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Linear and sublinear operators on generalized Morrey spaces with non-doubling measures

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Abstract. By using a geometric structure of the Euclidean space, the theory of generalized Morrey spaces is shown to be available in the non-doubling setting. Some classical operators are established to be bounded in the generalized spaces defined in the present paper.

1. Introduction

Morrey spaces are function spaces that appear not only in harmonic analysis but also in partial differential equations. In this paper, we can and do modify, generalize and extend the definition so that the definition fits the setting of non-doubling measures. As examples in the present paper below, our framework covers many existing function spaces related to Morrey spaces. The aim of the present paper is to define generalized Morrey spaces associated to Radon measures in general. Actually, we present the following definition.

Definition 1.1. Let $k \geq 1$ and let μ be a Radon measure on $\mathbb{R}^d$.

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(1) Let $x \in \mathbb{R}^d$ and $r > 0$. Then define

$$Q(x,r) := \{y \in \mathbb{R}^d : \max(|x_1 - y_1|, |x_2 - y_2|, \dots, |x_d - y_d|) \le r\},$$

where one wrote $x = (x_1, x_2, \dots, x_d)$ and $y = (y_1, y_2, \dots, y_d)$.

(2) The set $\mathcal{Q}(\mu)$ denotes the totality of the cubes $Q(x,r)$ with positive μ-measure. Given $Q \in \mathcal{Q}(\mu)$, denote by kQ the k-times expansion of Q.

(3) Let $1 \le p < \infty$ and $\varphi : \mathbb{R}^d \times (0,\infty) \to (0,\infty]$ be a function. One defines the generalized Morrey space $M_{p,\varphi}(k,\mu)$ for μ by the norm; for $f \in L^p_{\rm loc}(\mu)$, define

$$\|f\|_{M_{p,\varphi}(k,\mu)} := \sup_{Q(x,r) \in \mathcal{Q}(\mu)} \frac{1}{\varphi(x,kr)\mu(Q(x,kr))^{1/p}} \|f\|_{L^p(Q(x,r))}.$$

Here and below, it is understood that $\frac{a}{\infty} = 0$ for all $a \in \mathbb{R}$. The Morrey space $M_{p,\varphi}(k,\mu)$ denotes the set of all $f \in L^p_{\rm loc}(\mathbb{R}^d)$ for which the norm $\|f\|_{M_{p,\varphi}(k,\mu)}$ is finite.

Note that its origin is, of course, the classical generalized Morrey norm $\|f\|_{M_{p,\varphi}}$ given by $\|f\|_{M_{p,\varphi}} := \sup_{x \in \mathbb{R}^d,\, r>0} \frac{1}{\varphi(x,r)|Q(x,r)|^{1/p}} \|f\|_{L^p(Q(x,r))}$ defined by Nakai [28], where $|Q(x,r)|$ is the Lebesgue measure of $Q(x,r)$. See [30], [31] for further details. Here and below, for a measurable subset E, we write $|E|$ for the volume of E. Based upon this definition, we are going to prove the following theorem, which is again fundamental in the non-doubling setting.

Theorem 1.2. *Let $1 \le p < \infty$.*

(1) *There exists a function $\varphi^\dagger : \mathbb{R}^d \times (0,\infty) \to (0,\infty]$ such that $M_{p,\varphi}(k,\mu)$ and $M_{p,\varphi^\dagger}(k,\mu)$ coincide as a set, that $\varphi^\dagger$ is independent of $k > 1$, and that*

$$\varphi^\dagger(x,r)\mu(Q(x,r))^{1/p} \gtrsim \varphi^\dagger(y,s)\mu(Q(y,s))^{1/p} \tag{1.1}$$

for all $x, y \in \mathbb{R}^d$ and $r, s > 0$ such that $Q(x,r) \supset Q(y,s)$, $Q(y,s) \in \mathcal{Q}(\mu)$.

(2) *The function space $M_{p,\varphi}(k,\mu)$ does not depend upon the parameter $k > 1$.*

Therefore, in view of this theorem, we can say that the theory of generalized Morrey spaces is extended to a large extent. An example in [37] shows that $M^p_q(1,\mu)$ and $M^p_q(2,\mu)$ are not always isomorphic. This result combines [29, p. 445], [39] and [40, Proposition 1.1]. Observe also that this extends the following Morrey norm $\|f\|_{M^p_q}$ for non-doubling measures. In [40], for $k > 1$ and $f \in L^q_{\rm loc}(\mu)$, the second author and Tanaka defined

$$\|f\|_{M^p_q(k,\mu)} := \sup_{Q \in \mathcal{Q}(\mu)} \mu(k\,Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(y)|^q \, d\mu(y) \right)^{\frac{1}{q}}, \tag{1.2}$$

whose origin is the classical Morrey norm $\|f\|_{M^p_q}$ given by

$$\|f\|_{M^p_q} := \sup_{x \in \mathbb{R}^d,\, l>0} |B(x,l)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(x,l)} |f(y)|^q \, dy \right)^{\frac{1}{q}}.$$

Note that (1.2) is a special case of $M_{q,\varphi}(k,\mu)$ which is obtained by letting $\varphi(x,t) \equiv t^{-1/p}$. Recently, there are many results on generalized Morrey spaces [8], [9], [10], [11], [28], [36], [39], [43]. We aim here to arrange and strengthen some of them to the non-doubling setting. Recently, generalized Morrey spaces, initiated by E. Nakai [28], have a significant meaning in harmonic analysis. Especially, it turned out that this is useful when we want to describe the endpoint case of the boundedness of operators. For example, we have the following embedding result, which also describes the boundedness of $(1-\Delta)^{-n/(2p)}$ from $M^p_q(\mathbb{R}^d)$ to a generalized Morrey space.

Proposition 1.3 ([43, Theorem 5.1]). *Let $1 < q < p < \infty$. Then there exists a positive constant $C_{p,q}$ such that*

$$\int_Q |f(x)| dx \le C_{p,q} |Q| (1+|Q|)^{-\frac{1}{p}} \log\left(e + \frac{1}{|Q|} \right) \|(1-\Delta)^{n/(2p)} f\|_{M^p_q}$$

holds for all $f \in M^p_q(\mathbb{R}^n)$ with $(1-\Delta)^{n/(2p)} f \in M^p_q(\mathbb{R}^n)$ and for all cubes Q.

In view of the integral kernel of $(1-\Delta)^{-\alpha/2}$ (see [42]) and the Adams theorem, we have

$$(1-\Delta)^{-\alpha/2} : M^p_q(\mathbb{R}^d) \to M^s_t(\mathbb{R}^d) \tag{1.3}$$

is bounded as long as

$$1 < q \le p < \infty, \quad 1 < t \le s < \infty, \quad \frac{1}{s} = \frac{1}{p} - \frac{\alpha}{d}, \quad \frac{t}{s} = \frac{q}{p}.$$

The operator norm of $(1-\Delta)^{\alpha/2} : M^p_q(\mathbb{R}^d) \to M^s_t(\mathbb{R}^d)$ is shown to blow up as $p \uparrow \frac{d}{\alpha}$. Hence Proposition 1.3 can be considered as a substitute of (1.3). We refer to [43] for a counterexample showing that (1.3) is no longer true for $\alpha = \frac{d}{p}$.

Generalized Morrey spaces are now studied by many researchers and nowadays they are recognized as a suitable tool to grasp the property of fractional integral operators [13], [16], [17], [44]. The definition given above in the present paper covers the one in [36].

One of the advantages of allowing φ to take the value ∞ is that the following examples fall under the scope of our new framework.

