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Research Article

On the Riesz Potential and Its Commutators on Generalized Orlicz-Morrey Spaces

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We consider generalized Orlicz-Morrey spaces $M_{\Phi,\varphi}(\mathbb{R}^n)$ including their weak versions $WM_{\Phi,\varphi}(\mathbb{R}^n)$. In these spaces we prove the boundedness of the Riesz potential from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}(\mathbb{R}^n)$ and from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM_{\Psi,\varphi_2}(\mathbb{R}^n)$. As applications of those results, the boundedness of the commutators of the Riesz potential on generalized Orlicz-Morrey space is also obtained. In all the cases the conditions for the boundedness are given either in terms of Zygmund-type integral inequalities on (φ_1, φ_2), which do not assume any assumption on monotonicity of $\varphi_1(x,r)$, $\varphi_2(x,r)$ in r.

1. Introduction

The theory of boundedness of classical operators of the real analysis, such as the maximal operator, fractional maximal operator, Riesz potential, and the singular integral operators, and so forth, has been extensively investigated in various function spaces. Results on weak and strong type inequalities for operators of this kind in Lebesgue spaces are classical and can be found for example in [1–3]. This boundedness extended to several function spaces which are generalizations of L_p-spaces, for example, Orlicz spaces, Morrey spaces, Lorentz spaces, Herz spaces, and so forth.

Orlicz spaces, introduced in [4, 5], are generalizations of Lebesgue spaces L_p. They are useful tools in harmonic analysis and its applications. For example, the Hardy-Littlewood maximal operator is bounded on L_p for $1 < p < \infty$, but not on L_1. Using Orlicz spaces, we can investigate the boundedness of the maximal operator near $p = 1$ more precisely (see [6–8]).

It is well known that the Riesz potential I_α of order α ($0 < \alpha < n$) plays an important role in harmonic analysis, PDE, and potential theory (see [2]). Recall that I_α is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \qquad x \in \mathbb{R}^n. \tag{1}$$

The classical result by Hardy-Littlewood-Sobolev states that, if $1 < p < q < \infty$, then the operator I_α is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if and only if $\alpha = n((1/p) - (1/q))$ and, for $p = 1 < q < \infty$, the operator I_α is bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ if and only if $\alpha = n(1 - (1/q))$. For boundedness of I_α on Morrey spaces $M_{p,\lambda}(\mathbb{R}^n)$, see Peetre (Spanne) [9] and Adams [10].

The boundedness of I_α from Orlicz space $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$ was studied by O'Neil [11] and Torchinsky [12] under some restrictions involving the growths and certain monotonicity properties of Φ and Ψ. Moreover Cianchi [6] gave a necessary and sufficient condition for the boundedness of I_α from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$ and from $L_\Phi(\mathbb{R}^n)$ to weak Orlicz space $WL_\Psi(\mathbb{R}^n)$, which contain results above.

In [13] the authors study the boundedness of the maximal operator M and the Calderón-Zygmund operator T from one generalized Orlicz-Morrey space $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Phi,\varphi_2}(\mathbb{R}^n)$ and from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to the weak space $WM_{\Phi,\varphi_2}(\mathbb{R}^n)$.

Our definition of Orlicz-Morrey spaces (see Section 3) is different from that of Sawano et al. [14] and Nakai [15, 16].

The main purpose of this paper is to find sufficient conditions on general Young functions Φ, Ψ and functions φ_1, φ_2 which ensure the boundedness of the Riesz potential I_α from one generalized Orlicz-Morrey spaces $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to another $M_{\Psi,\varphi_2}(\mathbb{R}^n)$ and from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to weak generalized Orlicz-Morrey spaces $WM_{\Psi,\varphi_2}(\mathbb{R}^n)$ and the boundedness of the commutator of the Riesz potential $[b, I_\alpha]$ from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}(\mathbb{R}^n)$.

In the next section we recall the definitions of Orlicz and Morrey spaces and give the definition of Orlicz-Morrey and generalized Orlicz-Morrey spaces in Section 3. In Section 4 and Section 5 the results on the boundedness of the Riesz potential and its commutator on generalized Orlicz-Morrey spaces are obtained.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Some Preliminaries on Orlicz and Morrey Spaces

In the study of local properties of solutions of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $M_{p,\lambda}(\mathbb{R}^n)$ play an important role; see [17]. Introduced by Morrey Jr. [18] in 1938, they are defined by the norm

$$\|f\|_{M_{p,\lambda}} := \sup_{x,r>0} r^{-\lambda/p} \|f\|_{L_p(B(x,r))}, \tag{2}$$

where $0 \leq \lambda \leq n$, $1 \leq p < \infty$. Here and everywhere in the sequel $B(x,r)$ stands for the ball in $\mathbb{R}^n$ of radius r centered at x. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$ and $|B(x,r)| = v_n r^n$, where v_n is the volume of the unit ball in $\mathbb{R}^n$.

Note that $M_{p,0} = L_p(\mathbb{R}^n)$ and $M_{p,n} = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $M_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

We also denote by $WM_{p,\lambda} \equiv WM_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\lambda/p} \|f\|_{WL_p(B(x,r))} < \infty, \tag{3}$$

where $WL_p(B(x,r))$ denotes the weak L_p-space.

We refer in particular to [19] for the classical Morrey spaces.

We recall the definition of Young functions.

Definition 1. A function $\Phi : [0,+\infty) \to [0,\infty]$ is called a Young function if Φ is convex and left-continuous, $\lim_{r\to+0}\Phi(r) = \Phi(0) = 0$, and $\lim_{r\to+\infty}\Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0,+\infty)$ such that $\Phi(s) = +\infty$, then $\Phi(r) = +\infty$ for $r \geq s$.

Let $\mathcal{Y}$ be the set of all Young functions Φ such that

$$0 < \Phi(r) < +\infty \quad \text{for } 0 < r < +\infty. \tag{4}$$

If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,+\infty)$ and bijective from $[0,+\infty)$ to itself.

Definition 2 (Orlicz space). For a Young function Φ, the set

$$L_\Phi(\mathbb{R}^n) = \left\{ f \in L_1^{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)\,dx < +\infty \text{ for some } k > 0 \right\} \tag{5}$$

is called Orlicz space. If $\Phi(r) = r^p$, $1 \leq p < \infty$, then $L_\Phi(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. If $\Phi(r) = 0$ $(0 \leq r \leq 1)$ and $\Phi(r) = \infty$ $(r > 1)$, then $L_\Phi(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$. The space $L_\Phi^{\mathrm{loc}}(\mathbb{R}^n)$ endowed with the natural topology is defined as the set of all functions f such that $f\chi_B \in L_\Phi(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$. We refer to the books [20–22] for the theory of Orlicz spaces.

