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Research Article

Multilinear Commutators of Calderón-Zygmund Operator on Generalized Weighted Morrey Spaces

Vagif S. Guliyev[1,2] and Farida Ch. Alizadeh[2]

[1] *Department of Mathematics, Ahi Evran University, 40200 Kirsehir, Turkey*
[2] *Institute of Mathematics and Mechanics, ANAS, AZ1141 Baku, Azerbaijan*

Correspondence should be addressed to Vagif S. Guliyev; vagif@guliyev.com

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The boundedness of multilinear commutators of Calderón-Zygmund operator $T_{\vec{b}}$ on generalized weighted Morrey spaces $M_{p,\varphi}(w)$ with the weight function w belonging to Muckenhoupt's class A_p is studied. When $1 < p < \infty$ and $\vec{b} = (b_1, \dots, b_m)$, $b_i \in \mathrm{BMO}$, $i = 1, \dots, m$, the sufficient conditions on the pair (φ_1, φ_2) which ensure the boundedness of the operator $T_{\vec{b}}$ from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ are found. In all cases the conditions for the boundedness of $T_{\vec{b}}$ are given in terms of Zygmund-type integral inequalities on (φ_1, φ_2), which do not assume any assumption on monotonicity of $\varphi_1(x,r)$, $\varphi_2(x,r)$ in r.

1. Introduction

Let T be a Calderón-Zygmund singular integral operator and $b \in \mathrm{BMO}(\mathbb{R}^n)$. A well known result of Coifman et al. [1] states that if $b \in \mathrm{BMO}(\mathbb{R}^n)$ and T is a Calderón-Zygmund operator, then the commutator operator $[b,T]f = T(bf) - bTf$ is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. The commutators of Calderón-Zygmund operator play an important role in studying the regularity of solutions of elliptic, parabolic and ultraparabolic partial differential equations of second order (see, [2–7]).

The classical Morrey spaces $M_{p,\lambda}$ were originally introduced by Morrey in [8] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [2–4, 8, 9].

Let $\vec{b} = (b_1, \dots, b_m)$, b_j, and $1 \le j \le m$ be locally integrable functions when we consider multilinear commutators as defined by

$$T_{\vec{b}} f(x) = \int_{R^n} \prod_{j=1}^{m} \left(b_j(x) - b_j(y)\right) K(x,y) f(y) \, dy, \qquad (1)$$

where $K(x,y)$ is Calderón-Zygmund kernel. That is, for all distinct $x, y \in \mathbb{R}^n$, and all z with $2|x-z| < |x-y|$, there exist positive constant C and γ such that

(i) $|K(x,y)| \le C|x-y|^{-n}$
(ii) $|K(x,y) - K(z,y)| \le C(|x-z|^{\gamma}/|x-y|^{n+\gamma})$; and
(iii) $|K(y,x) - K(y,z)| \le C(|x-z|^{\gamma}/|x-y|^{n+\gamma})$

when $m = 1$, it is the classical commutator which was introduced by Coifman et al. in [1]. It is well known that Calderón-Zygmund operators play an important role in harmonic analysis (see [10–12]).

We define the generalized weighed Morrey spaces as follows.

Definition 1. Let $1 \le p < \infty$, φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and w be non-negative measurable function on $\mathbb{R}^n$. We denote by $M_{p,\varphi}(w)$ the generalized weighted Morrey space, the space of all functions $f \in L^{\mathrm{loc}}_{p,w}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{M_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} w(B(x,r))^{-1/p} \|f\|_{L_{p,w}(B(x,r))}, \qquad (2)$$

where $L_{p,w}(B(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{L_{p,w}(\mathbb{R}^n)} = \left(\int_{B(x,r)} |f(y)|^p w(y)dy\right)^{1/p}. \quad (3)$$

Furthermore, by $WM_{p,\varphi}(w)$ we denote the weak generalized weighted Morrey space of all functions $f \in WL^{\text{loc}}_{p,w}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} w(B(x,r))^{-1/p} \|f\|_{WL_{p,w}(B(x,r))} < \infty, \quad (4)$$

where $WL_{p,w}(B(x,r))$ denotes the weak $L_{p,w}$-space of measurable functions f for which

$$\|f\|_{WL_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_{p,w}(\mathbb{R}^n)} = \sup_{t>0} t\left(\int_{\{y\in B(x,r):|f(y)|>t\}} w(y)dy\right)^{1/p}. \quad (5)$$

Remark 2. (1) If $w \equiv 1$, then $M_{p,\varphi}(1) = M_{p,\varphi}$ is the generalized Morrey space.

(2) If $\varphi(x,r) \equiv w(B(x,r))^{(\kappa-1)/p}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(w)$ is the weighted Morrey space.

(3) If $\varphi(x,r) \equiv \nu(B(x,r))^{\kappa/p} w(B(x,r))^{-1/p}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(\nu, w)$ is the two weighted Morrey space.

(4) If $w \equiv 1$ and $\varphi(x,r) = r^{(\lambda-n)/p}$ with $0 < \lambda < n$, then $M_{p,\varphi}(w) = L_{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space and $WM_{p,\varphi}(w) = WL_{p,\lambda}(\mathbb{R}^n)$ is the weak Morrey space.

(5) If $\varphi(x,r) \equiv w(B(x,r))^{-1/p}$, then $M_{p,\varphi}(w) = L_{p,w}(\mathbb{R}^n)$ is the weighted Lebesgue space.