Example 1.4. (1) Let $\mu(x) = e^{-\pi|x|^2}\,dx$ be the Gaussian measure. Let us set

$$\mathcal{B}_a = \{B(x,r) > r \le a\min(1,|x|^{-1})\}$$

be the set of locally doubling balls. Let $1 \le q \le p < \infty$. Recently, in [23] the second author, LIGUANG LIU and DACHUN YANG considered Morrey spaces given by

$$\|f\|_{\mathcal{M}^{p,q}_{\mathcal{B}_a}(\mu)} := \sup_{B\in\mathcal{B}_a} \frac{1}{[\mu(B)]^{1/q-1/p}} \left\{\int_B |f(y)|^q\,d\mu(y)\right\}^{1/q} < \infty.$$

In [23, Proposition 2.6], the space $\mathcal{M}^{p,q}_{\mathcal{B}_a}(\mu)$ is shown to be independent of the parameter $a > 0$. Note that there exists a constant $C_a > 0$ such that $\mu(B(x,2r)) \le C_a\mu(B(x,r))$ for all $B(x,r) \in \mathcal{B}_a$. This is a concrete example of our new framework, where

$$\varphi(x,r) := \begin{cases} \mu(B(x,r))^{-1/p} & B(x,r) \in \mathcal{B}_a, \\ \infty & \text{otherwise.} \end{cases}$$

(2) Let $G \subset \mathbb{R}^d$ be an open set and $(p,\nu) \in [1,\infty) \times (0,\infty)$. In [27] MIZUTA, SHIMOMURA and SOBUKAWA considered the Morrey norm $\|f\|_{L^{p,k,\nu}(\mu)}$ given by

$$\|f\|_{L^{p,k,\nu}(\mu)} := \sup \left(\frac{r^\nu}{\mu(B(x,kr))}\int_{B(x,r)} |f(y)|^p\,d\mu(y)\right)^{1/p}$$

for μ-measurable functions f, where sup is over $x \in G$, $r \in (0,\mathrm{diam}(G))$ and $\mu(B(x,r)) > 0$. This is again a concrete example of our new framework, where

$$\varphi(x,r) := r^{\nu/p} + \infty\chi_{[\mathrm{diam}(G),\infty)}(r).$$

In the present paper we also consider the weak-type function spaces. Let $k \ge 1$ and $1 \le p < \infty$. For a function $\varphi : \mathbb{R}^d \times (0,\infty) \to (0,\infty]$, we also define

$$\|f\|_{WM_{p,\varphi}(k,\mu)} := \sup_{Q(x,r)\in\mathcal{Q}(\mu),\ \lambda>0} \frac{\lambda(\mu\{y \in Q(x,r) > |f(y)| > \lambda\})^{1/p}}{\varphi(x,kr)\mu(Q(x,kr))^{1/p}}$$

and $WM_{p,\varphi}(k,\mu)$ denotes the set of all $f \in L^p_{\mathrm{loc}}(k,\mu)$ for which the quasi-norm $\|f\|_{WM_{p,\varphi}(k,\mu)}$ is finite. When $\varphi(x,t) \equiv t^{-1/u}$, then this definition coincides with the one appearing in [34]. Indeed, in this case, $WM_{p,\varphi}(k,\mu) = WM_{p,u}(k,\mu)$ with norm coincidence.

An idea similar to Theorem 1.2 yields the following: The proof being close to that in Theorem 1.2, we omit the proof.

Theorem 1.5. *Let $1 \le p < \infty$ and $\varphi : \mathbb{R}^d \times (0,\infty) \to (0,\infty]$ be a function. Then the space $WM_{p,\varphi}(k,\mu)$ does not depend upon the parameter $k > 1$ as a set.*

The remaining part of this paper is structured as follows: In Section 2 we study the fundamental strucuture of our Morrey spaces. The result will amount to the combination of [29, p. 445] and [40, Proposition 1.1]. Section 3 is devoted to investigating the boundedness of the operators. We take up maximal operators, singular integral operators fractional integral operators and commutators in Section 4–Section 6, where we formulated the main results for these operators in the beginning of each section.

2. Fundamental structure of the function space $M_{p,\varphi}(k,\mu)$

Now we prove Theorems 1.2 and 1.5. The following is the first step for this purpose.

Proposition 2.1. *Let $1 \le p < \infty$, $k > 1$ and $\varphi : \mathbb{R}^{d+1}_+ = \mathbb{R}^d \times (0,\infty) \to (0,\infty]$ be a function. Then define a function $\varphi^\dagger : \mathbb{R}^{d+1}_+ \to (0,\infty]$ by*

$$\varphi^\dagger(x,t) := \begin{cases} \mu(Q(x,t))^{-1/p} \inf\limits_{Q(y,s) \supset Q(x,t)} \varphi(y,s)\mu(Q(y,s))^{1/p} & (Q(x,t) \in \mathcal{Q}(\mu)), \\ \infty & (\textit{otherwise}) \end{cases}$$

for $(x,t) \in \mathbb{R}^{d+1}_+$. Then

$$\|f\|_{M_{p,\varphi^\dagger}(k,\mu)} = \|f\|_{M_{p,\varphi}(k,\mu)}$$

for all $f \in L^p_{\rm loc}(\mu)$.

Proof. Since $\varphi^\dagger(x,t) \le \varphi(x,t)$, it is easy from Definition 1.1 to see that

$$\|f\|_{M_{p,\varphi}(k,\mu)} \le \|f\|_{M_{p,\varphi^\dagger}(k,\mu)}. \tag{2.1}$$

Let us prove the reverse inequality of (2.1). To this end we take $Q = Q(x,r) \in \mathcal{Q}(\mu)$ and consider

$$\mathrm{I}(x,r) = \frac{1}{\varphi^\dagger(x,kr)\mu(Q(x,kr))^{1/p}} \|f\|_{L^p(Q(x,r))}.$$

It follows from the definition of $\varphi^\dagger$ that we have

$$\mathrm{I}(x,r) = \sup\left\{ \frac{1}{\varphi(y,ks)\mu(Q(y,ks))^{1/p}} \|f\|_{L^p(Q(x,r))} > Q(y,s) \supset Q(x,r) \right\}.$$

Observe that

$$\frac{1}{\varphi(y,ks)\mu(Q(y,ks))^{1/p}}\|f\|_{L^p(Q(x,r))} \le \frac{1}{\varphi(y,ks)\mu(Q(y,ks))^{1/p}}\|f\|_{L^p(Q(y,s))}$$

when $Q(y,s) \supset Q(x,r)$. Consequently, it follows that

$$\mathrm{I}(x,r) \le \|f\|_{M_{p,\varphi}(k,\mu)}.$$

Since $Q = Q(x,r)$ being arbitrary, we obtain the reverse inequality of (2.1). $\square$

Here and below $A \lesssim B$ means that $A \le CB$ for some constant $C > 0$ depending only on parameters. We also write $A \sim B$ to indicate $A \lesssim B \lesssim A$.

In view of Proposition 2.1 we obtain (1) of Theorem 1.2. Let us now assume $\varphi = \varphi^\dagger$ to satisfy (1.1). Once we make this change, a similar argument in [40, Proposition 1.1] works to prove Theorem 1.2. For the sake of convenience, we provide the detail. Let $1 < k_1 \le k_2$. Then the inclusion $M_{p,\varphi}(k_1,\mu) \hookrightarrow M_{p,\varphi}(k_2,\mu)$ is obvious by that fact that φ satisfies (1.1). Let us show the reverse inclusion. Let $f \in M_{p,\varphi}(k_2,\mu)$ and $Q = Q(x,r) \in \mathcal{Q}(\mu)$ be fixed. Then we have to estimate

$$\mathrm{I} := \frac{1}{\varphi(x,k_1 r)}\left(\frac{1}{\mu(Q(x,k_1 r))}\int_Q |f(y)|^p \, d\mu(y)\right)^{\frac{1}{p}}.$$

A simple geometric observation shows that there exists a collection of N cubes $Q_1 = Q(x_1,s)$, $Q_2 = Q(x_2,s), \ldots, Q_N = Q(x_N,s)$ with the same sidelength such that

$$Q(x,r) \subset \bigcup_{i=1}^N Q(x_i,s), \quad Q(x_i,k_2 s) \subset Q(x,k_1 r) \quad (i = 1,2,\ldots,N)$$

and that the number N of cubes has a bound

$$N \lesssim \left(\frac{k_2-1}{k_1-1}\right)^d.$$

Using this covering and the fact that $\varphi = \varphi^\dagger$, we easily obtain

$$\begin{aligned}
\mathrm{I}^p &\le \sum_{i=1}^N \frac{1}{\varphi(x,k_1 r)^p \mu(Q(x,k_1 r))}\int_{Q(x_i,s)} |f(y)|^p \, d\mu(y) \\
&\le \sum_{i:Q(x_i,s)\in\mathcal{Q}(\mu)} \frac{1}{\varphi(x,k_2 r)^p \mu(Q(x,k_2 r))}\int_{Q(x_i,s)} |f(y)|^p \, d\mu(y)
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i:Q(x_i,s)\in\mathcal{Q}(\mu)} \frac{1}{\varphi(x_i,k_2 s)^p \mu(Q(x_i,k_2 s))} \int_{Q(x_i,s)} |f(y)|^p \, d\mu(y) \\
&\le N(\|f\|_{M_{p,\varphi}(k_2,\mu)})^p.
\end{aligned}
$$

Thus, Theorem 1.2 is proved.