$L_\Phi(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L_\Phi} = \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\}. \tag{6}$$

We note that

$$\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\|f\|_{L_\Phi}}\right) dx \leq 1. \tag{7}$$

For a measurable set $\Omega \subset \mathbb{R}^n$, a measurable function f, and $t > 0$, let

$$m(\Omega, f, t) = \left|\{x \in \Omega : |f(x)| > t\}\right|. \tag{8}$$

In the case $\Omega = \mathbb{R}^n$, we shortly denote it by $m(f,t)$.

Definition 3. The weak Orlicz space

$$WL_\Phi(\mathbb{R}^n) := \left\{ f \in L_{\mathrm{loc}}^1(\mathbb{R}^n) : \|f\|_{WL_\Phi} < +\infty \right\} \tag{9}$$

is defined by the norm

$$\|f\|_{WL_\Phi} = \inf\left\{ \lambda > 0 : \sup_{t>0} \Phi(t)\, m\left(\frac{f}{\lambda}, t\right) \leq 1 \right\}. \tag{10}$$

For a Young function Φ and $0 \leq s \leq +\infty$, let

$$\Phi^{-1}(s) = \inf\{r \geq 0 : \Phi(r) > s\} \quad (\inf \emptyset = +\infty). \tag{11}$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. We note that

$$\Phi\left(\Phi^{-1}(r)\right) \leq r \leq \Phi^{-1}(\Phi(r)) \quad \text{for } 0 \leq r < +\infty. \tag{12}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted by $\Phi \in \Delta_2$, if

$$\Phi(2r) \leq k\Phi(r) \quad \text{for } r > 0 \tag{13}$$

for some $k > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2k}\Phi(kr), \quad r \geq 0, \tag{14}$$

for some $k > 1$. The function $\Phi(r) = r$ satisfies the Δ_2-condition but does not satisfy the ∇_2-condition. If $1 < p < \infty$, then $\Phi(r) = r^p$ satisfies both the conditions. The function $\Phi(r) = e^r - r - 1$ satisfies the ∇_2-condition but does not satisfy the Δ_2-condition.

For a Young function Φ, the complementary function $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\}, & r \in [0,\infty), \\ +\infty, & r = +\infty. \end{cases} \quad (15)$$

The complementary function $\widetilde{\Phi}$ is also a Young function and $\widetilde{\widetilde{\Phi}} = \Phi$. If $\Phi(r) = r$, then $\widetilde{\Phi}(r) = 0$ for $0 \le r \le 1$ and $\widetilde{\Phi}(r) = +\infty$ for $r > 1$. If $1 < p < \infty$, $1/p + 1/p' = 1$, and $\Phi(r) = r^p/p$, then $\widetilde{\Phi}(r) = r^{p'}/p'$. If $\Phi(r) = e^r - r - 1$, then $\widetilde{\Phi}(r) = (1+r)\log(1+r) - r$. Note that $\Phi \in \nabla_2$ if and only if $\widetilde{\Phi} \in \Delta_2$. It is known that

$$r \le \Phi^{-1}(r)\,\widetilde{\Phi}^{-1}(r) \le 2r \quad \text{for } r \ge 0. \quad (16)$$

Note that Young functions satisfy the properties

$$\begin{aligned} \Phi(\alpha t) &\le \alpha\Phi(t), \quad \text{if } 0 \le \alpha \le 1, \\ \Phi(\alpha t) &\ge \alpha\Phi(t), \quad \text{if } \alpha > 1, \\ \Phi^{-1}(\alpha t) &\ge \alpha\Phi^{-1}(t), \quad \text{if } 0 \le \alpha \le 1, \\ \Phi^{-1}(\alpha t) &\le \alpha\Phi^{-1}(t), \quad \text{if } \alpha > 1. \end{aligned} \quad (17)$$

The following analogue of the Hölder inequality is known; see [23].

Theorem 4 (see [23]). *For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid:*

$$\|fg\|_{L_1(\mathbb{R}^n)} \le 2\|f\|_{L_\Phi}\|g\|_{L_{\widetilde{\Phi}}}. \quad (18)$$

The following lemma is valid.

Lemma 5 (see [1, 24]). *Let Φ be a Young function and B a set in $\mathbb{R}^n$ with finite Lebesgue measure. Then*

$$\|\chi_B\|_{WL_\Phi(\mathbb{R}^n)} = \|\chi_B\|_{L_\Phi(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}\left(|B|^{-1}\right)}. \quad (19)$$

In the next sections where we prove our main estimates, we use the following lemma, which follows from Theorem 4, Lemma 5, and (16).

Lemma 6. *For a Young function Φ and $B = B(x,r)$, the following inequality is valid:*

$$\|f\|_{L_1(B)} \le 2|B|\,\Phi^{-1}\left(|B|^{-1}\right)\|f\|_{L_\Phi(B)}. \quad (20)$$

3. Orlicz-Morrey and Generalized Orlicz-Morrey Spaces

Definition 7 (Orlicz-Morrey space). For a Young function Φ and $0 \le \lambda \le n$, one denotes by $M_{\Phi,\lambda}(\mathbb{R}^n)$ the Orlicz-Morrey space, the space of all functions $f \in L_\Phi^{\text{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{\Phi,\lambda}} = \sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}\left(r^{-\lambda}\right)\|f\|_{L_\Phi(B(x,r))}. \quad (21)$$

Note that $M_{\Phi,0} = L_\Phi(\mathbb{R}^n)$ and if $\Phi(r) = r^p$, $1 \le p < \infty$, then $M_{\Phi,\lambda}(\mathbb{R}^n) = M_{p,\lambda}(\mathbb{R}^n)$.

We also denote by $WM_{\Phi,\lambda}(\mathbb{R}^n)$ the weak Orlicz-Morrey space of all functions $f \in WL_\Phi^{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{\Phi,\lambda}} = \sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}\left(r^{-\lambda}\right)\|f\|_{WL_\Phi(B(x,r))} < \infty, \quad (22)$$

where $WL_\Phi(B(x,r))$ denotes the weak L_Φ-space of measurable functions f for which

$$\|f\|_{WL_\Phi(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_\Phi(\mathbb{R}^n)}. \quad (23)$$

Definition 8 (generalized Orlicz-Morrey space). Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and Φ any Young function. One denotes by $M_{\Phi,\varphi}(\mathbb{R}^n)$ the generalized Orlicz-Morrey space, the space of all functions $f \in L_\Phi^{\text{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\Phi^{-1}\left(|B(x,r)|^{-1}\right)\|f\|_{L_\Phi(B(x,r))}. \quad (24)$$

Also by $WM_{\Phi,\varphi}(\mathbb{R}^n)$ one denotes the weak generalized Orlicz-Morrey space of all functions $f \in WL_\Phi^{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\Phi^{-1}\left(|B(x,r)|^{-1}\right)\|f\|_{WL_\Phi(B(x,r))} < \infty. \quad (25)$$

