MORPHIC MODULES

W.K. Nicholson
Department of Mathematics
University of Calgary
Calgary T2N 1N4, Canada
wknichol@ucalgary.ca

E. Sánchez Campos
Department of Algebra
University of Málaga
29080-Málaga, Spain
esperanz@agt.cie.uma.es

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Abstract

A module M is called morphic if $M/M\alpha \cong ker(\alpha)$ for all endomorphisms α in $end(M)$, and a ring R is called a left morphic ring if ${}_RR$ is a morphic module. We consider the open question when the matrix ring $M_n(R)$ is left morphic by relating it to when R^n is morphic as a left R-module. More generally, we investigate when M being morphic implies that $end(M)$ is left morphic, and conversely. Finally, we relate the morphic condition to internal cancellation in the module.

A module M is called morphic if $M/M\alpha \cong ker(\alpha)$ for every endomorphism α of M. We show that a module M is morphic if and only if $M/N \cong K \subseteq M$ implies that $M/K \cong N$, and use this to show that every uniserial module of finite length over a commutative (in fact left duo) ring is morphic. It is an open problem when the matrix ring $M_n(R)$ is left morphic, and we show that this holds if and only if R^n is morphic as a left module. Hence the rings R for which this happens for every $n \geq 1$ are a Morita invariant class containing the left FP-injective rings. An example is given of a commutative, FP-injective ring with simple, essential socle, in which $J(R)^3 = 0$, but which is not morphic.

Direct summands of a morphic module are again morphic, and the study of when a direct sum is morphic leads to a characterization of the morphic, finitely generated abelian groups, and to a characterization of when semisimple modules are morphic. If $E = end({}_RM)$, we say that M is self-projective if $M\gamma \subseteq M\alpha$, γ, $\alpha \in E$, implies that $\gamma \in E\alpha$. We then show that: (1) M is morphic and self-projective if and only if E is left morphic and M generates $ker(\beta)$ for each $\beta \in E$; (2) E is unit regular if and only if M is morphic and $ker(\alpha)$ is a direct summand of M for all $\alpha \in E$; and (3) M has internal cancellation if and only if every regular endomorphism in E is morphic.

Throughout this paper every ring R is associative with unity and all modules are unitary. We often abbreviate $J(R) = J$. A submodule $N \subseteq M$ is said to be an essential submodule (written $N \subseteq^{ess} M$) if $N \cap K \neq 0$ for every nonzero submodule K of M. We write $K \subseteq^{\oplus} M$ to mean that K is a direct summand of the module M, and the "length" of a module means the composition length. We denote left and right annihilators of a subset $X \subseteq R$ by $\mathtt{l}(X)$ and $\mathtt{r}(X)$ respectively, and we write $\mathbb{Z}$ for the ring of integers and $\mathbb{Z}_n$ for the ring of integers modulo n. Modules are left modules unless otherwise stated, and module homomorphisms are written on the right of their arguments.

1. Examples

An endomorphism α of a module M will be called **morphic** if $M/M\alpha \cong ker(\alpha)$, that is if the dual of the Noether isomorphism theorem holds for α. The module M is called a **morphic module** if every endomorphism is morphic. If R is a ring, an element a in R is called **left morphic** if right multiplication $\cdot a : {}_RR \to {}_RR$ is a morphic endomorphism, that is if $R/Ra \cong \mathtt{l}(a)$. The ring itself is called a **left morphic ring** if every element is left morphic, that is if ${}_RR$ is a morphic module. These rings were studied in [10].

Lemma 1. *The following conditions are equivalent for an endomorphism α of a module M:*

(1) α is morphic, that is $M/M\alpha \cong ker(\alpha)$.
(2) There exists $\beta \in end(M)$ such that $M\beta = ker(\alpha)$ and $ker(\beta) = M\alpha$.
(3) There exists $\beta \in end(M)$ such that $M\beta \cong ker(\alpha)$ and $ker(\beta) = M\alpha$.

Proof. Given (1), let $\sigma : M/M\alpha \to ker(\alpha)$ be an isomorphism and define $\beta : M \to M$ by $m\beta = (m+M\alpha)\sigma$. Then $M\beta = (M/M\alpha)\sigma = ker(\alpha)$, and $ker(\beta) = \{m \mid (m+M\alpha)\sigma = 0\} = M\alpha$. Hence (1)⇒(2), and (2)⇒(3) is clear. But if (3) holds then $M/M\alpha = M/ker(\beta) \cong M\beta \cong ker(\alpha)$, proving (1). □

A ring R is called *directly finite* if $ab = 1$ in R implies $ba = 1$.