Remark that the proof of Theorem 1.5 is identical to that of Theorem 1.2. So we omit the proof.

In view of the proof of Theorem 1.2 we see that the definition of Morrey spaces can be made with cubes replaced by balls and that the Morrey norms are equivalent. Denote by $\mathcal{B}(\mu)$ the set of all open balls with positive μ-measure and for $B = B(x,r) \in \mathcal{B}(\mu)$ and $k > 0$, where $B(x,r)$ denotes the open ball centered at x and of radius $r > 0$, define $kB := B(x,kr)$. We repeat Theorem 1.2 in terms of the definition by open balls. The next observation is sometimes helpful in Section 3.

Theorem 2.2. *Let $k > 1$, $1 \le p < \infty$ and $\varphi : \mathbb{R}^d \times (0,\infty) \to (0,\infty]$ be a function. Define*

$$
\begin{aligned}
\|f\|_{M_{p,\varphi}(k,\mu)_{\rm cube}} &:= \sup_{Q(x,r)\in\mathcal{Q}(\mu)} \frac{1}{\varphi(x,kr)\mu(Q(x,kr))^{1/p}} \|f\|_{L^p(Q(x,r))} \\
\|f\|_{M_{p,\varphi}(k,\mu)_{\rm ball}} &:= \sup_{B(x,r)\in\mathcal{B}(\mu)} \frac{1}{\varphi(x,kr)\mu(B(x,kr))^{1/p}} \|f\|_{L^p(B(x,r))}.
\end{aligned}
$$

(1) *There exists a function $\varphi_\dagger : \mathbb{R}^{d+1}_+ \to (0,\infty]$ such that*

$$
\varphi_\dagger(x,r)\mu(B(x,r))^{1/p} \ge \varphi_\dagger(y,s)\mu(B(y,s))^{1/p} \tag{2.2}
$$

for all $(x,r),(y,s) \in \mathbb{R}^{d+1}_+$ with $B(x,r) \in \mathcal{B}(\mu)$ and $B(x,r) \supset B(y,s)$ and that the norms $\|f\|_{M_{p,\varphi_\dagger}(k,\mu)_{\rm ball}}$ and $\|f\|_{M_{p,\varphi}(k,\mu)_{\rm ball}}$ are equivalent.

(2) *Let $\varphi : \mathbb{R}^d \times (0,\infty) \to (0,\infty]$ be a function satisfying* (1.1). *Assume also that, for every $1 < k < \infty$, there exist $\kappa \in (1/k,\infty)$ and $C > 0$ such that*

$$
\varphi(x,r)\mu(B(x,r))^{1/p} \le C\varphi(x,\kappa r)\mu(Q(x,\kappa r))^{1/p}. \tag{2.3}
$$

Then the norms $\|f\|_{M_{p,\varphi}(k,\mu)_{\rm cube}}$ and $\|f\|_{M_{p,\varphi}(k,\mu)_{\rm ball}}$ are equivalent.

For example, if $\varphi(x,t) \equiv t^{-1/u}$ with some $u \in (p,\infty)$, then the assumption is satisfied. Hence, if (2.3) holds, then in (1.2) one may replace cubes with balls to obtain equivalent norms.

Proof. The proof of (1) is analogous to that of Theorem 1.2, since we can arrange that $B(x,r) \in \mathcal{B}(\mu)$ and $B(x,r) \subset B(y,s)$ imply

$$\mu(B(x,r))^{1/p}\varphi(x,r) \le \mu(B(y,s))^{1/p}\varphi(y,s)$$

whenever (x,r) and $(y,s) \in \mathbb{R}^{d+1}_+$ satisfy $B(x,r) \in \mathcal{B}(\mu)$ and $B(x,r) \subset B(y,s)$. A geometric observation shows

$$\begin{aligned} \|f\|_{M_{p,\varphi}(\sqrt{d}k,\mu)_{\rm cube}} &= \sup_{Q(x,r)\in\mathcal{Q}(\mu)} \frac{1}{\varphi(x,\sqrt{d}kr)\mu(Q(x,\sqrt{d}kr))^{1/p}}\|f\|_{L^p(Q(x,r))} \\ &\le \sup_{B(x,r)\in\mathcal{B}(\mu)} \frac{1}{\varphi(x,\sqrt{d}kr)\mu(B(x,\sqrt{d}kr))^{1/p}}\|f\|_{L^p(B(x,\sqrt{d}r))} \\ &= \|f\|_{M_{p,\varphi}(k,\mu)_{\rm ball}}. \end{aligned}$$

Meanwhile, by (2.3),

$$\begin{aligned} \|f\|_{M_{p,\varphi}(k,\mu)_{\rm ball}} &:= \sup_{B(x,r)\in\mathcal{B}(\mu)} \frac{1}{\varphi(x,kr)\mu(B(x,kr))^{1/p}}\|f\|_{L^p(B(x,r))} \\ \lesssim \sup_{Q(x,r)\in\mathcal{Q}(\mu)} &\frac{1}{\varphi(x,k\kappa r)\mu(Q(x,k\kappa r))^{1/p}}\|f\|_{L^p(Q(x,r))} = \|f\|_{M_{p,\varphi}(k\kappa,\mu)_{\rm cube}}. \end{aligned}$$

Since φ satisfies (1.1), we obtain

$$\|f\|_{M_{p,\varphi}(k,\mu)_{\rm ball}} \lesssim \|f\|_{M_{p,\varphi}(k\kappa,\mu)_{\rm cube}} \sim \|f\|_{M_{p,\varphi}(\sqrt{d}k,\mu)_{\rm cube}} \le \|f\|_{M_{p,\varphi}(k,\mu)_{\rm cube}}.$$

This is the desired result. □

3. Boundedness of the modified maximal operators

Until the end of this paper, we consider a class Φ: Denote by Φ the set of all functions $\varphi : \mathbb{R}^d \times (0,\infty) \to (0,\infty]$ satisfying (2.2) and (2.3) with $\varphi^\dagger$ replaced by φ as well as

$$\varphi(y,s) \gtrsim \varphi(x,r)$$

for all $x, y \in \mathbb{R}^n$ and $r > 0$ such that $Q(x,r) \supset Q(y,s)$, $Q(y,s) \in \mathcal{Q}(\mu)$. Note that $t^{-1/v} \in \Phi$ for $v > p \ge 1$. Here and below we do not distinguish the norms $\|\cdot\|_{M_{p,\varphi}(2\kappa^{-1},\mu)_{\rm cube}}$ and $\|\cdot\|_{M_{p,\varphi}(2\kappa^{-1},\mu)_{\rm ball}}$ and we denote them simply by $\|\cdot\|_{M_{p,\varphi}(\mu)}$. Also, we abuse a notation: $\varphi(Q) = \varphi(Q(x,r)) := \varphi(x,r)$, where $Q = Q(x,r) \in \mathcal{Q}(\mu)$.

Based now upon Theorems 1.2 and 1.5, we denote $M_{p,\varphi}(\mu) := M_{p,\varphi}(2,\mu)$ and $WM_{p,\varphi}(\mu) := WM_{p,\varphi}(2,\mu)$, where the norms are both based on cubes.

For the sake of convenience, we assume (2.3) in the remaining part of the present paper.

For $\kappa > 1$ we define the modified maximal operator M_κ by

$$M_\kappa f(x) := \sup_{Q \in \mathcal{Q}(\mu;\{x\})} \frac{1}{\mu(\kappa Q)} \int_Q |f(y)| \, d\mu(y),$$

where we defined $\mathcal{Q}(\mu; E)$ as the set of all cubes in $\mathcal{Q}(\mu)$ that contain a set $E \subset \mathbb{R}^d$.

Proposition 3.1 ([33], [45]). *If $\kappa > 1$ and $1 < p \le \infty$, then*

$$\|M_\kappa f\|_{L^p(\mu)} \lesssim \|f\|_{L^p(\mu)},$$

where the implicit constant depend on d, p and κ.

Remark 3.2. Remark that, according to [33], the "so called" growth condition

$$\mu(B(x,r)) \le cr^n, \quad (x \in \mathrm{supp}(\mu),\ r > 0)$$

is not necessary.

The following boundedness of M_κ will be used in the proof of the main theorem of this section.