According to this definition, we recover the spaces $M_{\Phi,\lambda}$ and $WM_{\Phi,\lambda}$ under the choice $\varphi(x,r) = (\Phi^{-1}(r^{-n})/\Phi^{-1}(r^{-\lambda}))$:

$$\begin{aligned} M_{\Phi,\lambda} &= M_{\Phi,\varphi}\Big|_{\varphi(x,r)=(\Phi^{-1}(r^{-n})/\Phi^{-1}(r^{-\lambda}))}, \\ WM_{\Phi,\lambda} &= WM_{\Phi,\varphi}\Big|_{(\Phi^{-1}(r^{-n})/\Phi^{-1}(r^{-\lambda}))}. \end{aligned} \quad (26)$$

According to this definition, we recover the generalized Morrey spaces $M_{p,\varphi}$ and weak generalized Morrey spaces $WM_{p,\varphi}$ under the choice $\Phi(r) = r^p$, $1 \le p < \infty$:

$$\begin{aligned} M_{p,\varphi} &= M_{\Phi,\varphi}\Big|_{\Phi(r)=r^p}, \\ WM_{p,\varphi} &= WM_{\Phi,\varphi}\Big|_{\Phi(r)=r^p}. \end{aligned} \quad (27)$$

Remark 9. There are different kinds of Orlicz-Morrey spaces in the literature. We want to make some comment about these spaces.

Let $\varphi : (0,\infty) \to (0,\infty)$ be a function and $\Phi : (0,\infty) \to (0,\infty)$ a Young function.

(1) For a cube Q, define (φ,Φ)-average over Q by

$$\|f\|_{(\varphi,\Phi);Q} := \inf\left\{\lambda > 0 : \frac{\varphi(|Q|)}{|Q|}\int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right)dx \le 1\right\} \quad (28)$$

and define its Φ-average over Q by

$$\|f\|_{\Phi;Q} := \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right)dx \le 1\right\}. \quad (29)$$

(2) Define

$$\|f\|_{\mathscr{L}_{\varphi,\Phi}} := \sup_{Q\in\mathcal{Q}} \|f\|_{(\varphi,\Phi);Q}. \quad (30)$$

The function space $\mathscr{L}_{\varphi,\Phi}$ is defined to be the Orlicz-Morrey space of the first kind as the set of all measurable functions f for which the norm $\|f\|_{\mathscr{L}_{\varphi,\Phi}}$ is finite.

(3) Define

$$\|f\|_{\mathscr{M}_{\varphi,\Phi}} := \sup_{Q\in\mathcal{Q}} \varphi(|Q|)\,\|f\|_{\Phi;Q}. \quad (31)$$

The function space $\mathscr{M}_{\varphi,\Phi}$ is defined to be the Orlicz-Morrey space of the second kind as the set of all measurable functions f for which the norm $\|f\|_{\mathscr{M}_{\varphi,\Phi}}$ is finite.

According to our best knowledge, it seems that $\mathscr{L}_{\varphi,\Phi}$ is more investigated than $\mathscr{M}_{\varphi,\Phi}$. The space $\mathscr{L}_{\varphi,\Phi}$ is investigated in [15, 16, 25–34] and the space $\mathscr{M}_{\varphi,\Phi}$ is investigated in [14, 35–37].

4. Boundedness of the Riesz Potential in Generalized Orlicz-Morrey Spaces

In this section sufficient conditions on the pairs (φ_1,φ_2) and (Φ,Ψ) for the boundedness of I_α from one generalized Orlicz-Morrey spaces $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to another $M_{\Psi,\varphi_2}(\mathbb{R}^n)$ and from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to the weak space $WM_{\Psi,\varphi_2}(\mathbb{R}^n)$ have been obtained.

Necessary and sufficient conditions on (Φ,Ψ) for the boundedness of I_α from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$ and $L_\Phi(\mathbb{R}^n)$ to $WL_\Psi(\mathbb{R}^n)$ have been obtained in [6, Theorem 2]. In the statement of the theorem, Ψ_p is the Young function associated with the Young function Ψ and $p \in (1,\infty]$ whose Young conjugate is given by

$$\widetilde{\Psi_p}(s) = \int_0^s r^{p'-1}\left(\mathscr{B}_p^{-1}\left(r^{p'}\right)\right)^{p'} dr, \quad (32)$$

where

$$\mathscr{B}_p(s) = \int_0^s \frac{\Psi(t)}{t^{1+p'}}dt, \quad (33)$$

and p', the Holder conjugate of p, equals either $p/(p-1)$ or 1, according to whether $p<\infty$ or $p=\infty$ and Φ_p denotes the Young function defined by

$$\Phi_p(s) = \int_0^s r^{p'-1}\left(\mathscr{A}_p^{-1}\left(r^{p'}\right)\right)^{p'} dr, \quad (34)$$

where

$$\mathscr{A}_p(s) = \int_0^s \frac{\widetilde{\Phi}(t)}{t^{1+p'}}dt. \quad (35)$$

Recall that, if Φ and Ψ are functions from $[0,\infty)$ into $[0,\infty]$, then Ψ is said to dominate Φ globally if a positive constant c exists such that $\Phi(s) \le \Psi(cs)$ for all $s \ge 0$.

Theorem 10 (see [6]). *Let $0<\alpha<n$. Let Φ and Ψ Young functions and let $\Phi_{n/\alpha}$ and $\Psi_{n/\alpha}$ be the Young functions defined as in* (34) *and* (32), *respectively. Then*

(i) *the Riesz potential I_α is bounded from $L_\Phi(\mathbb{R}^n)$ to $WL_\Psi(\mathbb{R}^n)$ if and only if*

$$\int_0^1 \frac{\widetilde{\Phi}(t)}{t^{1+n/(n-\alpha)}}dt < \infty \text{ and } \Phi_{n/\alpha} \text{ dominates } \Psi \text{ globally}. \quad (36)$$

(ii) *The Riesz potential I_α is bounded from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$ if and only if*

$$\int_0^1 \frac{\widetilde{\Phi}(t)}{t^{1+n/(n-\alpha)}}dt < \infty, \qquad \int_0^1 \frac{\Psi(t)}{t^{1+n/(n-\alpha)}}dt < \infty,$$

$$\Phi \text{ dominates } \Psi_{n/\alpha} \text{ globally, and } \Phi_{n/\alpha} \text{ dominates } \Psi \text{ globally}. \quad (37)$$

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_t^\infty g(s)\,w(s)\,ds, \quad 0<t<\infty, \quad (38)$$

where w is a weight.

The following theorem was proved in [38] (see, also [13]).

Theorem 11. *Let v_1, v_2, and w be weights on $(0,\infty)$ and $v_1(t)$ bounded outside a neighborhood of the origin. The inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t)\,H_w g(t) \le C \operatorname*{ess\,sup}_{t>0} v_1(t)\,g(t) \quad (39)$$

holds for some $C>0$ for all nonnegative and nondecreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)\,ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty. \quad (40)$$

Moreover, the value $C=B$ is the best constant for (39).