Corollary 2. *A morphic endomorphism is monic if and only if it is epic. Hence every left morphic ring is directly finite.*

Every idempotent $\alpha : M \to M$ is morphic (take $\beta = 1 - \alpha$ in Lemma 1) as is every automorphism α (take $\beta = 0$). Hence every simple module is morphic by Schur's lemma. An endomorphism $\alpha : M \to M$ is called *unit regular* if $\alpha\sigma\alpha = \alpha$ for some automorphism σ of M, equivalently if $\alpha = \pi\sigma$ where $\pi^2 = \pi$ and σ is an automorphism.

Example 3. *Let $\alpha \in end(M)$ be a morphic element. If $\sigma : M \to M$ is an automorphism, then $\sigma\alpha$ and $\alpha\sigma$ are both morphic. In particular, every unit regular endomorphism is morphic.*

Proof. By Lemma 1, choose $\beta \in end(M)$ such that $M\beta = ker(\alpha)$ and $ker(\beta) = M\alpha$. Then $M\sigma\alpha = M\alpha = ker(\beta) = ker(\beta\sigma^{-1})$, and $ker(\sigma\alpha) = (ker(\alpha))\sigma^{-1} = M(\beta\sigma^{-1})$, so $\sigma\alpha$ is morphic. Similarly, $M\alpha\sigma = (ker(\beta))\sigma = ker(\sigma^{-1}\beta)$ and $ker(\alpha\sigma) = ker(\alpha) = M\beta = M(\sigma^{-1}\beta)$, so σa is morphic. □

Example 4. *In Lemma 1 we cannot replace (3) by "There exists $\beta \in end(M)$ such that $M\beta = ker(\alpha)$ and $ker(\beta) \cong M\alpha$". Moreover, the composite of morphic endomorphisms need not be morphic.*

Proof. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_2 \oplus \mathbb{Z}_4$. Define $\sigma : \mathbb{Z}_2 \to \mathbb{Z}_4$ by $(n+2\mathbb{Z})\sigma = (2n+4\mathbb{Z})$ for $n \in \mathbb{Z}$, and then define α and β in $end(M)$ by $(x,y)\alpha = (0, x\sigma)$ and $(x,y)\beta = (0,y)$. Then $M\beta = ker(\alpha)$ and $ker\beta \cong M\alpha$, but $M/M\alpha \ncong ker(\alpha)$ so α is not morphic.

Next define $\theta : \mathbb{Z}_4 \to \mathbb{Z}_2$ by $(n+4\mathbb{Z})\theta = n+2\mathbb{Z}$. Then define $\pi : M \to M$ and $\gamma : M \to M$ by $(x,y)\pi = (x,0)$ and $(x,y)\gamma = (y\theta, x\sigma)$ where $\sigma : \mathbb{Z}_2 \to \mathbb{Z}_4$ is the above map. Then π is morphic (it is an idempotent) and γ is morphic as $M/M\gamma \cong \mathbb{Z}_2 \cong ker(\gamma)$, but $\pi\gamma = \alpha$ is not morphic. □

We now give a characterization of a morphic module in terms of the lattice of submodules.

Theorem 5. *The following are equivalent for a module M :*

(1) M is morphic.

(2) If $M/K \cong N$ where K and N are submodules of M, then $M/N \cong K$.

Proof. (1)$\Rightarrow$(2). If $\sigma : M/K \to N$ is an isomorphism, define $\alpha : M \to M$ by $m\alpha = (m+K)\sigma$. Then $N = M\alpha$ and $K = ker(\alpha)$. By (1) there exists $\beta : M \to M$ such that $M\beta = ker(\alpha) = K$ and $ker(\beta) = M\alpha = N$. Hence $M/N = M/ker(\beta) \cong M\beta = K$, proving (2).

(2)$\Rightarrow$(1). Given $\alpha : {}_RM \to {}_RM$, write $K = ker(\alpha)$ and $N = M\alpha$. Hence (2) provides an isomorphism $\tau : M/N \to K$. If we define $\beta : M \to M$ by $m\beta = (m+N)\tau$, then $M\beta = K = ker(\alpha)$ and $ker(\beta) = N = M\alpha$. This proves (1). □

Corollary 6. *Let M be morphic and $K \subseteq M$. If $M \cong K$ then $K = M$; if $M \cong M/K$ then $K = 0$.*

Proof. If $M \cong K$ then $M/0 \cong K$ so $M/K \cong 0$ by Theorem 5, that is $K = M$. If $M \cong M/K$ then $K \cong M/M$ by Theorem 5, so $K = 0$. □

Corollary 6 shows that the infinite cyclic group $\mathbb{Z}$ is not morphic as a $\mathbb{Z}$-module. However, $\mathbb{Z}_n$ is morphic by Theorem 5 because finite cyclic groups are isomorphic if and only if they have the same order. We will characterize the morphic, finitely generated abelian groups in Theorem 26.