Theorem 3.3. *Let $1 < p, \kappa < \infty$ and $\varphi_1, \varphi_2 \in \Phi$. Assume in particular that*

$$\varphi_1(y,s) \gtrsim \varphi_1(x,r) \tag{3.1}$$

for all $x, y \in \mathbb{R}^n$ and $r > 0$ such that $Q(x,r) \supset Q(y,s)$, $Q(y,s) \in \mathcal{Q}(\mu)$. If

$$\varphi_1(Q) \lesssim \varphi_2(Q) \tag{3.2}$$

for all $Q \in \mathcal{Q}(\mu)$, then M_κ is bounded from $M_{p,\varphi_1}(\mu)$ to $M_{p,\varphi_2}(\mu)$ and from $M_{1,\varphi_1}(\mu)$ to $WM_{1,\varphi_2}(\mu)$.

Here and below we denote by χ_E the indicator function of a set E.

Proof. We can assume that $\varphi_1 = \varphi_2$ because we have embedding $M_{p,\varphi_1}(\mu) \subset M_{p,\varphi_2}(\mu)$ by (3.2). For simplicity, we let $\kappa = 3$. We omit the proof of the weak boundedness, that is, the fact that M_3 is bounded from $M_{1,\varphi_1}(\mu)$ to $WM_{1,\varphi_1}(\mu)$: The proof is similar to the boundedness of M_3 from $M_{p,\varphi_1}(\mu)$ to $M_{p,\varphi_1}(\mu)$.

Let us now prove that M_3 is actually bounded on $M_{p,\varphi_1}(\mu) = M_{p,\varphi_1}(10/9,\mu)$ to $M_{p,\varphi_1}(\mu) = M_{p,\varphi_1}(10,\mu)$, which is sufficient by virtue of Theorem 1.2. Let Q be a fixed cube. Then, since $f = \chi_{3Q}f + \chi_{\mathbb{R}^d\setminus 3Q}f$, we have

$$\frac{\|M_3 f\|_{L^p(Q)}}{\varphi_1(10Q)\mu(10Q)^{1/p}} \le \frac{\|M_3(\chi_{9Q}f)\|_{L^p(Q)}}{\varphi_1(10Q)\mu(10Q)^{1/p}} + \frac{\|M_3(\chi_{\mathbb{R}^d\setminus 9Q}f)\|_{L^p(Q)}}{\varphi_1(10Q)\mu(10Q)^{1/p}}$$

by the triangle inequality. As for the first term we use Proposition 3.1 and (3.2) to obtain

$$\frac{\|M_3(\chi_{9Q}f)\|_{L^p(Q)}}{\varphi_1(10Q)\mu(10Q)^{1/p}} \lesssim \frac{\|f\|_{L^p(9Q)}}{\varphi_1(10Q)\mu(10Q)^{1/p}} \lesssim \|f\|_{M_{p,\varphi_1}(10/9,\mu)} \tag{3.3}$$

A geometric observation shows that

$$M_3(\chi_{\mathbb{R}^d\setminus 9Q}f)(x) \le \sup_{Q'\in\mathcal{Q}(\mu;2Q)} \frac{1}{\mu(2Q')}\int_{Q'} |f(y)|\,d\mu(y) \tag{3.4}$$

on Q. Hence we have

$$\frac{\|M_3(\chi_{\mathbb{R}^d\setminus 9Q}f)\|_{L^p(Q)}}{\varphi_1(10Q)\mu(10Q)^{1/p}} \le \frac{\mu(Q)^{1/p}}{\varphi_1(10Q)\mu(10Q)^{1/p}} \sup_{Q'\in\mathcal{Q}(\mu;2Q)} \frac{1}{\mu(2Q')}\int_{Q'} |f(y)|\,d\mu(y).$$

Now let $Q' \in \mathcal{Q}(\mu;2Q)$ and we distinguish two cases.

Case 1. $\frac{10}{9}Q' \subset 10Q$.

Case 2. $2Q'$ engulfs $10Q$.

Note that at least one of (Case 1) and (Case 2) holds.

Let us consider Case 1. We recall that

$$\varphi_1(10Q)\mu(10Q)^{1/p} \lesssim \varphi_1\left(\frac{10}{9}Q'\right)\mu\left(\frac{10}{9}Q'\right)^{1/p}.$$

Thus, we have

$$\begin{aligned}
&\frac{\mu(Q)^{1/p}}{\varphi_1(10Q)\mu(10Q)^{1/p}} \times \frac{1}{\mu(2Q')}\int_{Q'} |f(y)|\,d\mu(y)\\
&\lesssim \frac{\mu(Q)^{1/p}}{\varphi_1(\frac{10}{9}Q')\mu(\frac{10}{9}Q')^{1/p}} \times \frac{1}{\mu(2Q')}\int_{Q'} |f(y)|\,d\mu(y)\\
&\le \frac{1}{\varphi_1(\frac{10}{9}Q')} \times \frac{1}{\mu(2Q')}\int_{Q'} |f(y)|\,d\mu(y) \le \|f\|_{M_{1,\varphi_1}(\frac{10}{9},\mu)} \le \|f\|_{M_{p,\varphi_1}(\frac{10}{9},\mu)}.
\end{aligned}$$

When we deal with Case 2, we use $\varphi_1(10Q) \geq \varphi_1(2Q')$ (see (3.1)) to have

$$\frac{\mu(Q)^{1/p}}{\varphi_1(10Q)\mu(10Q)^{1/p}} \times \frac{1}{\mu(2Q')} \int_{Q'} |f(y)| d\mu(y) \lesssim \frac{1}{\varphi_1(2Q')} \times \frac{1}{\mu(2Q')} \int_{Q'} |f(y)| d\mu(y)$$
$$\leq \|f\|_{M_{1,\varphi_1}(2,\mu)} \leq \|f\|_{M_{p,\varphi_1}(2,\mu)}.$$

Hence from Case 1 and Case 2, we obtain

$$\frac{\|M_3(\chi_{\mathbb{R}^d \setminus 9Q} f)\|_{L^p(Q)}}{\varphi_1(10Q)\mu(10Q)^{1/p}} \lesssim \|f\|_{M_{p,\varphi_1}(10/9,\mu)} \tag{3.5}$$

Using (3.2) and (3.4), we can estimate the second term. Thus, in view of (3.3) and (3.5) the proof is complete. □

Remark 3.4. If we reexamine the proof and we use Proposition 3.1, then we see that M_κ with $\kappa > 1$ is bounded.

Here and below in the rest of the present paper, we let $M = M_3$ for definiteness.

4. Singular integral operators

Here and below, we assume that μ is a (positive) Radon measure on $\mathbb{R}^d$ satisfying the growth condition;

$$\mu(B(x,\ell)) \leq c_0 \, \ell^n \quad \text{for all } x \in \operatorname{supp}(\mu) \text{ and } \ell > 0, \tag{4.1}$$

where c_0 and n, $0 < n \leq d$, are some fixed numbers.

4.1. Main results. We employ the definition of singular integral operators due to Nazarov, Treil and Volberg [32].

Definition 4.1. Let μ and n be as above. The singular integral operator S is a bounded linear operator from $L^2(\mu)$ to $L^2(\mu)$ for which there exists a measurable function $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ that satisfies three properties listed below.

(1) There exists $C > 0$ such that $|K(x,y)| \leq \frac{C}{|x-y|^n}$ for all $x \neq y$.

(2) There exist $\varepsilon > 0$ and $C > 0$ such that

$$|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \leq C \frac{|x-z|^\varepsilon}{|x-y|^{n+\varepsilon}},$$

if $|x-y| \geq 2|x-z|$ with $x \neq y$.

(3) If $f \in L^2(\mu)$ is a bounded μ-measurable function with a compact support, then we have

$$Sf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)\,d\mu(y) \quad \text{for all } x \notin \operatorname{supp}(f).$$

As for this singular integral operator S, the following result is due to Nazarov, Treil and Volberg.

Proposition 4.2 ([32]). *In Definition 4.1, S extends to a bounded linear operator on $L^p(\mu)$ for $1 < p < \infty$ and S extends to a bounded linear operator from $L^1(\mu)$ to $WL^1(\mu)$.*

In this section we prove;

Theorem 4.3. *Assume that a pair $(\varphi_1, \varphi_2) \in \Phi \times \Phi$ satisfies*

$$\int_r^\infty \frac{\varphi_1(x,2t)\mu(B(x,2t))^{1/p}}{t^{n/p}}\frac{dt}{t} \le \frac{\varphi_2(x,2r)\mu(B(x,2r))^{1/p}}{r^{n/p}} \tag{4.2}$$

for all $x \in \mathbb{R}^d$ and $r > 0$. Assume in particular that

$$\varphi_1(y,s) \gtrsim \varphi_1(x,r) \tag{4.3}$$

for all $x, y \in \mathbb{R}^n$ and $r > 0$ such that $Q(x,r) \supset Q(y,s)$, $Q(y,s) \in \mathcal{Q}(\mu)$. Assume in addition that T is a sublinear operator satisfying

$$|Tf(x)| \lesssim \int_{\mathbb{R}^d \setminus B} \frac{|f(y)|}{|x-y|^n}\,d\mu(y) \quad (\mu - \text{a.e.}\, x \in B) \tag{4.4}$$

for all balls $B \in \mathcal{B}(\mu)$ and all functions $f \in L^\infty(\mu)$ with compact support in $\mathbb{R}^d \setminus B$.