The commutators are useful in many nondivergence elliptic equations with discontinuous coefficients, [2–5]. In the recent development of commutators, Pérez and Trujillo-González [13] generalized these multilinear commutators and proved the weighted Lebesgue estimates. The weighted Morrey spaces $L_{p,\kappa}(w)$ was introduced by Komori and Shirai [14]. Moreover, they showed that some classical integral operators and corresponding commutators are bounded in weighted Morrey spaces. Feng in [15] obtained the boundedness of the multilinear commutators in weighted Morrey spaces $L_{p,\kappa}(w)$ for $1 < p < \infty$ and $0 < \kappa < 1$, where the symbol $\vec{b}$ belongs to bounded mean oscillation $(\text{BMO})^n$. Furthermore, was given the weighted weak type estimate of these operators in weighted Morrey spaces of $L_{p,\kappa}(w)$ for $p = 1$ and $0 < \kappa < 1$.

Recently, the generalized weighted Morrey spaces $M_{p,\varphi}(w)$ introduced by Guliyev [16, 17]. Moreover, in [16, 17] he studied the boundedness of the sublinear operators and their higher order commutators generated by Calderón-Zygmund operators and Riesz potentials in these spaces (see, also [18–20]).

The following statement was proved in [18].

Theorem A. *Let $1 \le p < \infty$, $w \in A_p$ and (φ_1, φ_2) satisfies the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\, w(B(x,s))^{1/p}}{w(B(x,t))^{1/p}} \frac{dt}{t} \le C\varphi_2(x,r), \quad (6)$$

where C does not depend on x and r. Then the operator T is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $p > 1$ and from $M_{1,\varphi_1}(w)$ to $WM_{1,\varphi_2}(w)$.

Remark 3. Note that, Theorem A was proved in the case $w \equiv 1$ in [21] and in the case $w \in A_p$ and $\varphi_1(x,r) = \varphi_2(x,r) \equiv w(B(x,r))^{(\kappa-1)/p}$ in [14].

Definition 4. $\text{BMO}(\mathbb{R}^n)$ is the Banach space modulo constants with the norm $\|\cdot\|_*$ defined by

$$\|b\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy < \infty, \quad (7)$$

where $b \in L_1^{\text{loc}}(\mathbb{R}^n)$ and

$$b_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} b(y)\, dy. \quad (8)$$

In this paper, we prove the boundedness of the multilinear commutators of Calderón-Zygmund operator $T_{\vec{b}}$ from one generalized weighted Morrey space $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$ for $1 < p < \infty$ and $\vec{b} = (b_1, \dots, b_m)$, $b_i \in \text{BMO}$, $i = 1, \dots, m$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Main Results

In the following, main results are given. First, we present some estimates which are the main tools to prove our theorems, for the boundedness of the multilinear commutator operators $T_{\vec{b}}$ on the generalized weighted Morrey spaces.

Theorem 5. *Let $1 < p < \infty$, $w \in A_p$, $\vec{b} = (b_1, \dots, b_m)$, $b_i \in BMO$, $i = 1, \dots, m$, and $T_{\vec{b}}$ be a multilinear commutators defined as* (6). *Then*

$$\|T_{\vec{b}} f\|_{L_{p,w}(B(x_0,r))} \le C \|\vec{b}\|_* w(B(x_0,r))^{1/p} \times \int_{2r}^\infty \ln^m\left(e + \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-1/p} \frac{dt}{t} \quad (9)$$

holds for any ball $B = B(x_0, r)$ and for all $f \in L^{\text{loc}}_{p,w}(\mathbb{R}^n)$, where C does not depend on f, $x_0 \in \mathbb{R}^n$ and $r > 0$.

Theorem 6. *Let $w \in A_1$, $\vec{b} = (b_1, \ldots, b_m)$, $b_i \in BMO$, $i = 1, \ldots, m$, and $T_{\vec{b}}$ be a multilinear commutators defined as* (6). *Then*

$$\|T_{\vec{b}} f\|_{WL_{1,w}(B(x_0,r))} \leq C\|\vec{b}\|_* w(B(x_0,r)) \times \int_{2r}^{\infty} \ln^m\left(e + \frac{t}{r}\right) \|f\|_{L_{\Phi,w}(B(x_0,t))} w(B(x_0,t))^{-1} \frac{dt}{t} \quad (10)$$

holds for any ball $B = B(x_0, r)$ and for all $f \in L_1^{\text{loc}}(\mathbb{R}^n)$, where C does not depend on f, $x_0 \in \mathbb{R}^n$ and $r > 0$, where $\Phi(t) = t \ln^m(e + t)$ and $\|f\|_{L_{\Phi,w}} = \|\Phi(|f|)\|_{L_{1,w}}$.

Now we give a theorem about the boundedness of the multilinear commutator operator $T_{\vec{b}}$ on the generalized weighted Morrey spaces.