A module is called *uniserial* if its submodule lattice is a chain. Hence the Prüfer group $\mathbb{Z}_{p^\infty}$ is uniserial, injective and artinian but it is not morphic by Corollary 6. Thus the injective hull of the (simple) morphic module $\mathbb{Z}_p$ is not morphic.

Corollary 7. *The following conditions are equivalent for a uniserial module with submodule lattice $0 = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_{n-2} \subset K_{n-1} \subset K_n = M$.*

(1) M is morphic.

(2) For each $t = 0, 1, \cdots, n$, if $M/K_t \cong K_{n-t}$ then $M/K_{n-t} \cong K_t$.

Proof. We have (1)$\Rightarrow$(2) by Theorem 5. Conversely, given (2) let $M/K \cong N$, $K \subseteq M$ and $N \subseteq M$. If $K = K_t$ then $N \cong M/K_t$ so N has length $n - t$. Hence $N = K_{n-t}$ so, by (2), $M/N \cong K_{n-(n-t)} = K_t = K$. □

A ring is called *left duo* if every left ideal is two-sided.

Proposition 8. *Let ${}_RM$ be a uniserial module of finite length.*

(1) If every submodule of M is an image of M then M is morphic.

(2) In particular, M is morphic if $M = Rm$ where $\mathtt{l}(m)$ is an ideal of R.

(3) Hence every uniserial left module of finite length over a left duo ring is morphic.

Proof. (1). Let $M/K \cong N$ where $K \subseteq M$ and $N \subseteq M$, and let $\alpha : M \to K$ be epic. If M and K have lengths n and t respectively, then N and $ker(\alpha)$ both have length $n - t$, so $N = ker(\alpha)$ because M is uniserial. Thus $M/N = M/ker(\alpha) \cong K$, so (1) follows by Theorem 5.

(2). If $K \subseteq M$ is a submodule then K is principal (it is uniserial), say $K = Rk$. Then the map $\alpha : M \to K$ given by $(rm)\alpha = rk$ is well defined [if $rm = 0$ then, writing $k = sm$, we have $rk = r(sm) = 0$ because $rs \in \mathtt{l}(m)$ by hypothesis]. Hence (2) follows from (1).

(3). This follows from (2). □

A ring R is called *left special* if it is left morphic, local and J is nilpotent; equivalently [10, Theorem 9] if ${}_RR$ is a uniserial module of finite length.

Corollary 9. *If ${}_RR$ is left special then every principal left module is morphic.*

Proof. Every principal module over a left uniserial ring of finite length is again uniserial of finite length, and R is left duo (indeed the only left ideals are $R \supset J \supset J^2 \supset \cdots \supset J^{n-1} \supset J^n = 0$—see [10, Theorem 9]). Hence Proposition 8 applies □

The converse of (2) in Proposition 8 is false: Take $M = R/L$ and $m = 1 + L$, where L is a maximal left ideal of R that is not an ideal. The next example shows that the converse of (1) in Proposition 8 is also false.

Example 10. *If D is a division ring and $R = \begin{bmatrix} D & D \\ 0 & D \end{bmatrix}$, let $M = \begin{bmatrix} 0 & D \\ 0 & D \end{bmatrix}$. Then ${}_RM$ is uniserial of length 2 that but not every submodule is an image.*

Proof. The only proper submodule of M is $K = \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix}$, so M is uniserial, and hence morphic by Example 13. But K is not an image of M since; if so $M/K \cong K$, contrary to the fact that $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ annihilates M/K but not K. □

We now look at pairs of dual conditions that imply that a module of finite length is morphic.

Lemma 11. *A module ${}_RM$ of finite length is morphic if either (1) or (2) holds:*
(1) (a) Every submodule of M is isomorphic to an image of M; and
(b) If $length(K) = length(K')$ where $K, K' \subseteq M$, then $M/K \cong M/K'$.
(2) (c) Every image of M is isomorphic to a submodule of M; and
(d) If $length(M/K) = length(M/K')$ where $K, K' \subseteq M$, then $K \cong K'$.