(1) *Let $1 < p < \infty$ and assume in addition that T is $L^p(\mu)$-bounded. Then T extends to a bounded sublinear operator from $M_{p,\varphi_1}(\mu)$ to $M_{p,\varphi_2}(\mu)$.*
(2) *Assume in addition that T is weak-$L^1(\mu)$-bounded. Then T extends to a bounded sublinear operator from $M_{1,\varphi_1}(\mu)$ to $WM_{1,\varphi_2}(\mu)$.*

Note that this condition is proposed in [25], [26].

Remark 4.4.

(1) In view of Proposition 4.2, singular integral operators defined in Definition 4.1 are examples of the operators of Theorem 4.3.
(2) The assumption (4.4) appears in [20], [21], [22]. In the case $d\mu(x) = dx$, Theorem 4.3 was proved in [10], [11].

4.2. Proof of Theorem 4.3. To prove Theorem 4.3, we need the following lemma.

Lemma 4.5. *Let T be as above with $p > 1$. Then we have*

$$\|Tf\|_{L^p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} t^{-\frac{n}{p}} \|f\|_{L^p(B(x_0,t))} \frac{dt}{t}$$

for all balls $B = B(x_0, r) \in \mathcal{B}(\mu)$.

Proof. Let $B = B(x_0, r)$ be a fixed ball. Then we have

$$\begin{aligned}
\|Tf\|_{L^p(B)} &\leq \|T(\chi_{2B} f)\|_{L^p(B)} + \|T(\chi_{\mathbb{R}^d \setminus 2B} f)\|_{L^p(B)} \\
&\lesssim \|f\|_{L^p(2B)} + \mu(B)^{1/p} \int_{\mathbb{R}^d \setminus 2B} \frac{|f(y)|}{|x_0 - y|^n} \, d\mu(y) \\
&= \|f\|_{L^p(2B)} + \mu(B)^{1/p} \left(\int_0^{\infty} \frac{n}{\ell^{n+1}} \int_{B(x_0,\ell)} \chi_{\mathbb{R}^d \setminus 2B}(y) |f(y)| \, d\mu(y) \right) d\ell \\
&\lesssim \|f\|_{L^p(2B)} + r^{n/p} \left(\int_{2r}^{\infty} \frac{1}{\ell^{n-n/p+1}} \left(\int_{B(x_0,\ell)} |f(y)|^p \, d\mu(y) \right)^{1/p} \right) d\ell.
\end{aligned}$$

Observe that

$$\|f\|_{L^p(2B)} \lesssim r^{n/p} \left(\int_{2r}^{3r} \frac{1}{\ell^{n-n/p+1}} \left(\int_{B(x_0,\ell)} |f(y)|^p \, d\mu(y) \right)^{1/p} \right) d\ell$$

and hence

$$\|Tf\|_{L^p(B)} \lesssim r^{n/p} \left(\int_{2r}^{\infty} \frac{1}{\ell^{n-n/p+1}} \left(\int_{B(x_0,\ell)} |f(y)|^p \, d\mu(y) \right)^{1/p} \right) d\ell$$

for all $B(x_0, r)$. This is the desired result. □

Proof of Theorem 4.3. We freeze a ball $B = B(x_0, r)$ and we let $p > 1$. Then we have

$$\begin{aligned}
&\frac{\|Tf\|_{L^p(B)}}{\varphi_2(2B)\mu(2B)^{1/p}} \lesssim \frac{r^{\frac{n}{p}}}{\varphi_2(2B)\mu(2B)^{1/p}} \int_{2r}^{\infty} t^{-\frac{n}{p}} \|f\|_{L^p(B(x_0,t))} \frac{dt}{t} \\
&\lesssim \frac{r^{\frac{n}{p}} \|f\|_{M_{p,\varphi_1}(\mu)}}{\varphi_2(2B)\mu(2B)^{1/p}} \int_{2r}^{\infty} \frac{\varphi_1(B(x_0,2t))\mu(B(x_0,2t))^{1/p}}{t^{n/p}} \frac{dt}{t} \lesssim \|f\|_{M_{p,\varphi_1}(\mu)}.
\end{aligned}$$

The ball B being arbitrary, this is the desired result.

The case when $p = 1$ can be proven similarly. Indeed, fixing $\lambda > 0$ arbitrarily, we just use

$$\begin{aligned}
&\frac{\lambda\mu\{y \in B : |Tf(y)| > \lambda\}}{\varphi_2(2B)\mu(2B)} \\
&\le \frac{\lambda\mu\{y \in B : |T(\chi_{\frac{3}{2}B}f)(y)| > \lambda/2\}}{\varphi_2(2B)\mu(2B)} + \frac{\lambda\mu\{y \in B > |T(\chi_{\mathbb{R}^n\setminus\frac{3}{2}B}f)(y)| > \lambda/2\}}{\varphi_2(2B)\mu(2B)} \\
&\le \frac{\lambda\mu\{y \in B : |T(\chi_{\frac{3}{2}B}f)(y)| > \lambda/2\}}{\varphi_2(2B)\mu(2B)} + \frac{\|T(\chi_{\mathbb{R}^n\setminus\frac{3}{2}B}f)\|_{L^1(B)}}{\varphi_2(2B)\mu(2B)} \\
&\lesssim \frac{\|f\|_{L^1(B)}}{\varphi_2(2B)\mu(2B)} + \frac{r^n}{\varphi_2(2B)\mu(2B)}\int_{2r}^{\infty} t^{-n}\|f\|_{L^1(B(x_0,t))}\frac{dt}{t} \\
&\lesssim \frac{r^n}{\varphi_2(2B)\mu(2B)}\int_{2r}^{\infty} t^{-n}\|f\|_{L^1(B(x_0,t))}\frac{dt}{t}.
\end{aligned}$$

Thus, the proof is complete. □

Remark 4.6. In the case $d\mu(x) = dx$, Lemma 4.5 was proved in [6], [7], see also [8].

5. Boundedness of fractional integral operators

Now, assuming that μ satisfies the growth condition (4.1), we shall consider the boundedness of I_α, which is given by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, d\mu(y)$$

for all positive μ-measurable functions f. Remark that I_α extends to a bounded linear operator from $L^p(\mu)$ and $L^q(\mu)$ if $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ [14].

5.1. Main results. Here we shall prove the following theorems.

Theorem 5.1 (Adams–Ding type result)**.** *Let the parameters p, q, α, b satisfy*

$$1 < p < \infty,\ 1 \le q < \infty,\ -\frac{1}{p} \le b < -\frac{\alpha}{n} < 0,\ q = \frac{bnp}{\alpha + bn}. \tag{5.1}$$

Assume in particular that

$$\varphi(y,s) \gtrsim \varphi(x,r) \tag{5.2}$$

for all $x, y \in \mathbb{R}^n$ and $r > 0$ such that $Q(x,r) \supset Q(y,s)$, $Q(y,s) \in \mathcal{Q}(\mu)$. If the function $\varphi \in \Phi$ is surjective and satisfies the inequality $\varphi(x,t) \le t^{bn}$, then I_α is bounded from $M_{p,\varphi^{1/p}}(\mu)$ to $M_{q,\varphi^{1/q}}(\mu)$.