Theorem 7. *Let $1 < p < \infty$, $w \in A_p$, and (φ_1, φ_2) satisfies the condition*

$$\int_r^{\infty} \ln^m\left(e + \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\, w(B(x,s))^{1/p}}{w(B(x,t))^{1/p}} \frac{dt}{t} \leq C\varphi_2(x,r), \quad (11)$$

where C does not depend on x and r. Let $\vec{b} = (b_1, \ldots, b_m)$, $b_i \in BMO$, $i = 1, \ldots, m$. Then the operator $T_{\vec{b}}$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$. Moreover,

$$\|T_{\vec{b}} f\|_{M_{p,\varphi_2}(w)} \lesssim \|\vec{b}\|_* \|f\|_{M_{p,\varphi_1}(w)}. \quad (12)$$

Theorem 8. *Let $w \in A_1$, and (φ_1, φ_2) satisfies the condition*

$$\int_r^{\infty} \ln^m\left(e + \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\, w(B(x,s))}{w(B(x,t))} \frac{dt}{t} \leq C\varphi_2(x,r), \quad (13)$$

where C does not depend on x and r. Let $\vec{b} = (b_1, \ldots, b_m)$, $b_i \in BMO$, $i = 1, \ldots, m$. Then the operator $T_{\vec{b}}$ is bounded from $M_{\Phi,\varphi_1}(w)$ to $WM_{1,\varphi_2}(w)$. Moreover,

$$\|T_{\vec{b}} f\|_{WM_{1,\varphi_2}(w)} \lesssim \|\vec{b}\|_* \|f\|_{M_{\Phi,\varphi_1}(w)}, \quad (14)$$

where $\Phi(t) = t \ln^m(e + t)$ and $\|f\|_{M_{\Phi,\varphi}(w)} = \|\Phi(|f|)\|_{M_{1,\varphi}(w)}$.

When $\varphi_1(x,r) = \varphi_2(x,r) \equiv w(B(x,r))^{(\kappa-1)/p}$, from Theorem 7 we also get the following new result.

Corollary 9. *Let $1 \leq p < \infty$, $0 < \kappa < 1$, $w \in A_p$, $\vec{b} = (b_1, \ldots, b_m)$, $b_i \in BMO$, $i = 1, \ldots, m$. Then the operator $T_{\vec{b}}$ is bounded on $L_{p,\kappa}(w)$ for $p > 1$ and from $L_{\Phi,\kappa}(w)$ to $WM_{1,\kappa}(w)$ for $p = 1$, where $\Phi(t) = t\ln^m(e+t)$ and $\|f\|_{L_{\Phi,\kappa}(w)} = \|\Phi(|f|)\|_{L_{1,\kappa}(w)}$.*

Proof. Let $1 \leq p < \infty$, $w \in A_p$, $0 < \kappa < 1$ and $\vec{b} = (b_1, \ldots, b_m)$, $b_i \in$ BMO, $i = 1, \ldots, m$. Then the pair $(w(B(x,r))^{(\kappa-1)/p}, w(B(x,r))^{(\kappa-1)/p})$ satisfies the condition (11) for $p > 1$ and the condition (13) for $p = 1$. Indeed, by Lemma 10 there exists $C > 0$ and $\delta > 0$ such that for all $x \in \mathbb{R}^n$ and $t > r$:

$$w(B(x,t)) \geq C\left(\frac{t}{r}\right)^{n\delta} w(B(x,r)). \quad (15)$$

Then

$$\begin{aligned}
&\int_r^{\infty} \ln^m\left(e + \frac{t}{r}\right) \frac{\operatorname{ess\,inf}_{t<s<\infty} w(B(x,s))^{\kappa/p}}{w(B(x,t))^{1/p}} \frac{dt}{t} \\
&= \int_r^{\infty} \ln^m\left(e + \frac{t}{r}\right) w(B(x,t))^{(\kappa-1)/p} \frac{dt}{t} \\
&\lesssim \int_r^{\infty} \ln^m\left(e + \frac{t}{r}\right) \left(\left(\frac{t}{r}\right)^{n\delta} w(B(x,r))\right)^{(\kappa-1)/p} \frac{dt}{t} \\
&= w(B(x,r))^{(\kappa-1)/p} \int_r^{\infty} \ln^m\left(e + \frac{t}{r}\right) \left(\frac{t}{r}\right)^{n\delta((\kappa-1)/p)} \frac{dt}{t} \\
&= w(B(x,r))^{(\kappa-1)/p} \int_1^{\infty} \ln^m(e+\tau)\, \tau^{n\delta((\kappa-1)/p)} \frac{d\tau}{\tau} \\
&\approx w(B(x,r))^{(\kappa-1)/p}.
\end{aligned} \quad (16)$$

□

Note that from Corollary 9 was proved in [15].

3. Some Lemmas

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space of points $x = (x_1, \ldots, x_n)$ with norm $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$. For $x \in \mathbb{R}^n$ and $r > 0$, denote $B(x,r)$ the open ball centered at x of radius r. Let ${}^{\complement}B(x,r)$ be the complement of the ball $B(x,r)$, and $|B(x,r)|$ be the Lebesgue measure of $B(x,r)$.

A weight function is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0, \infty)$ almost everywhere. For a weight w and a measurable set E, we define $w(E) = \int_E w(x)dx$, the Lebesgue measure of E by $|E|$, and the characteristic function of E by χ_E. Given a weight w, we say that w satisfies the doubling condition if there is a constant $D > 0$ such that $w(2B) \leq Dw(B)$ for any ball B. When w satisfies the doubling condition, we denote $w \in \Delta_2$, for short.