Lemma 12. *Let Φ and Ψ Young functions and Φ_p, $p\in(1,\infty]$, Young function defined as in* (34). *If $\int_0^1 \widetilde{\Phi}(t)/t^{1+p'}dt < \infty$ and Φ_p dominates Ψ globally, then*

$$\Phi^{-1}(r) \lesssim r^{1/p}\Psi^{-1}(r), \quad \text{for } r>0. \quad (41)$$

Proof. If $\int_0^1 \widetilde{\Phi}(t)/t^{1+p'}\,dt < \infty$, then

$$1 \le 2r^{-1/p'}\widetilde{\Phi}^{-1}(r)\,\Phi_p^{-1}(r), \quad \text{for } r > 0. \tag{42}$$

For the proof of this claim see [39, page 50].

If Φ_p dominates Ψ globally, then a positive constant C exists such that

$$\Phi_p^{-1}(r) \le C\Psi^{-1}(r), \quad \text{for } r > 0. \tag{43}$$

Indeed,

$$\begin{aligned} \Psi^{-1}(r) &= \inf\{t \ge 0 : \Psi(t) > r\} \\ &\ge \inf\{t \ge 0 : \Phi_p(Ct) > r\} \\ &= \frac{1}{C}\inf\{Ct \ge 0 : \Phi_p(Ct) > r\} \\ &= \frac{1}{C}\Phi_p^{-1}(r). \end{aligned} \tag{44}$$

Thus, (41) follows from (42), (43), and (16). □

The following lemma is valid.

Lemma 13. *Let $0 < \alpha < n$, Φ and Ψ Young functions, $f \in L_\Phi^{\mathrm{loc}}(\mathbb{R}^n)$, and $B = B(x_0, r)$. If (Φ, Ψ) satisfy the conditions* (37)*, then*

$$\|I_\alpha f\|_{L_\Psi(B)} \lesssim \frac{1}{\Psi^{-1}(r^{-n})}\int_{2r}^{\infty}\|f\|_{L_\Phi(B(x_0,t))}\Psi^{-1}(t^{-n})\frac{dt}{t} \tag{45}$$

and if (Φ, Ψ) satisfy the conditions (36)*, then*

$$\|I_\alpha f\|_{WL_\Psi(B)} \lesssim \frac{1}{\Psi^{-1}(r^{-n})}\int_{2r}^{\infty}\|f\|_{L_\Phi(B(x_0,t))}\Psi^{-1}(t^{-n})\frac{dt}{t}. \tag{46}$$

Proof. Suppose that the conditions (37) are satisfied. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r, $2B = B(x_0, 2r)$. We represent f as

$$\begin{aligned} f = f_1 + f_2, \qquad & f_1(y) = f(y)\,\chi_{2B}(y), \\ f_2(y) = f(y)\,\chi_{\complement(2B)}(y), \quad & r > 0, \end{aligned} \tag{47}$$

and have

$$\|I_\alpha f\|_{L_\Psi(B)} \le \|I_\alpha f_1\|_{L_\Psi(B)} + \|I_\alpha f_2\|_{L_\Psi(B)}. \tag{48}$$

Since $f_1 \in L_\Phi(\mathbb{R}^n)$, $I_\alpha f_1 \in L_\Psi(\mathbb{R}^n)$, and from the boundedness of I_α from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$ (see Theorem 10) it follows that

$$\begin{aligned} \|I_\alpha f_1\|_{L_\Psi(B)} &\le \|I_\alpha f_1\|_{L_\Psi(\mathbb{R}^n)} \\ &\le C\|f_1\|_{L_\Phi(\mathbb{R}^n)} = C\|f\|_{L_\Phi(2B)}, \end{aligned} \tag{49}$$

where constant $C > 0$ is independent of f.

It is clear that $x \in B$, $y \in \complement(2B)$ implies $(1/2)|x_0 - y| \le |x - y| \le (3/2)|x_0 - y|$. We get

$$|I_\alpha f_2(x)| \le 2^{n-\alpha}\int_{\complement(2B)}\frac{|f(y)|}{|x_0 - y|^{n-\alpha}}dy. \tag{50}$$

By Fubini's theorem we have

$$\begin{aligned} \int_{\complement(2B)}\frac{|f(y)|}{|x_0 - y|^{n-\alpha}}dy &\approx \int_{\complement(2B)}|f(y)|\int_{|x_0-y|}^{\infty}\frac{dt}{t^{n+1-\alpha}}dy \\ &\approx \int_{2r}^{\infty}\int_{2r\le|x_0-y|<t}|f(y)|\,dy\frac{dt}{t^{n+1-\alpha}} \\ &\lesssim \int_{2r}^{\infty}\int_{B(x_0,t)}|f(y)|\,dy\frac{dt}{t^{n+1-\alpha}}. \end{aligned} \tag{51}$$

By Lemmas 6 and 12 for $p = n/\alpha$ we get

$$\begin{aligned} &\int_{\complement(2B)}\frac{|f(y)|}{|x_0 - y|^{n}}dy \\ &\quad\lesssim \int_{2r}^{\infty}\|f\|_{L_\Phi(B(x_0,t))}\Phi^{-1}(t^{-n})\,t^{\alpha-1}dt \\ &\quad\lesssim \int_{2r}^{\infty}\|f\|_{L_\Phi(B(x_0,t))}\Psi^{-1}(t^{-n})\frac{dt}{t}. \end{aligned} \tag{52}$$

Moreover,

$$\|I_\alpha f_2\|_{L_\Psi(B)} \lesssim \frac{1}{\Psi^{-1}(r^{-n})}\int_{2r}^{\infty}\|f\|_{L_\Phi(B(x_0,t))}\Psi^{-1}(t^{-n})\frac{dt}{t} \tag{53}$$

is valid. Thus

$$\begin{aligned} &\|I_\alpha f\|_{L_\Psi(B)} \\ &\quad\lesssim \|f\|_{L_\Phi(2B)} + \frac{1}{\Psi^{-1}(r^{-n})} \\ &\qquad\times\int_{2r}^{\infty}\|f\|_{L_\Phi(B(x_0,t))}\Psi^{-1}(t^{-n})\frac{dt}{t}. \end{aligned} \tag{54}$$

On the other hand, using the property of Young function as it is mentioned in (16)

$$\begin{aligned} \Psi^{-1}(r^{-n}) &\approx \Psi^{-1}(r^{-n})\,r^n\int_{2r}^{\infty}\frac{dt}{t^{n+1}} \\ &\lesssim \int_{2r}^{\infty}\Psi^{-1}(t^{-n})\frac{dt}{t} \end{aligned} \tag{55}$$

and we get

$$\|f\|_{L_\Phi(2B)} \lesssim \frac{1}{\Psi^{-1}(r^{-n})}\int_{2r}^{\infty}\|f\|_{L_\Phi(B(x_0,t))}\Psi^{-1}(t^{-n})\frac{dt}{t}. \tag{56}$$