Theorem 5.2 (Adams–Guliyev type result)**.** *Let $1 \le p < \infty$, $0 < \alpha < \frac{n}{p}$, $q > p$ and let $\varphi \in \Phi$ satisfy conditions*

$$\int_r^\infty \varphi(x,t)^{\frac{1}{p}} \frac{dt}{t} \lesssim \varphi(x,r)^{\frac{1}{p}}, \tag{5.3}$$

and

$$\int_r^\infty t^\alpha \varphi(x,t)^{\frac{1}{p}} \frac{dt}{t} \lesssim r^{-\frac{\alpha p}{q-p}}. \tag{5.4}$$

Assume in particular that

$$\varphi(y,s) \gtrsim \varphi(x,r) \tag{5.5}$$

for all $x, y \in \mathbb{R}^n$ and $r > 0$ such that $Q(x,r) \supset Q(y,s)$, $Q(y,s) \in \mathcal{Q}(\mu)$. Then the operator I_α extends to a bounded linear operator from $M_{p,\varphi^{1/p}}(\mu)$ to $M_{q,\varphi^{1/q}}(\mu)$ for $p > 1$. Furthermore, for $p = 1$, the operator I_α extends to a bounded linear operator from $M_{1,\varphi}(\mu)$ to $WM_{q,\varphi^{1/q}}(\mu)$.

The following is a result of Spanne type.

Theorem 5.3. *Let $0 < \alpha < n$ and $1 < p < \frac{n}{\alpha}$. Define q by $\frac{1}{q} = \frac{1}{p} - \frac{n}{\alpha}$. Assume that $\varphi_1, \varphi_2 \in \Phi$ satisfy*

$$r^{\frac{n}{q}} \int_r^\infty \frac{\varphi_1(x,2t)\mu(B(x,2t))^{1/q}}{\varphi_2(x,2r)\mu(B(x,2r))^{1/q}} t^{-\frac{n}{q}-1}\, dt \lesssim 1 \quad (x \in \mathbb{R}^n). \tag{5.6}$$

Assume in particular that

$$\varphi_1(y,s) \ge \varphi_1(x,r) \tag{5.7}$$

for all $x, y \in \mathbb{R}^n$ and $r > 0$ such that $Q(x,r) \supset Q(y,s)$, $Q(y,s) \in \mathcal{Q}(\mu)$. Then I_α extends to a bounded linear operator from $M_{p,\varphi_1}(\mu)$ to $M_{q,\varphi_2}(\mu)$.

There is another variant of the Adams–Gunawan theorem. See [15] for the case of the Lebesgue measure.

Theorem 5.4. *Let $1 < p \le q < \infty$ and $0 < \alpha < \frac{n}{p}$. Assume that $\omega \in \Phi$ satisfies*

$$r^\alpha \omega(x,r) + \int_r^\infty t^\alpha \omega(x,t) \frac{dt}{t} \lesssim \omega(x,r)^{\frac{p}{q}} \quad (x \in \mathbb{R}^n) \tag{5.8}$$

and that $\omega(x,\cdot) : (0,\infty) \to (0,\infty]$ is surjective. Assume in particular that

$$\omega(y,s) \ge \omega(x,r) \tag{5.9}$$

for all $x, y \in \mathbb{R}^n$ and $r > 0$ such that $Q(x,r) \supset Q(y,s)$, $Q(y,s) \in \mathcal{Q}(\mu)$. Then I_α extends to a bounded linear operator from $M_{p,\omega}(\mu)$ to $M_{q,\omega^{p/q}}(\mu)$.

Remark 5.5. In the case $d\mu(x) = dx$, Theorem 5.2 was proved in [12]. In the case $\varphi(x,t) \equiv t^{\lambda-n}$, $0 < \lambda < n$ from Theorem 5.2 we get the Adams theorem [1].

Remark 5.6. In the case $d\mu(x) = dx$, Theorem 5.3 and Lemma 5.7 were proved in [6], [7], see also [8] and Theorem 5.4 was established in [8]. In the case $\varphi(x,t) \equiv t^{\lambda-n}$, $0 < \lambda < n$ from Theorem 5.4 we get the Adams theorem [1].

5.2. Proof of Theorems 5.1, 5.2 and 5.4.

Proof of Theorem 5.1. First, let us remark again that we have the following maximal operator estimate

$$\|(Mf)^{p/q}\|_{M_{q,\varphi^{q/p}}(\mu)} = (\|Mf\|_{M_{q,\varphi}(\mu)})^{p/q} \lesssim (\|f\|_{M_{p,\varphi}(\mu)})^{q/p} \tag{5.10}$$

in view of the conditions (1.1), (3.2) and (5.2).

We rewrite

$$|I_\alpha f(x)| \leqq (n-\alpha)\int_0^\infty \left(\int_{B(x,\ell)} |f(y)|\,d\mu(y)\right)\frac{d\ell}{\ell^{n-\alpha+1}} \tag{5.11}$$

by using the Fubini theorem. Then we notice that

$$|I_\alpha f(x)| \lesssim \int_0^\infty \left(\int_{B(x,\ell)} |f(y)|\,d\mu(y)\right)\frac{d\ell}{\ell^{n-\alpha+1}}. \tag{5.12}$$

We decompose (5.12) into two parts;

$$|I_\alpha f(x)| \lesssim \int_0^{\ell_0} \left(\int_{B(x,\ell)} |f(y)|\,d\mu(y)\right)\frac{d\ell}{\ell^{n-\alpha+1}} + \int_{\ell_0}^\infty \left(\int_{B(x,\ell)} |f(y)|\,d\mu(y)\right)\frac{d\ell}{\ell^{n-\alpha+1}}, \tag{5.13}$$

where ℓ_0 is a constant specified by (5.16) later. Notice that

$$\frac{1}{\ell^{n-\alpha+1}}\left(\int_{B(x,\ell)} |f(y)|\,d\mu(y)\right) \lesssim \ell^{\alpha-1} Mf(x) \tag{5.14}$$

and that

$$\frac{1}{\ell^{n-\alpha+1}}\left(\int_{B(x,\ell)} |f(y)|\,d\mu(y)\right) \lesssim \frac{\varphi(x,2\ell)}{\ell^{-\alpha+1}}\|f\|_{M_{p,\varphi^{1/p}}(\mu)} \lesssim \frac{\|f\|_{M_{p,\varphi^{1/p}}(\mu)}}{\ell^{-bn-\alpha+1}}. \tag{5.15}$$

Now choose $\ell_0 > 0$ so that

$$\ell_0{}^{-bn} = \frac{\|f\|_{M_{p,\varphi^{1/p}}(\mu)}}{Mf(x)} \tag{5.16}$$

An arithmetic shows that

$$Mf(x)\ell_0{}^{\alpha} = Mf(x)\left(\frac{\|f\|_{M_{p,\varphi^{1/p}}(\mu)}}{Mf(x)}\right)^{-\frac{\alpha}{bn}} = Mf(x)^{1+\frac{\alpha}{bn}}\|f\|_{M_{p,\varphi^{1/p}}(\mu)}{}^{-\frac{\alpha}{bn}}.$$

If we insert (5.15), (5.16) to (5.13), then we have

$$|I_\alpha f(x)| \lesssim Mf(x)^{1+\frac{\alpha}{bn}}\|f\|_{M_{p,\varphi^{1/p}}(\mu)}{}^{-\frac{\alpha}{bn}}. \tag{5.17}$$

If we use (5.10) and (5.16), then we obtain the desired result. □

Proof of Theorem 5.2. First, we remark again that we have the following maximal operator estimate

$$\|(Mf)^{\frac{p}{q}}\|_{M_{q,\varphi^{q/p}}(\mu)} \lesssim \left(\|f\|_{M_{p,\varphi}(\mu)}\right)^{q/p} \tag{5.18}$$

in view of the conditions (1.1), (3.2) and (5.5).

Let $f \in M_{p,\varphi}(\mu)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r. We represent f as

$$f = f_1 + f_2, \text{ where } f_1 := f \cdot \chi_{2B}, \quad f_2 := f - f_1. \tag{5.19}$$

We use

$$|I_\alpha f(x)| \le |I_\alpha f_1(x)| + |I_\alpha f_2(x)|.$$

The estimate of $I_\alpha f_1$ is simple. Just recall that μ satisfies the growth condition and use a crude estimate

$$\begin{aligned}|I_\alpha f_1(x)| &\le \int_{2B} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,d\mu(y) = \sum_{j=1}^{\infty}\int_{2^{-j+2}B\setminus 2^{-j+1}B} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,d\mu(y)\\ &\lesssim \sum_{j=1}^{\infty}\frac{1}{(2^{-j}r)^{n-\alpha}}\int_{2^{-j+2}B\setminus 2^{-j+1}B}|f(y)|\,d\mu(y) \lesssim r^{\alpha}Mf(x).\end{aligned}$$

Here the equality above is valid because μ does not charge a point x by virtue of the growth condition (4.1).