If w is a weight function, then we denote the weighted Lebesgue space by $L_p(w) \equiv L_p(\mathbb{R}^n, w)$ with the norm

$$\|f\|_{L_{p,w}} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)dx\right)^{1/p} < \infty, \quad \text{when } 1 \leq p < \infty \quad (17)$$

and $\|f\|_{L_{\infty,w}} = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} |f(x)| w(x)$ when $p = \infty$.

We recall that a weight function w is in the Muckenhoupt's class A_p, $1 < p < \infty$, if

$$[w]_{A_p} := \sup_B [w]_{A_p(B)} = \sup_B \left(\frac{1}{|B|} \int_B w(x)\,dx \right) \times \left(\frac{1}{|B|} \int_B w(x)^{1-p'}\,dx \right)^{p-1} < \infty, \tag{18}$$

where the sup is taken with respect to all the balls B and $1/p + 1/p' = 1$. Note that, for all balls B we have

$$[w]^{1/p}_{A_p(B)} = |B|^{-1} \|w\|^{1/p}_{L_1(B)} \left\| w^{-1/p} \right\|_{L_{p'}(B)} \geq 1 \tag{19}$$

by Hölder's inequality. For $p = 1$, the class A_1 is defined by the condition $Mw(x) \leq Cw(x)$ with $[w]_{A_1} = \sup_{x\in\mathbb{R}^n}(Mw(x)/w(x))$, and for $p = \infty$ we define $A_\infty = \bigcup_{1\leq p<\infty} A_p$.

Lemma 10 (see [22]). *We have the following:*

(1) *If $w \in A_p$ for some $1 \leq p < \infty$, then $w \in \Delta_2$. Moreover, for all $\lambda > 1$ we have*

$$w(\lambda B) \leq \lambda^{np} [w]_{A_p} w(B). \tag{20}$$

(2) *If $w \in A_\infty$, then $w \in \Delta_2$. Moreover, for all $\lambda > 1$ we have*

$$w(\lambda B) \leq 2^{\lambda^n} [w]^{\lambda^n}_{A_\infty} w(B). \tag{21}$$

(3) *If $w \in A_p$ for some $1 \leq p \leq \infty$, then there exist $C > 0$ and $\delta > 0$ such that for any ball B and a measurable set $S \subset B$,*

$$\frac{w(S)}{w(B)} \leq C \left(\frac{|S|}{|B|} \right)^{\delta}. \tag{22}$$

The following results are proved by Pérez and Trujillo-González [13].

Lemma 11. *Let $1 < p < \infty$ and $w \in A_p$ and suppose that $\vec{b} = (b_1, \ldots, b_m)$, $b_i \in BMO$, $i = 1, \ldots, m$, then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^n} \left| T_{\vec{b}} f(x) \right|^p w(x)\,dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx. \tag{23}$$

Although the commutators with BMO function are not of weak type $(1,1)$, they have the following inequality.

Lemma 12. *Let $w \in A_\infty$ and suppose that $\vec{b} = (b_1, \ldots, b_m)$, $b_i \in BMO$, $i = 1, \ldots, m$, then there exists a constant $C > 0$ such that*

$$\sup_{t>0} \frac{1}{\Phi(1/t)} w\left(x \in \mathbb{R}^n : \left| T_{\vec{b}} f(x) \right| > t \right) \leq C \sup_{t>0} \frac{1}{\Phi(1/t)} w\left(x \in \mathbb{R}^n : \left| M_\Phi \left(\left\| \vec{b} \right\| f \right)(x) \right| > t \right), \tag{24}$$

where $\Phi(t) = t \ln^m (e + t)$.

Lemma 13. *Let $w \in A_1$ and suppose that $\vec{b} = (b_1, \ldots, b_m)$, $b_i \in BMO$, $i = 1, \ldots, m$, then there exists a constant $C > 0$ such that*

$$w\left(x \in \mathbb{R}^n : \left| T_{\vec{b}} f(x) \right| > \lambda \right) \leq C \int_{\mathbb{R}^n} \Phi\left(|f(x)| \right) w(x)\,dx, \tag{25}$$

where $\Phi(t) = t \ln^m (e + t)$.

In this paper, we need the following statement on the boundedness of the Hardy type operator

$$(H_1 g)(t) := \frac{1}{t} \int_0^t \ln^m \left(e + \frac{t}{r} \right) g(r)\,d\mu(r), \quad 0 < t < \infty, \tag{26}$$

where μ be a non-negative Borel measure on $(0, \infty)$.

Theorem 14. *The inequality*

$$\operatorname*{ess\,sup}_{t>0} w(t) H_1 g(t) \leq c \operatorname*{ess\,sup}_{t>0} v(t) g(t) \tag{27}$$

holds for all non-negative and non-increasing g on $(0, \infty)$ if and only if

$$A_1 := \sup_{t>0} \frac{w(t)}{t} \int_0^t \ln^m \left(e + \frac{t}{r} \right) \frac{d\mu(r)}{\operatorname*{ess\,sup}_{0<s<r} v(s)} < \infty, \tag{28}$$

and $c \approx A_1$.

Note that, Theorem 14 is proved analogously to Theorem 4.3 in [21].