Thus

$$\|I_\alpha f\|_{L_\Psi(B)} \lesssim \frac{1}{\Psi^{-1}(r^{-n})}\int_{2r}^{\infty}\|f\|_{L_\Phi(B(x_0,t))}\Psi^{-1}(t^{-n})\frac{dt}{t}. \tag{57}$$

Suppose that the conditions (37) are satisfied. From the boundedness of I_α from $L_\Phi(\mathbb{R}^n)$ to $WL_\Psi(\mathbb{R}^n)$ (see Theorem 10) and (56) it follows that

$$\begin{aligned}\|I_\alpha f_1\|_{WL_\Psi(B)} &\le \|I_\alpha f_1\|_{WL_\Psi(\mathbb{R}^n)} \lesssim \|f_1\|_{L_\Phi(\mathbb{R}^n)} \\ &= \|f\|_{L_\Phi(2B)} \\ &\lesssim \frac{1}{\Psi^{-1}(r^{-n})}\int_{2r}^{\infty}\|f\|_{L_\Phi(B(x_0,t))}\Psi^{-1}(t^{-n})\frac{dt}{t}.\end{aligned} \tag{58}$$

Then by (53) and (58) we get the inequality (46). □

Theorem 14. *Let $0<\alpha<n$ and the functions (φ_1,φ_2) and (Φ,Ψ) satisfy the condition*

$$\int_r^\infty \operatorname*{ess\,inf}_{t<s<\infty}\frac{\varphi_1(x,s)}{\Phi^{-1}(s^{-n})}\Psi^{-1}(t^{-n})\frac{dt}{t}\le C\varphi_2(x,r), \tag{59}$$

where C does not depend on x and r. Then for the conditions (37), I_α is bounded from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}(\mathbb{R}^n)$ and for the conditions (36), I_α is bounded from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM_{\Psi,\varphi_2}(\mathbb{R}^n)$.

Proof. By Lemma 13 and Theorem 11 we get

$$\begin{aligned}\|I_\alpha f\|_{M_{\Psi,\varphi_2}} &\lesssim \sup_{x\in\mathbb{R}^n,r>0}\varphi_2(x,r)^{-1}\int_r^\infty \Psi^{-1}(t^{-n})\|f\|_{L_\Phi(B(x,t))}\frac{dt}{t} \\ &\lesssim \sup_{x\in\mathbb{R}^n,r>0}\varphi_1(x,r)^{-1}\Phi^{-1}(r^{-n})\|f\|_{L_\Phi(B(x,r))} \\ &= \|f\|_{M_{\Phi,\varphi_1}},\end{aligned} \tag{60}$$

if (37) is satisfied and

$$\begin{aligned}\|I_\alpha f\|_{WM_{\Psi,\varphi_2}} &\lesssim \sup_{x\in\mathbb{R}^n,r>0}\varphi_2(x,r)^{-1}\int_r^\infty \Psi^{-1}(t^{-n})\|f\|_{L_\Phi(B(x,t))}\frac{dt}{t} \\ &\lesssim \sup_{x\in\mathbb{R}^n,r>0}\varphi_1(x,r)^{-1}\Phi^{-1}(r^{-n})\|f\|_{L_\Phi(B(x,r))} \\ &= \|f\|_{M_{\Phi,\varphi_1}},\end{aligned} \tag{61}$$

if (36) is satisfied. □

Remark 15. Recall that, for $0<\alpha<n$,

$$M_\alpha f(x)\le v_n^{(\alpha/n)-1}I_\alpha(|f|)(x); \tag{62}$$

hence Theorem 14 implies the boundedness of the fractional maximal operator M_α from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}(\mathbb{R}^n)$ and from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM_{\Psi,\varphi_2}(\mathbb{R}^n)$.

If we take $\Phi(t)=t^p$, $\Psi(t)=t^q$, $1\le p,q<\infty$, at Theorem 14 we get following corollary which was proved in [40] and containing results obtained in [41–45].

Corollary 16. *Let $0<\alpha<n$, $1\le p<n/\alpha$, $1/q=(1/p)-(\alpha/n)$ and (φ_1,φ_2) satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x,s)s^{n/p}}{t^{(n/q)+1}}dt\le C\varphi_2(x,r), \tag{63}$$

where C does not depend on x and r. Then I_α is bounded from M_{p,φ_1} to M_{q,φ_2} for $p>1$ and from M_{1,φ_1} to WM_{q,φ_2} for $p=1$.

In the case $\varphi_1(x,r)=(\Phi^{-1}(r^{-n})/\Phi^{-1}(r^{-\lambda_1}))$, $\varphi_2(x,r)=\Psi^{-1}(r^{-n})/\Psi^{-1}(r^{-\lambda_2})$ from Theorem 14 we get the following Spanne type theorem for the boundedness of the Riesz potential on Orlicz-Morrey spaces.

Corollary 17. *Let $0<\alpha<n$, Φ and Ψ Young functions, $0\le\lambda_1,\lambda_2<n$, and (Φ,Ψ) satisfy the condition*

$$\int_r^\infty \frac{\Psi^{-1}(t^{-n})}{\Phi^{-1}(t^{-\lambda_1})}\frac{dt}{t}\le C\frac{\Psi^{-1}(r^{-n})}{\Psi^{-1}(r^{-\lambda_2})}, \tag{64}$$

where C does not depend on r. Then for the conditions (37), I_α is bounded from $M_{\Phi,\lambda_1}(\mathbb{R}^n)$ to $M_{\Psi,\lambda_2}(\mathbb{R}^n)$ and for the conditions (36), I_α is bounded from $M_{\Phi,\lambda_1}(\mathbb{R}^n)$ to $WM_{\Psi,\lambda_2}(\mathbb{R}^n)$.

Remark 18. If we take $\Phi(t)=t^p$, $\Psi(t)=t^q$, $1\le p,q<\infty$, at Corollary 17 we get Spanne type boundedness of I_α; that is, if $0<\alpha<n$, $1<p<n/\alpha$, $0<\lambda<n-\alpha p$, $(1/p)-(1/q)=\alpha/n$, and $\lambda/p=\mu/q$, then for $p>1$ the Riesz potential I_α is bounded from $M_{p,\lambda}(\mathbb{R}^n)$ to $M_{q,\mu}(\mathbb{R}^n)$ and for $p=1$, I_α is bounded from $M_{1,\lambda}(\mathbb{R}^n)$ to $WM_{q,\mu}(\mathbb{R}^n)$.

