamental properties of the class of generalized hypergeometric functions introduced here.

In their special cases when $p = 0$, the results obtained in this paper would reduce to the corresponding well-known results for the generalized hypergeometric function ${}_rF_s$ and for its various related functions (see, for details, [8,19,22,25].

The closed-form expressions of the integrals, which we have evaluated in this paper, are presumably not available in the existing literature. With the help of the generalized Pochhammer symbol $(\lambda; p)_\nu$ defined by (8), various other families of special functions can also be generalized similarly and the closed-form expressions of integrals, which may not be expressible in terms of known functions, can be obtained.

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