Lemma 15 (see [23, Theorem 5, page 236]). *Let $w \in A_\infty$. Then the norm of $BMO(\mathbb{R}^n)$ is equivalent to the norm of $BMO(w)$, where*

$$BMO(w) = \left\{ b : \|b\|_{*,w} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{w(B(x,r))} \times \int_{B(x,r)} \left| b(y) - b_{B(x,r),w} \right| w(y)\,dy < \infty \right\},$$
$$b_{B(x,r),w} = \frac{1}{w(B(x,r))} \int_{B(x,r)} b(y) w(y)\,dy. \tag{29}$$

Remark 16. (1) The John-Nirenberg inequality: there are constants $C_1, C_2 > 0$, such that for all $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $\beta > 0$

$$\left| \left\{ x \in B : \left| b(x) - b_B \right| > \beta \right\} \right| \leq C_1 |B| e^{-C_2 \beta / \|b\|_*}, \quad \forall B \subset \mathbb{R}^n. \tag{30}$$

(2) For $1 < p < \infty$ the John-Nirenberg inequality implies that

$$\|b\|_* \approx \sup_B \left(\frac{1}{|B|} \int_B \left| b(y) - b_B \right|^p dy \right)^{1/p} \tag{31}$$

and for $1 \le p < \infty$ and $w \in A_\infty$

$$\|b\|_* \approx \sup_B \left(\frac{1}{w(B)} \int_B |b(y) - b_B|^p w(y) dy \right)^{1/p}. \quad (32)$$

Indeed, it follows from the John-Nirenberg inequality and using Lemma 10 (3), we get

$$w(\{x \in B : |b(x) - b_B| > \beta\}) \le Cw(B) e^{-C_2 \beta\delta/\|b\|_*} \quad (33)$$

for some $\delta > 0$. Hence, this inequality implies that

$$\begin{aligned} &\int_B |b(y) - b_B|^p w(y) dy \\ &= p \int_0^\infty \beta^{p-1} w(\{x \in B : |b(x) - b_B| > \beta\}) d\beta \\ &\le Cw(B) \int_0^\infty \beta^{p-1} e^{-C_2\beta\delta/\|b\|_*} d\beta = Cw(B) \|b\|_*^p. \end{aligned} \quad (34)$$

To prove the requested equivalence we also need to have the right inequality, that is easily obtained using Hölder inequality, then we get (32). Note that (31) follows from (32) in the case $w \equiv 1$.

The following lemma was proved in [24].

Lemma 17. *Let b be a function in $BMO(\mathbb{R}^n)$. Let also $1 \le p < \infty$, $x \in \mathbb{R}^n$, and $r_1, r_2 > 0$. Then*

$$\begin{aligned} &\left(\frac{1}{|B(x, r_1)|} \int_{B(x,r_1)} |b(y) - b_{B(x,r_2)}|^p dy \right)^{1/p} \\ &\quad \le C \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right) \|b\|_*, \end{aligned} \quad (35)$$

where $C > 0$ is independent of f, x, r_1 and r_2.

The following lemma was proved in [17].

Lemma 18. *(i) Let $w \in A_\infty$ and b be a function in $BMO(\mathbb{R}^n)$. Let also $1 \le p < \infty$, $x \in \mathbb{R}^n$, and $r_1, r_2 > 0$. Then*

$$\begin{aligned} &\left(\frac{1}{w(B(x, r_1))} \int_{B(x,r_1)} |b(y) - b_{B(x,r_2),w}|^p w(y) dy \right)^{1/p} \\ &\quad \le C \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right) \|b\|_*, \end{aligned} \quad (36)$$

where $C > 0$ is independent of f, x, r_1 and r_2.

(ii) Let $w \in A_p$ and b be a function in $BMO(\mathbb{R}^n)$. Let also $1 < p < \infty$, $x \in \mathbb{R}^n$, and $r_1, r_2 > 0$. Then

$$\begin{aligned} &\left(\frac{1}{w^{1-p'}(B(x, r_1))} \int_{B(x,r_1)} |b(y) - b_{B(x,r_2),w}|^{p'} \right. \\ &\qquad \left. \times w(y)^{1-p'} dy \right)^{1/p'} \end{aligned} \quad (37)$$

$$\le C \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right) \|b\|_*,$$

where $C > 0$ is independent of f, x, r_1 and r_2.

4. Proof of the Theorems

Proof of Theorem 5. Let $p \in (1, \infty)$. For arbitrary $x_0 \in \mathbb{R}^n$ and $r > 0$, set $B = B(x_0, r)$. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement(2B)}$. Hence

$$\|T_{\vec{b}} f\|_{L_{p,w}(B)} \le \|T_{\vec{b}} f_1\|_{L_{p,w}(B)} + \|T_{\vec{b}} f_2\|_{L_{p,w}(B)}. \quad (38)$$

From the boundedness of $T_{\vec{b}}$ in $L_p(w)$ (see Lemma 11) it follows that:

$$\|T_{\vec{b}} f_1\|_{L_{p,w}(B)} \le \|T_{\vec{b}} f_1\|_{L_{p,w}} \lesssim \|\vec{b}\|_* \|f_1\|_{L_{p,w}} = \|\vec{b}\|_* \|f\|_{L_{p,w}(2B)}. \quad (39)$$