5. Commutators of Riesz Potential in the Spaces $M_{\Phi,\varphi}$

For a function $b\in L_1^{\text{loc}}(\mathbb{R}^n)$, let M_b be the corresponding multiplication operator defined by $M_b f=bf$ for measurable function f. Let T be the classical Calderón-Zygmund singular integral operator; then the commutator between T and M_b is denoted by $[b,T]:=M_bT-TM_b$. A famous theorem of Coifman et al. [46] gave a characterization of L_p-boundedness of $[b,T]$ when T are the Riesz transforms R_j $(j=1,\ldots,n)$. Using this characterization, the authors of [46] got a decomposition theorem of the real Hardy spaces. The boundedness result was generalized to other contexts and important applications to some nonlinear PDEs were given by Coifman et al. [47].

We recall the definition of the space of $\mathrm{BMO}(\mathbb{R}^n)$.

Definition 19. Suppose that $f\in L_1^{\text{loc}}(\mathbb{R}^n)$; let

$$\|f\|_*=\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)-f_{B(x,r)}|dy<\infty, \tag{65}$$

where

$$f_{B(x,r)}=\frac{1}{|B(x,r)|}\int_{B(x,r)}f(y)dy. \tag{66}$$

Define

$$\mathrm{BMO}(\mathbb{R}^n) = \left\{ f \in L_1^{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_* < \infty \right\}. \quad (67)$$

Modulo constants, the space $\mathrm{BMO}(\mathbb{R}^n)$ is a Banach space with respect to the norm $\|\cdot\|_*$.

Remark 20. (1) The John-Nirenberg inequality: there are constants $C_1, C_2 > 0$, such that for all $f \in \mathrm{BMO}(\mathbb{R}^n)$ and $\beta > 0$

$$\begin{aligned} &\left|\{x \in B : |f(x) - f_B| > \beta\}\right| \\ &\quad \leq C_1 |B| e^{-C_2\beta/\|f\|_*}, \quad \forall B \subset \mathbb{R}^n. \end{aligned} \quad (68)$$

(2) The John-Nirenberg inequality implies that

$$\|f\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}|^p dy \right)^{1/p} \quad (69)$$

for $1 < p < \infty$.

(3) Let $f \in \mathrm{BMO}(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$\left| f_{B(x,r)} - f_{B(x,t)} \right| \leq C \|f\|_* \ln \frac{t}{r} \quad \text{for } 0 < 2r < t, \quad (70)$$

where C is independent of f, x, r, and t.

Definition 21. A Young function Φ is said to be of upper type p (resp., lower type p) for some $p \in [0, \infty)$, if there exists a positive constant C such that, for all $t \in [1, \infty)$ (resp., $t \in [0, 1]$) and $s \in [0, \infty)$,

$$\Phi(st) \leq C t^p \Phi(s). \quad (71)$$

Remark 22. We know that if Φ is lower type p_0 and upper type p_1 with $1 < p_0 \leq p_1 < \infty$, then $\Phi \in \Delta_2 \cap \nabla_2$. Conversely if $\Phi \in \Delta_2 \cap \nabla_2$, then Φ is lower type p_0 and upper type p_1 with $1 < p_0 \leq p_1 < \infty$ (see [20]).

Lemma 23 (see [48]). *Let Φ be a Young function which is lower type p_0 and upper type p_1 with $1 \leq p_0 \leq p_1 < \infty$. Let $\widetilde{C}$ be a positive constant. Then there exists a positive constant C such that for any ball B of $\mathbb{R}^n$ and $\mu \in (0, \infty)$*

$$\int_B \Phi\left(\frac{|f(x)|}{\mu} \right) dx \leq \widetilde{C} \quad (72)$$

implies that $\|f\|_{L_\Phi(B)} \leq C\mu$.

In the following lemma we provide a generalization of the property (69) from L_p-norms to Orlicz norms.

Lemma 24. *Let $f \in \mathrm{BMO}(\mathbb{R}^n)$ and Φ a Young function. Let Φ is lower type p_0 and upper type p_1 with $1 \leq p_0 \leq p_1 < \infty$; then*

$$\|f\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}(r^{-n}) \left\| f(\cdot) - f_{B(x,r)} \right\|_{L_\Phi(B(x,r))}. \quad (73)$$

Proof. By Hölder's inequality, we have

$$\|f\|_* \lesssim \sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}(r^{-n}) \left\| f(\cdot) - f_{B(x,r)} \right\|_{L_\Phi(B(x,r))}. \quad (74)$$

Now we show that

$$\sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}(r^{-n}) \left\| f(\cdot) - f_{B(x,r)} \right\|_{L_\Phi(B(x,r))} \lesssim \|f\|_*. \quad (75)$$

Without loss of generality, we may assume that $\|f\|_* = 1$; otherwise, we replace f by $f/\|f\|_*$. By the fact that Φ is lower type p_0 and upper type p_1 and (12) it follows that

$$\begin{aligned} &\int_{B(x,r)} \Phi\left(\frac{|f(y) - f_{B(x,r)}| \Phi^{-1}\left(|B(x,r)|^{-1}\right)}{\|f\|_*} \right) dy \\ &\quad = \int_{B(x,r)} \Phi\left(|f(y) - f_{B(x,r)}| \Phi^{-1}\left(|B(x,r)|^{-1}\right) \right) dy \\ &\quad \lesssim \frac{1}{|B(x,r)|} \\ &\quad\quad \times \int_{B(x,r)} \left[|f(y) - f_{B(x,r)}|^{p_0} + |f(y) - f_{B(x,r)}|^{p_1} \right] dy \\ &\quad \lesssim 1. \end{aligned} \quad (76)$$

By Lemma 23 we get the desired result. □

Remark 25. Note that statements of type of Lemma 24 are known in a more general case of rearrangement invariant spaces and also variable exponent Lebesgue spaces $L^{p(\cdot)}$, see [49, 50], but we gave a short proof of Lemma 24 for completeness of presentation.

Definition 26. Let Φ be a Young function. Let

$$a_\Phi := \inf_{t\in(0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}, \qquad b_\Phi := \sup_{t\in(0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}. \quad (77)$$

Remark 27. It is known that $\Phi \in \Delta_2 \cap \nabla_2$ if and only if $1 < a_\Phi \leq b_\Phi < \infty$ (see [21]).

Remark 28. Remarks 27 and 22 show us that a Young function Φ is lower type p_0 and upper type p_1 with $1 < p_0 \leq p_1 < \infty$ if and only if $1 < a_\Phi \leq b_\Phi < \infty$.

The characterization of (L_p, L_q) boundedness of the commutator $[b, I_\alpha]$ between M_b and I_α was given by Chanillo [51].

Theorem 29 (see [51]). *Let $0 < \alpha < n$, $1 < p < n/\alpha$, and $1/q = (1/p) - (\alpha/n)$. Then $[b, I_\alpha]$ is a bounded operator from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$.*

The (L_Φ, L_Ψ) boundedness of the commutator $[b, I_\alpha]$ was given by Fu et al. [52].