For $I_\alpha f_2(x)$ we have

$$\left|I_\alpha f_2(x)\right| \le \int_{\mathbb{R}^d\setminus B(x,2r)} |x-y|^{\alpha-n}|f(y)|\,d\mu(y)$$

$$\lesssim \int_{\mathbb{R}^d\setminus B(x,2r)} \left(\int_{|x-y|}^{\infty} t^{\alpha-n-1} dt \right) |f(y)|\, d\mu(y)$$
$$\lesssim \int_{2r}^{\infty} \left(\int_{2r<|x-y|<t} |f(y)|\, d\mu(y) \right) t^{\alpha-n-1} dt$$
$$\lesssim \int_{r}^{\infty} t^{\alpha-\frac{n}{p}-1} \|f\|_{L^p(B(x,t))} dt. \tag{5.20}$$

Then from conditions (5.3) and (5.4) we get

$$\begin{aligned} |I_\alpha f(x)| &\lesssim r^\alpha\, Mf(x) + \int_r^\infty t^{\alpha-\frac{n}{p}-1} \|f\|_{L^p(B(x,t))} dt \\ &\le r^\alpha\, Mf(x) + \|f\|_{M_{p,\varphi}(\mu)} \int_r^\infty t^\alpha \varphi(x,t)^{\frac{1}{p}} \frac{dt}{t} \\ &\lesssim r^\alpha\, Mf(x) + r^{-\frac{\alpha p}{q-p}} \|f\|_{M_{p,\varphi^{1/p}}(\mu)}. \end{aligned} \tag{5.21}$$

Hence choosing $r = \left(\frac{\|f\|_{M_{p,\varphi^{1/p}}(\mu)}}{Mf(x)} \right)^{\frac{q-p}{\alpha q}}$ for each given $x \in \mathbb{R}^n$, we have

$$|I_\alpha f(x)| \lesssim (Mf(x))^{\frac{p}{q}} \, \|f\|_{M_{p,\varphi^{1/p}}(\mu)}^{1-\frac{p}{q}}.$$

Hence, the theorem follows in view of (5.3), (5.18) and Theorem 3.3, that is, the boundedness of the maximal operator M in $M_{p,\varphi^{1/p}}(\mu)$. Indeed,

$$\begin{aligned} \|I_\alpha f\|_{M_{q,\varphi^{1/q}}(\mu)} &= \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} \mu(B(x,2t))^{-\frac{1}{q}} \|I_\alpha f\|_{L^q(B(x,t))} \\ &\lesssim \|f\|_{M_{p,\varphi^{1/p}}(\mu)}^{1-\frac{p}{q}} \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} \mu(B(x,2t))^{-\frac{1}{q}} \|Mf\|_{L^p(B(x,t))}^{\frac{p}{q}} \\ &\lesssim \|f\|_{M_{p,\varphi^{1/p}}(\mu)}^{1-\frac{p}{q}} \|Mf\|_{M_{p,\varphi^{1/p}}(\mu)}^{\frac{p}{q}} \lesssim \|f\|_{M_{p,\varphi^{1/p}}(\mu)}, \end{aligned}$$

if $1 < p < q < \infty$ and

$$\begin{aligned} \|I_\alpha f\|_{WM_{q,\varphi^{1/q}}(\mu)} &= \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} \mu(B(x,2t))^{-\frac{1}{q}} \|I_\alpha f\|_{WL^q(B(x,t))} \\ &\lesssim \|f\|_{M_{1,\varphi}(\mu)}^{1-\frac{1}{q}} \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} \mu(B(x,2t))^{-\frac{1}{q}} \|Mf\|_{WL^1(B(x,t))}^{\frac{1}{q}} \\ &\lesssim \|f\|_{M_{1,\varphi}(\mu)}^{1-\frac{1}{q}} \|Mf\|_{WM_{1,\varphi^{1/p}}(\mu)}^{\frac{p}{q}} \lesssim \|f\|_{M_{1,\varphi}(\mu)}, \end{aligned}$$

if $1 = p < q < \infty$. □

To prove Theorem 5.3 we need the following lemma.

Lemma 5.7. *Let $0 < \alpha < n$ and $1 < p < \frac{n}{\alpha}$. Define q by $\frac{1}{q} = \frac{1}{p} - \frac{n}{\alpha}$. Then we have*

$$\|I_\alpha f\|_{L^q(B(x,r))} \lesssim r^{\frac{n}{q}} \int_r^\infty t^{-\frac{n}{q}-1} \|f\|_{L^p(B(x,t))}\, dt$$

for all positive μ-measurable functions f.

Proof. We proceed as follows:

$$\begin{aligned}
&\|I_\alpha f\|_{L^q(B(x,r))}\\
&\lesssim \|I_\alpha(\chi_{B(x,2r)} f)\|_{L^q(B(x,r))} + \sum_{j=1}^\infty \|I_\alpha(\chi_{B(x,2^{j+1}r)\setminus B(x,2^j r)} f)\|_{L^q(B(x,r))}\\
&\lesssim \|f\|_{L^p(B(x,2r))} + \sum_{j=1}^\infty \frac{2^{-jn+j\alpha}}{r^{n-\alpha}} \mu(B(x,r))^{\frac{1}{q}} r^{-n+\alpha} \|f\|_{L^1(B(x,2^{j+1}r))}\\
&\lesssim \|f\|_{L^p(B(x,2r))} + \sum_{j=1}^\infty \frac{2^{-jn+j\alpha}}{r^{n-\alpha}} \mu(B(x,r))^{\frac{1}{q}} \mu(B(x,2^{j+1}r))^{1-\frac{1}{p}} \|f\|_{L^p(B(x,2^{j+1}r))}.
\end{aligned}$$

If we use the growth condition (4.1), then we have

$$\begin{aligned}
\|I_\alpha f\|_{L^q(B(x,r))} &\lesssim \|f\|_{L^p(B(x,2r))} + r^{\frac{n}{q}} \int_r^\infty t^{-\frac{n}{q}-1} \|f\|_{L^p(B(x,t))}\, dt\\
&\lesssim r^{\frac{n}{q}} \int_r^\infty t^{-\frac{n}{q}-1} \|f\|_{L^p(B(x,t))}\, dt.
\end{aligned}$$

Thus, the proof is complete. □

Proof of Theorem 5.3. By Lemma 5.7 and (5.6), we have

$$\begin{aligned}
&\frac{\|I_\alpha f\|_{L^q(B(x,r))}}{\varphi_2(x,2r)\mu(B(x,2r))^{1/q}} \lesssim r^{\frac{n}{q}} \int_r^\infty \frac{t^{-\frac{n}{q}-1}\|f\|_{L^p(B(x,t))}}{\varphi_2(x,2r)\mu(B(x,2r))^{1/q}}\, dt\\
&\lesssim \|f\|_{M_{p,\varphi_1}(\mu)} r^{\frac{n}{q}} \int_r^\infty \frac{\varphi_1(x,2t)\mu(B(x,2t))^{1/q}}{\varphi_2(x,2r)\mu(B(x,2r))^{1/q}} t^{-\frac{n}{q}-1}\, dt \lesssim \|f\|_{M_{p,\varphi_1}(\mu)}.
\end{aligned}$$

Thus, the proof is complete. □

To prove Theorem 5.4, we need the following pointwise estimate:

Lemma 5.8. *Let $1 \le p < \infty$. Then*

$$|I_\alpha f(x)| \lesssim \left(\int_0^R \frac{\mu(B(x,2t))}{t^{n-\alpha+1}}\, dt\right) Mf(x) + \int_R^\infty t^{\alpha-n/p-1} \|f\|_{L^p(B(x,t))}\, dt$$

for all $R > 0$ and $x \in \operatorname{supp}(\mu)$.

Proof. Just use (5.11) and

$$\frac{1}{\ell^{n-\alpha+1}}\int_{B(x,\ell)}|f(y)|\,d\mu(y) \lesssim \min\left(\frac{\mu(B(x,2\ell))}{\ell^{n-\alpha+1}}Mf(x), \frac{1}{\ell^{n/p-\alpha+1}}\|f\|_{L^p(\mu)}\right). \qquad \square$$

Now we refer back to the proof of Theorem 5.4.

Proof of Theorem 5.4. By virtue of Lemma 5.8 with $R = \omega(x,r)$ and (5.8) we have

$$I_\alpha f(x) \lesssim \omega(x,r)^{p/q-1}Mf(x) + \omega(x,r)^{p/q}\|f\|_{L^p(B(x,r))}.$$

Since $\omega(x,\cdot)$ is surjective for all $x \in \mathbb{R}^d$, if we optimize this inequality, then we obtain

$$I_\alpha f(x) \lesssim Mf(x)^{\frac{p}{q}}\|f\|_{L^p(B(x,r))}^{1-\frac{p}{q}}.$$

and Theorem 3.3. This is the desired result. $\square$

Remark 5.9. When $d\mu(x) = dx$, then $\omega(x,t)$ is a doubling function with respect to t and the doubling constant can be taken uniformly over x. Therefore, we do not need to assume that $\omega(x,\cdot)$ is surjective for all x and we can modify the argument above.