For the term $\|T_{\vec{b}} f_2\|_{L_{p,w}(B)}$, without loss of generality, we can assume $m = 2$. Thus, the operator $T_{\vec{b}} f_2$ can be divided into four parts

$$\begin{aligned} T_{\vec{b}} f_2(x) &= (b_1(x) - (b_1)_{B,w})(b_2(x) - (b_2)_{B,w}) \\ &\quad \times \int_{\mathbb{R}^n} K(x,y) f_2(y) dy \\ &\quad + \int_{\mathbb{R}^n} K(x,y)(b_1(y) - (b_1)_{B,w}) \\ &\qquad \times (b_2(y) - (b_2)_{B,w}) f_2(y) dy \\ &\quad - (b_1(x) - (b_1)_{B,w}) \\ &\quad \times \int_{\mathbb{R}^n} K(x,y)(b_2(y) - (b_2)_{B,w}) f_2(y) dy \\ &\quad - (b_2(x) - (b_2)_{B,w}) \\ &\quad \times \int_{\mathbb{R}^n} K(x,y)(b_1(y) - (b_1)_{B,w}) f_2(y) dy \\ &= I_1(x) + I_2(x) + I_3(x) + I_4(x). \end{aligned} \quad (40)$$

For $x \in B$ we have

$$\begin{aligned} |T_{\vec{b}} f_2(x)| &\le |I_1(x)| + |I_2(x)| + |I_3(x)| + |I_4(x)| \\ &\lesssim |b_1(x) - (b_1)_{B,w}| |b_2(x) - (b_2)_{B,w}| \\ &\quad \times \int_{\complement(2B)} \frac{|f(y)|}{|x_0 - y|^n} dy \\ &\quad + \int_{\complement(2B)} |b_1(y) - (b_1)_{B,w}| \\ &\qquad \times |b_2(y) - (b_2)_{B,w}| \frac{|f(y)|}{|x_0 - y|^n} dy \\ &\quad + |b_1(x) - (b_1)_{B,w}| \\ &\quad \times \int_{\complement(2B)} |b_2(y) - (b_2)_{B,w}| \frac{|f(y)|}{|x_0 - y|^n} dy \end{aligned}$$

$$+\left|b_2(x)-(b_2)_{B,w}\right| \times \int_{\complement(2B)} \left|b_1(y)-(b_1)_{B,w}\right| \frac{|f(y)|}{|x_0-y|^n} dy. \tag{41}$$

Then

$$\begin{aligned}
&\|T_{\vec{b}} f_2\|_{L_{p,w}(B)} \\
&\lesssim \left(\int_B \left(\int_{\complement(2B)} \frac{\prod_{j=1}^2 \left|b_i(y)-(b_i)_{B,w}\right|}{|x_0-y|^n} \times |f(y)| dy \right)^p w(x)\,dx \right)^{1/p} \\
&+ \left(\int_B \left|b_1(x)-(b_1)_{B,w}\right| \times \left(\int_{\complement(2B)} \frac{\left|b_2(y)-(b_2)_{B,w}\right|}{|x_0-y|^n} |f(y)| dy \right)^p \times w(x)\,dx \right)^{1/p} \\
&+ \left(\int_B \left|b_2(x)-(b_2)_{B,w}\right| \times \left(\int_{\complement(2B)} \frac{\left|b_1(y)-(b_1)_{B,w}\right|}{|x_0-y|^n} |f(y)| dy \right)^p \times w(x)\,dx \right)^{1/p} \\
&+ \left(\int_B \left(\int_{\complement(2B)} \frac{\prod_{j=1}^2 \left|b_i(x)-(b_i)_{B,w}\right|}{|x_0-y|^n} \times |f(y)| dy \right)^p w(x)\,dx \right)^{1/p} \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{42}$$

Let us estimate I_1

$$\begin{aligned}
I_1 &= w(B)^{1/p} \int_{\complement(2B)} \frac{\prod_{j=1}^2 \left|b_i(y)-(b_i)_{B,w}\right|}{|x_0-y|^n} |f(y)| dy \\
&\approx w(B)^{1/p} \int_{\complement(2B)} \prod_{j=1}^2 \left|b_i(y)-(b_i)_{B,w}\right| \times |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy \\
&\approx w(B)^{1/p} \int_{2r}^{\infty} \int_{2r\le|x_0-y|\le t} \prod_{j=1}^2 \left|b_i(y)-(b_i)_{B,w}\right| \times |f(y)| dy \frac{dt}{t^{n+1}} \\
&\lesssim w(B)^{1/p} \int_{2r}^{\infty} \int_{B(x_0,t)} \prod_{j=1}^2 \left|b_i(y)-(b_i)_{B,w}\right| \times |f(y)| dy \frac{dt}{t^{n+1}}.
\end{aligned} \tag{43}$$

Applying Hölder's inequality and by Lemma 18, we get

$$\begin{aligned}
I_1 &\lesssim w(B)^{1/p} \times \int_{2r}^{\infty} \prod_{j=1}^2 \left(\int_{B(x_0,t)} \left|b_i(y)-(b_i)_{B,w}\right|^{2p'} w(y)^{1-2p'} dy \right)^{1/2p'} \times \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&\lesssim \prod_{j=1}^2 \|b_j\|_* w(B)^{1/p} \times \int_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)^2 \|w^{-1/p}\|_{L_{p'}(B(x_0,t))} \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&\lesssim \|\vec{b}\|_* w(B)^{1/p} \int_{2r}^{\infty} \ln^2\left(e+\frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0,t))} \times w(B(x_0,t))^{-1/p} \frac{dt}{t}.
\end{aligned} \tag{44}$$