Theorem 30 (see [52]). *Let $0 < \alpha < n$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Let Φ be a Young function and Ψ defined, via its inverse, by setting,* *for all* $t \in (0,\infty)$, $\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n}$. *If* $1 < a_\Phi \le b_\Phi < \infty$, *and* $1 < a_\Psi \le b_\Psi < \infty$ *then* $[b, I_\alpha]$ *is bounded from* $L_\Phi(\mathbb{R}^n)$ *to* $L_\Psi(\mathbb{R}^n)$.

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w^* g(r) := \int_r^\infty \left(1 + \ln\frac{t}{r}\right) g(t)\, w(t)\, dt, \quad r \in (0,\infty), \tag{78}$$

where w is a weight.

The following theorem was proved in [53].

Theorem 31. *Let* v_1, v_2, *and* w *be weights on* $(0,\infty)$ *and* $v_1(t)$ *bounded outside a neighborhood of the origin. The inequality*

$$\operatorname*{ess\,sup}_{r>0} v_2(r) H_w^* g(r) \le C \operatorname*{ess\,sup}_{r>0} v_1(r) g(r) \tag{79}$$

holds for some $C > 0$ *for all nonnegative and nondecreasing* g *on* $(0,\infty)$ *if and only if*

$$B := \sup_{r>0} v_2(r) \int_r^\infty \left(1 + \ln\frac{t}{r}\right) \frac{w(t)\, dt}{\operatorname*{ess\,sup}_{t<s<\infty} v_1(s)} < \infty. \tag{80}$$

Moreover, the value $C = B$ *is the best constant for* (79).

Remark 32. In (79) and (80) it is assumed that $1/\infty = 0$ and $0 \cdot \infty = 0$.

The following lemma is valid.

Lemma 33. *Let* $0 < \alpha < n$ *and* $b \in \mathrm{BMO}(\mathbb{R}^n)$. *Let* Φ *be a Young function and* Ψ *defined, via its inverse, by setting, for all* $t \in (0,\infty)$, $\Psi^{-1}(t) := \Phi^{-1}(t)t^{-\alpha/n}$, *and* $1 < a_\Phi \le b_\Phi < \infty$ *and* $1 < a_\Psi \le b_\Psi < \infty$; *then the inequality*

$$\begin{aligned} &\left\| [b, I_\alpha] f \right\|_{L_\Psi(B(x_0,r))} \\ &\quad\lesssim \|b\|_* \frac{1}{\Psi^{-1}(r^{-n})} \\ &\quad\times \int_{2r}^\infty \left(1 + \ln\frac{t}{r}\right) \|f\|_{L_\Phi(B(x_0,t))} \Psi^{-1}(t^{-n}) \frac{dt}{t} \end{aligned} \tag{81}$$

holds for any ball $B(x_0, r)$ *and for all* $f \in L_\Phi^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement(2B)}$. Hence

$$\left\|[b, I_\alpha] f\right\|_{L_\Psi(B)} \le \left\|[b, I_\alpha] f_1\right\|_{L_\Psi(B)} + \left\|[b, I_\alpha] f_2\right\|_{L_\Psi(B)}. \tag{82}$$

From the boundedness of $[b, I_\alpha]$ from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$ (see Theorem 30) it follows that

$$\begin{aligned} \left\|[b, I_\alpha] f_1\right\|_{L_\Psi(B)} &\le \left\|[b, I_\alpha] f_1\right\|_{L_\Psi(\mathbb{R}^n)} \\ &\lesssim \|b\|_* \left\|f_1\right\|_{L_\Phi(\mathbb{R}^n)} \\ &= \|b\|_* \|f\|_{L_\Phi(2B)}. \end{aligned} \tag{83}$$

For $x \in B$ we have

$$\begin{aligned} \left|[b, I_\alpha] f_2(x)\right| &\lesssim \int_{\mathbb{R}^n} \frac{|b(y) - b(x)|}{|x - y|^{n-\alpha}} |f(y)|\, dy \\ &\approx \int_{\complement(2B)} \frac{|b(y) - b(x)|}{|x_0 - y|^{n-\alpha}} |f(y)|\, dy. \end{aligned} \tag{84}$$

Then

$$\begin{aligned} &\left\|[b, I_\alpha] f_2\right\|_{L_\Psi(B)} \\ &\quad\lesssim \left\| \int_{\complement(2B)} \frac{|b(y) - b(\cdot)|}{|x_0 - y|^{n-\alpha}} |f(y)|\, dy \right\|_{L_\Psi(B)} \\ &\quad\lesssim \left\| \int_{\complement(2B)} \frac{|b(y) - b_B|}{|x_0 - y|^{n-\alpha}} |f(y)|\, dy \right\|_{L_\Psi(B)} \\ &\quad\quad + \left\| \int_{\complement(2B)} \frac{|b(\cdot) - b_B|}{|x_0 - y|^{n-\alpha}} |f(y)|\, dy \right\|_{L_\Psi(B)} \\ &\quad= J_1 + J_2. \end{aligned} \tag{85}$$

Let us estimate J_1:

$$\begin{aligned} J_1 &= \frac{1}{\Psi^{-1}(r^{-n})} \int_{\complement(2B)} \frac{|b(y) - b_B|}{|x_0 - y|^{n-\alpha}} |f(y)|\, dy \\ &\approx \frac{1}{\Psi^{-1}(r^{-n})} \int_{\complement(2B)} |b(y) - b_B| |f(y)| \int_{|x_0 - y|}^\infty \frac{dt}{t^{n+1-\alpha}} dy \\ &\approx \frac{1}{\Psi^{-1}(r^{-n})} \int_{2r}^\infty \int_{2r \le |x_0 - y| \le t} |b(y) - b_B| |f(y)|\, dy \frac{dt}{t^{n+1-\alpha}} \\ &\lesssim \frac{1}{\Psi^{-1}(r^{-n})} \int_{2r}^\infty \int_{B(x_0,t)} |b(y) - b_B| |f(y)|\, dy \frac{dt}{t^{n+1-\alpha}}. \end{aligned} \tag{86}$$