6. Boundedness of commutators generated by RBMO functions

Finally we shall consider the boundedness of commutators generated by RBMO functions.

6.1. Main results. Now to describe the definition of commutators we recall the definition of RBMO due to Tolsa [45].

Definition 6.1 ([45, Sections 2.2 and 2.3]).

(1) Given two cubes $Q, R \in \mathcal{Q}(\mu)$ with $Q \subset R$, one defines

$$K_{Q,R} := 1 + \sum_{k=1}^{N_{Q,R}} \frac{\mu(2^kQ)}{\ell(2^kQ)^n},$$

where $N_{Q,R}$ is the least integer $k \geq 1$ such that $2^kQ \supset R$.

(2) One says that Q is a doubling cube if $\mu(2Q) \leq 2^{d+1}\mu(Q)$. One denotes by $\mathcal{Q}(\mu,2)$ the set of all doubling cubes.

(3) Given $Q \in \mathcal{Q}(\mu)$, we set Q^* as the smallest doubling cube R of the form $R = 2^j Q$ with $j \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

(4) One says that $f \in L^1_{\rm loc}(\mu)$ is an element of RBMO if it satisfies

$$\sup_{Q \in \mathcal{Q}(\mu)} \frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |f(y) - m_{Q^*}(f)|\, d\mu(y) + \sup_{\substack{Q \subset R \\ Q, R \in \mathcal{Q}(\mu,2)}} \frac{|m_Q(f) - m_R(f)|}{K_{Q,R}} < \infty, \tag{6.1}$$

where $m_E(f) := \frac{1}{\mu(E)} \int_E f(x)\, d\mu(x)$ denotes the average of the function f over a μ-measurable set E. Denote this quantity (6.1) by $\|f\|_*$.

Commutators in Morrey spaces with non-doubling measures are investigated in [35], [38], [41]. Also, mutlicommutators are investigated in [18], [19], [24], [38]. Here we shall prove the following result.

Theorem 6.2. *Let $1 < p < \infty$. Assume that a pair $(\varphi_1, \varphi_2) \in \Phi \times \Phi$ satisfies*

$$\int_r^\infty \varphi_1(x, 2t) \log\left(2 + \frac{t}{r}\right) \frac{\mu(B(x,2t))^{1/p}}{t^{n/p}} \frac{dt}{t} \le \frac{\varphi_2(x,2r)\mu(B(x,2r))^{1/p}}{r^{n/p}} \tag{6.2}$$

for all $x \in \mathbb{R}^d$ and $r > 0$. Assume in addition that there exists $a \in$ RBMO such that

$$|Tf(x)| \lesssim \int_{\mathbb{R}^d \setminus B} \frac{|(a(x) - a(y))f(y)|}{|x-y|^n}\, d\mu(y) \quad (\mu - \text{a.e. } x \in B)$$

for all balls $B \in \mathcal{B}(\mu)$. Assume in addition that T is $L^p(\mu)$-bounded. Then T extends to a bounded sublinear operator from $M_{p,\varphi_1}(\mu)$ to $M_{p,\varphi_2}(\mu)$.

Proposition 6.3 ([45, Theorem 9.1])**.** *Suppose that $a \in$ RBMO. Let $1 < p < \infty$ and T be a singular integral operator with associated kernel K. Then*

$$[a,T]f(x) := \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} (a(x) - a(y))K(x,y)f(y)\, d\mu(y)$$

defines a bounded operator on $L^p(\mu)$.

Remark 6.4.

(1) Proposition 6.3 is an example of the operator T in Theorem 6.2.

(2) In the case $d\mu(x) = dx$, Theorem 6.2 and Lemma 6.6 was covered in [10], [11].

6.2. Proof of Theorem 6.2. To prove Theorem 6.2, we need lemmas. We first record the John–Nirenberg lemma for RBMO due to Tolsa.

Lemma 6.5 ([45, Corollary 3.5]). *For all* $1 \le r < \infty$ *and* $a \in \mathrm{RBMO}$,

$$\sup_{Q \in \mathcal{Q}(\mu)} \left(\frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |a(y) - m_{Q^*}(a)|^r \, d\mu(y) \right)^{\frac{1}{r}} \lesssim \|a\|_*.$$

Lemma 6.6. *Under the condition of Theorem 6.2, we have*

$$\|Tf\|_{L^p(B)} \lesssim r^{\frac{n}{p}} \int_r^\infty \log\left(1 + \frac{t}{r}\right) t^{-\frac{n}{p}} \|f\|_{L^p(B)} \frac{dt}{t} \tag{6.3}$$

when $1 < p < \infty$.

Proof. We shall prove (6.3) first. The proof is similar to the one for T. Fix a ball $B = B(x_0, r) \in \mathcal{B}(\mu)$. We just use

$$\begin{aligned}
&\int_{\mathbb{R}^d \setminus 2B} \frac{|a(x) - a(y)|}{|x-y|^n} |f(y)| \, d\mu(y) \\
&\quad = \int_0^\infty \frac{n}{\ell^{n+1}} \left(\int_{B(x_0,\ell) \setminus 2B} |(a(x) - a(y)) f(y)| \, d\mu(y) \right) d\ell \\
&\quad \lesssim \int_r^\infty \left(\int_{B(x_0,\ell)} |(a(x) - m_B(a)) f(y)| + |(m_B(a) - a(y)) f(y)| \, d\mu(y) \right) \frac{d\ell}{\ell^{n+1}} \\
&\quad \lesssim |a(x) - m_B(a)| r^{\frac{n}{p}} \int_{2r}^\infty \log\left(1 + \frac{t}{r}\right) t^{-\frac{n}{p}} \|f\|_{L^p(B(x_0,t))} \frac{dt}{t} \\
&\qquad + \int_r^\infty \frac{1}{t^{n+1}} \left(\int_{B(x_0,t)} |f(y)|^p \, d\mu(y) \right)^{1/p} \\
&\qquad \times \left(\int_{B(x_0,t)} |m_B(a) - a(y)|^{p'} \, d\mu(y) \right)^{1/p'} dt.
\end{aligned} \tag{6.4}$$

By the John–Nirenberg inequality (see Lemma 6.5) and the growth condition (4.1), we have

$$\begin{aligned}
&\int_r^\infty \frac{1}{t^{n+1}} \left(\int_{B(x_0,t)} |f(y)|^p \, d\mu(y) \right)^{1/p} \left(\int_{B(x_0,t)} |m_B(a) - a(y)|^{p'} \, d\mu(y) \right)^{1/p'} dt \\
&\qquad \lesssim \|a\|_* \int_r^\infty \log\left(2 + \frac{t}{r}\right) \frac{1}{t^{n/p+1}} \left(\int_{B(x_0,t)} |f(y)|^p \, d\mu(y) \right)^{1/p} dt.
\end{aligned} \tag{6.5}$$

Putting toghther Lemma 6.5, (6.4) and (6.5), we obtain

$$\int_B \left(\int_{\mathbb{R}^d \setminus 2B} \frac{|a(x)-a(y)|}{|x-y|^n} |f(y)|\, d\mu(y) \right)^p d\mu(x)$$

$$\lesssim \|a\|_* r^{\frac{n}{p}} \int_{2r}^{\infty} \log\left(1+\frac{t}{r}\right) t^{-\frac{n}{p}} \|f\|_{L^p(B(x_0,t))} \frac{dt}{t}$$

$$+ \|a\|_* \mu(B)^{1/p} \int_r^{\infty} \log\left(2+\frac{t}{r}\right) \frac{1}{t^{n/p+1}} \left(\int_{B(x_0,t)} |f(y)|^p \, d\mu(y) \right)^{1/p} dt$$

$$\lesssim \|a\|_* r^{\frac{n}{p}} \int_r^{\infty} \log\left(2+\frac{t}{r}\right) t^{-\frac{n}{p}} \|f\|_{L^p(B(x_0,t))} \frac{dt}{t}.$$

Thus, (6.3) is proved. □

We can prove Theorem 6.2 by using (6.2) analogously to Theorem 4.3, once Lemma 6.6 is proven.

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