Let us estimate I_2

$$\begin{aligned}
I_2 &= \left(\int_B \left|b_1(x)-(b_1)_{B,w}\right|^p w(x)\,dx \right)^{1/p} \times \int_{\complement(2B)} \frac{\left|b_2(y)-(b_2)_{B,w}\right|}{|x_0-y|^n} |f(y)| dy \\
&\lesssim \|b_1\|_* w(B)^{1/p} \int_{\complement(2B)} \left|b_2(y)-(b_2)_{B,w}\right| |f(y)| \times \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy \\
&\approx \|b_1\|_* w(B)^{1/p} \times \int_{2r}^{\infty} \int_{2r\le|x_0-y|\le t} \left|b_2(y)-(b_2)_{B,w}\right| |f(y)| dy \frac{dt}{t^{n+1}} \\
&\lesssim \|b_1\|_* w(B)^{1/p} \times \int_{2r}^{\infty} \int_{B(x_0,t)} \left|b_2(y)-(b_2)_{B,w}\right| |f(y)| dy \frac{dt}{t^{n+1}}.
\end{aligned} \tag{45}$$

Applying Hölder's inequality and by Lemma 18, we get

$$I_2 \lesssim \|b_1\|_* w(B)^{1/p} \times \int_{2r}^{\infty} \left(\int_{B(x_0,t)} \left|b_2(y) - (b_2)_{B,w}\right|^{p'} w(y)^{1-p'} dy \right)^{1/p'} \times \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}}$$
$$\lesssim \prod_{j=1}^{2} \left\|b_j\right\|_* w(B)^{1/p} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \left\|w^{-1/p}\right\|_{L_{p'}(B(x_0,t))} \times \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}}$$
$$\lesssim \left\|\vec{b}\right\|_* w(B)^{1/p} \times \int_{2r}^{\infty} \ln^2\left(e + \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-1/p} \frac{dt}{t}. \quad (46)$$

In the same way, we shall get the result of I_3

$$I_3 \lesssim \left\|\vec{b}\right\|_* w(B)^{1/p} \int_{2r}^{\infty} \ln^2\left(e + \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0,t))} \times w(B(x_0,t))^{-1/p} \frac{dt}{t}. \quad (47)$$

In order to estimate I_4 note that

$$I_4 = \left(\int_B \prod_{j=1}^{2} \left|b_i(x) - (b_i)_{B,w}\right|^p w(x) dx \right)^{1/p} \times \int_{\complement(2B)} \frac{|f(y)|}{|x_0 - y|^n} dy$$
$$\leq \prod_{j=1}^{2} \left(\int_B \left|b_i(x) - (b_i)_{B,w}\right|^{2p} w(x) dx \right)^{1/2p} \times \int_{\complement(2B)} \frac{|f(y)|}{|x_0 - y|^n} dy. \quad (48)$$

By Lemma 18, we get

$$I_4 \lesssim \left\|\vec{b}\right\|_* w(B)^{1/p} \int_{\complement(2B)} \frac{|f(y)|}{|x_0 - y|^n} dy. \quad (49)$$

Applying Hölder's inequality, we get

$$\int_{\complement(2B)} \frac{|f(y)|}{|x_0 - y|^n} dy \lesssim \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \left\|w^{-1/p}\right\|_{L_{p'}(B(x_0,t))} \frac{dt}{t^{n+1}}$$
$$\lesssim [w]_{A_p}^{1/p} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-1/p} \frac{dt}{t}. \quad (50)$$

Thus, by (50)

$$I_4 \lesssim \left\|\vec{b}\right\|_* w(B)^{1/p} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-1/p} \frac{dt}{t}. \quad (51)$$

Summing up I_1 and I_4, for all $p \in [1, \infty)$ we get

$$\left\|T_{\vec{b}} f_2\right\|_{L_{p,w}(B)} \lesssim \left\|\vec{b}\right\|_* w(B)^{1/p} \times \int_{2r}^{\infty} \ln^2\left(e + \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0,t))} \times w(B(x_0,t))^{-1/p} \frac{dt}{t}. \quad (52)$$

On the other hand,

$$\|f\|_{L_{p,w}(2B)} \approx |B| \, \|f\|_{L_{p,w}(2B)} \int_{2r}^{\infty} \frac{dt}{t^{n+1}}$$
$$\leq |B| \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}}$$
$$\leq w(B)^{1/p} \left\|w^{-1/p}\right\|_{L_{p'}(B)} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \frac{dt}{t^{n+1}}$$
$$\leq w(B)^{1/p} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \left\|w^{-1/p}\right\|_{L_{p'}(B(x_0,t))} \frac{dt}{t^{n+1}}$$
$$\leq [w]_{A_p}^{1/p} w(B)^{1/p} \times \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-1/p} \frac{dt}{t}. \quad (53)$$

Finally,

$$\left\|T_{\vec{b}} f\right\|_{L_{p,w}(B)} \lesssim \left\|\vec{b}\right\|_* \|f\|_{L_{p,w}(2B)} + \left\|\vec{b}\right\|_* w(B)^{1/p} \times \int_{2r}^{\infty} \ln^m\left(e + \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0,t))} \times w(B(x_0,t))^{-1/p} \frac{dt}{t}, \quad (54)$$

and the statement of Theorem 5 follows by (53). □

Proof of Theorem 6. Let $p = 1$. To deal with this result, we split f as above by $f = f_1 + f_2$, which yields

$$\left\|T_{\vec{b}} f\right\|_{WL_{1,w}(B)} \leq \left\|T_{\vec{b}} f_1\right\|_{WL_{1,w}(B)} + \left\|T_{\vec{b}} f_2\right\|_{WL_{1,w}(B)}. \quad (55)$$