Applying Hölder's inequality, by Lemma 24 and (70), we get

$$\begin{aligned} J_1 &\lesssim \frac{1}{\Psi^{-1}(r^{-n})} \int_{2r}^\infty \int_{B(x_0,t)} \left|b(y) - b_{B(x_0,t)}\right| |f(y)|\, dy \frac{dt}{t^{n+1-\alpha}} \\ &\quad+ \frac{1}{\Psi^{-1}(r^{-n})} \int_{2r}^\infty \left|b_{B(x_0,r)} - b_{B(x_0,t)}\right| \int_{B(x_0,t)} |f(y)|\, dy \frac{dt}{t^{n+1-\alpha}} \\ &\lesssim \frac{1}{\Psi^{-1}(r^{-n})} \int_{2r}^\infty \left\|b(\cdot) - b_{B(x_0,t)}\right\|_{L_{\widetilde{\Phi}}(B(x_0,t))} \|f\|_{L_\Phi(B(x_0,t))} \frac{dt}{t^{n+1-\alpha}} \\ &\quad+ \frac{1}{\Psi^{-1}(r^{-n})} \int_{2r}^\infty \left|b_{B(x_0,r)} - b_{B(x_0,t)}\right| \|f\|_{L_\Phi(B(x_0,t))} \Phi^{-1}(t^{-n}) \frac{dt}{t^{1-\alpha}} \\ &\lesssim \|b\|_* \frac{1}{\Psi^{-1}(r^{-n})} \int_{2r}^\infty \left(1 + \ln\frac{t}{r}\right) \|f\|_{L_\Phi(B(x_0,t))} \Psi^{-1}(t^{-n}) \frac{dt}{t}. \end{aligned} \tag{87}$$

In order to estimate J_2 note that

$$J_2 \approx \left\|b(\cdot) - b_B\right\|_{L_\Psi(B)} \int_{\complement(2B)} \frac{|f(y)|}{|x_0 - y|^{n-\alpha}} dy. \tag{88}$$

By Lemma 24, we get

$$J_2 \lesssim \|b\|_* \frac{1}{\Psi^{-1}(r^{-n})} \int_{\complement(2B)} \frac{|f(y)|}{|x_0 - y|^{n-\alpha}} dy. \quad (89)$$

Thus, by (52)

$$J_2 \lesssim \|b\|_* \frac{1}{\Psi^{-1}(r^{-n})} \int_{2r}^{\infty} \|f\|_{L_\Phi(B(x_0,t))} \Psi^{-1}(t^{-n}) \frac{dt}{t}. \quad (90)$$

Summing J_1 and J_2 we get

$$\begin{aligned} &\|[b, I_\alpha] f_2\|_{L_\Psi(B)} \\ &\quad \lesssim \|b\|_* \frac{1}{\Psi^{-1}(r^{-n})} \\ &\quad\quad \times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_\Phi(B(x_0,t))} \Psi^{-1}(t^{-n}) \frac{dt}{t}. \end{aligned} \quad (91)$$

Finally,

$$\begin{aligned} &\|[b, I_\alpha] f\|_{L_\Psi(B)} \\ &\quad \lesssim \|b\|_* \|f\|_{L_\Phi(2B)} + \|b\|_* \frac{1}{\Psi^{-1}(r^{-n})} \\ &\quad\quad \times \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_\Phi(B(x_0,t))} \Psi^{-1}(t^{-n}) \frac{dt}{t}, \end{aligned} \quad (92)$$

and the statement of Lemma 33 follows by (56). □

Theorem 34. *Let $0 < \alpha < n$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Let Φ be a Young function and Ψ defined, via its inverse, by setting, for all $t \in (0, \infty)$, $\Psi^{-1}(t) := \Phi^{-1}(t) t^{-\alpha/n}$, and $1 < a_\Phi \le b_\Phi < \infty$ and $1 < a_\Psi \le b_\Psi < \infty$. (φ_1, φ_2) and (Φ, Ψ) satisfy the condition*

$$\int_r^{\infty} \left(1 + \ln \frac{t}{r}\right) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x, s)}{\Phi^{-1}(s^{-n})} \Psi^{-1}(t^{-n}) \frac{dt}{t} \le C \varphi_2(x, r), \quad (93)$$

where C does not depend on x and r.

Then the operator $[b, I_\alpha]$ is bounded from $M_{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_{\Psi,\varphi_2}(\mathbb{R}^n)$. Moreover

$$\|[b, I_\alpha] f\|_{M_{\Psi,\varphi_2}} \lesssim \|b\|_* \|f\|_{M_{\Phi,\varphi_1}}. \quad (94)$$

Proof. The statement of Theorem 34 follows by Lemma 33 and Theorem 31 in the same manner as in the proof of Theorem 14. □

If we take $\Phi(t) = t^p$, $\Psi(t) = t^q$, $1 < p, q < \infty$, at Theorem 34 we get following corollary which was proved in [40] (see, also [54]).

Corollary 35. *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = (1/p) - (\alpha/n)$, $b \in \mathrm{BMO}(\mathbb{R}^n)$, and (φ_1, φ_2) satisfy the condition*

$$\int_r^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x, s)\, s^{n/p}}{t^{(n/q)+1}} dt \le C \varphi_2(x, r), \quad (95)$$

where C does not depend on x and r. Then $[b, I_\alpha]$ is bounded from M_{p,φ_1} to M_{q,φ_2}.

In the case $\varphi_1(x, r) = \Phi^{-1}(r^{-n})/\Phi^{-1}(r^{-\lambda_1})$, $\varphi_2(x, r) = \Psi^{-1}(r^{-n})/\Psi^{-1}(r^{-\lambda_2})$ from Theorem 34 we get the following Spanne type theorem for the boundedness of the operator $[b, I_\alpha]$ on Orlicz-Morrey spaces.

Corollary 36. *Let $0 < \alpha < n$, $0 \le \lambda_1, \lambda_2 < n$, and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Let also Φ be a Young function and Ψ defined, via its inverse, by setting, for all $t \in (0, \infty)$, $\Psi^{-1}(t) := \Phi^{-1}(t) t^{-\alpha/n}$, $1 < a_\Phi \le b_\Phi < \infty$, $1 < a_\Psi \le b_\Psi < \infty$, and (Φ, Ψ) satisfy the condition*

$$\int_r^{\infty} \left(1 + \ln \frac{t}{r}\right) \frac{\Psi^{-1}(t^{-n})}{\Phi^{-1}(t^{-\lambda_1})} \frac{dt}{t} \le C \frac{\Psi^{-1}(r^{-n})}{\Psi^{-1}(r^{-\lambda_2})}, \quad (96)$$

where C does not depend on r. Then $[b, I_\alpha]$ is bounded from $M_{\Phi,\lambda_1}(\mathbb{R}^n)$ to $M_{\Psi,\lambda_2}(\mathbb{R}^n)$.

Remark 37. If we take $\Phi(t) = t^p$, $\Psi(t) = t^q$, $1 \le p, q < \infty$, at Corollary 36 we get Spanne type boundedness of $[b, I_\alpha]$; that is, if $0 < \alpha < n$, $1 < p < n/\alpha$, $0 < \lambda < n - \alpha p$, $(1/p) - (1/q) = \alpha/n$, and $\lambda/p = \mu/q$, then for $p > 1$ the operator $[b, I_\alpha]$ is bounded from $M_{p,\lambda}(\mathbb{R}^n)$ to $M_{q,\mu}(\mathbb{R}^n)$ and for $p = 1$, $[b, I_\alpha]$ is bounded from $M_{1,\lambda}(\mathbb{R}^n)$ to $WM_{q,\mu}(\mathbb{R}^n)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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