From the boundedness of $T_{\vec{b}}$ from $L_{\Phi}(w)$ to $WL_{1,w}$ (see Lemma 13) it follows that:

$$\left\|T_{\vec{b}} f_1\right\|_{WL_{1,w}(B)} \leq \left\|T_{\vec{b}} f_1\right\|_{WL_{1,w}} \lesssim \left\|\vec{b}\right\|_* \|f_1\|_{L_{\Phi,w}} = \left\|\vec{b}\right\|_* \|f\|_{L_{\Phi,w}(2B)}. \quad (56)$$

For the last term $\|T_{\vec{b}} f_2\|_{WL_{1,w}(B)}$, without loss of generality, we still assume $m = 2$. By homogeneity it is enough to assume

$\lambda/2 = \|b_1\|_* = \|b_2\|_* = 1$ and hence, we only need to prove that

$$\begin{aligned} w\left(\left\{x \in B : \left|T_{\vec{b}} f_2(x)\right| > 1\right\}\right) \\ \lesssim w(B) \int_{2r}^{\infty} \|f\|_{L_{\Phi,w}(B(x_0,t))} w(B(x_0,t))^{-1} \frac{dt}{t} \end{aligned} \tag{57}$$

for all $B = B(x_0, r)$. In fact, by Lemma 12, we get

$$\begin{aligned} & w\left(x \in B : \left|T_{\vec{b}} f_2(x)\right| > 1\right) \\ & \leq \sup_{t>0} \frac{1}{\Phi(1/t)} w\left(x \in \mathbb{R}^n : \left|T_{\vec{b}} f_2(x)\right| > t\right) \\ & \lesssim \sup_{t>0} \frac{1}{\Phi(1/t)} w\left(x \in B : M_{\Phi}(f_2)(x) > t\right) \\ & = \sup_{t>0} \frac{1}{\Phi(1/t)} w\left(x \in B : M(\Phi(f_2))(x) > t\right), \end{aligned} \tag{58}$$

where $\Phi(t) = t \ln^m(e + t)$. We use the Fefferman-Stein maximal inequality

$$\int_{\{x\in\mathbb{R}^n : Mf(x)>t\}} \phi(t)\, dx \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)|\, M\phi(x)\, dx, \tag{59}$$

for any functions f and $\phi \geq 0$. This yields

$$\begin{aligned} & w\left(\left\{x \in B : M(\Phi(f_2))(x) > t\right\}\right) \\ & \lesssim \frac{1}{t} \int_{\{x\in\mathbb{R}^n : \Phi(f_2)(x)>t\}} \chi_B(x)\, w(x)\, dx \\ & \lesssim \frac{1}{t} \int_{\mathbb{R}^n} \Phi(f_2)(x)\, M(w\chi_B)(x). \end{aligned} \tag{60}$$

Then

$$\begin{aligned} & w\left(x \in B : \left|T_{\vec{b}} f_2(x)\right| > 1\right) \\ & \lesssim \sup_{t>0} \frac{1}{\Phi(1/t)} w\left(x \in B : M(\Phi(f_2))(x) > t\right) \\ & \lesssim \sup_{t>0} \frac{1}{t\Phi(1/t)} \int_{\mathbb{R}^n} \Phi(f_2)(x)\, M(w\chi_B)(x). \end{aligned} \tag{61}$$

□

Proof of Theorem 7. By Theorem 5 and Theorem 14 we have for $p > 1$

$$\begin{aligned} \left\|T_{\vec{b}} f\right\|_{M_{p,\varphi_2}(w)} & \lesssim \left\|\vec{b}\right\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \\ & \quad \times \int_r^{\infty} \ln^m\left(e + \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x,t))} \\ & \quad \times w(B(x,t))^{-1/p} \frac{dt}{t} \\ & = \left\|\vec{b}\right\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \\ & \quad \times \int_0^{r^{-1}} \ln^m\left(e + \frac{1}{tr}\right) \|f\|_{L_{p,w}(B(x,t^{-1}))} \\ & \quad \times w\left(B\left(x,t^{-1}\right)\right)^{-1/p} \frac{dt}{t} \\ & = \left\|\vec{b}\right\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_2\left(x,r^{-1}\right)^{-1} r \frac{1}{r} \\ & \quad \times \int_0^{r} \ln^m\left(e + \frac{r}{t}\right) \|f\|_{L_{p,w}(B(x,t^{-1}))} \\ & \quad \times w\left(B\left(x,t^{-1}\right)\right)^{-1/p} \frac{dt}{t} \\ & \lesssim \left\|\vec{b}\right\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_1\left(x,r^{-1}\right)^{-1} \\ & \quad \times w\left(B\left(x,r^{-1}\right)\right)^{-1/p} \|f\|_{L_{p,w}(B(x,r^{-1}))} \\ & = \left\|\vec{b}\right\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} w(B(x,r))^{-1/p} \\ & \quad \times \|f\|_{L_{p,w}(B(x,r))} \\ & = \left\|\vec{b}\right\|_* \|f\|_{M_{p,\varphi_1}(w)}. \end{aligned} \tag{62}$$

□

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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