Proof. Let $M/K \cong N$ where $K, N \subseteq M$; we show $M/N \cong K$. Write $n = length(M)$ and $t = length(K)$. To prove (1), write $K \cong M/N'$ by (a) where $N' \subseteq M$. Then $length(N') = n - t = length(N)$ so $M/N \cong M/N' \cong K$ using (b). For (2), write $M/N \cong K'$ by (c) where $K' \subseteq M$. Then $length(M/K') = n - t = length(M/K)$ so $K \cong K' \cong M/N$ using (d). □

Note that both (b) and (d) in Lemma 11 hold in a uniserial module of finite length. The module ${}_{\mathbb{Z}}M = \mathbb{Z}_2 \oplus \mathbb{Z}_3$ is morphic (as we shall see in Example 13 below) and has finite length, but (b) and (d) both fail for M.

Example 12. *There exists a non-morphic module M of length 8 in which both (a) and (c) hold (and so both (b) and (d) fail).*

Proof. Let $M = \mathbb{Z}_2 \oplus \mathbb{Z}_4$. By the fundamental theorem of finite abelian groups, the only images of M are M, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbb{Z}_2$, $\mathbb{Z}_4$ and 0, each is isomorphic to a submodule, and these are the only submodules. However M is not morphic. In fact, if $K = \mathbb{Z}_2 \oplus 2\mathbb{Z}_4$ and $N = \mathbb{Z}_2 \oplus 0$ then $M/K \cong \mathbb{Z}_2 \cong N$ but $M/N \cong \mathbb{Z}_4 \ncong K$. □

It follows by Lemma 11 that every homogeneous semisimple module M of finite length is morphic. In fact we will show (see Theorem 35) that *every* semisimple module of finite length is morphic.

Example 13. *Every module M of length 2 is morphic.*

Proof. Let K and N be submodules of M with $M/K \cong N$; by Theorem 5 we must show that $M/N \cong K$. Since M has length 2 this holds if $K = 0, M$ or if $N = 0, M$. So we may assume that K and N are simple (and hence maximal in M). Since M has length 2 it is either uniserial or semisimple. In the first case M has submodule lattice $0 \subset P \subset M$, so $K = P = N$ and we have $M/N = M/K \cong N = K$, as required.

On the other hand, suppose that M is semisimple, say $M = X \oplus Y$ where X and Y are simple. If $X \cong Y$ then M is homogeneous and so is morphic by Lemma 11. So assume that $X \ncong Y$. Since $K \neq M$, either $X \nsubseteq K$ or $Y \nsubseteq K$, so $M = K \oplus X$ or $M = K \oplus Y$ because K is maximal in M. We assume that $M = K \oplus X$ (the other case is analogous). Similarly, $M = N \oplus X$ or $M = N \oplus Y$. If $M = N \oplus X$ then $Y \cong M/X \cong N \cong M/K \cong X$, contrary to our assumption. So $M = N \oplus Y$ and we obtain $M/N \cong Y \cong M/X \cong K$, as required. □

Example 14. *There exists a non-morphic module M with submodule lattice $0 \subset Q \subset P \subset M$.*

Proof. Let $R = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \mid a, b, c \in F \right\}$ where F is a field. If $M = \begin{bmatrix} F \\ F \\ F \end{bmatrix}$, $P = \begin{bmatrix} F \\ F \\ 0 \end{bmatrix}$, and $Q = \begin{bmatrix} F \\ 0 \\ 0 \end{bmatrix}$, then $M/P \cong Q$ via $\begin{bmatrix} c \\ d \\ a \end{bmatrix} \mapsto \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$ but $M/Q \ncong P$ because $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ annihilates M/Q but not P. It is routine to verify that 0, P, Q and M are the only submodules of M. □

Note that $\mathbb{Z}_8$ and the module M in Example 14 are uniserial modules with isomorphic submodule lattices, but $\mathbb{Z}_8$ is morphic while M is not.

It would be interesting to see an example of a non-morphic module of finite length in which every submodule is isomorphic to an image and every image is isomorphic to a submodule. However, if the module *is* morphic (possibly not of finite length), these two conditions are equivalent, and we obtain a satisfying symmetry.

Theorem 15. *The following are equivalent for a morphic module ${}_RM$:*

(1) Every submodule of M is isomorphic to an image of M.

(2) Every image of M is isomorphic to a submodule of M.

In this case, the following hold:

(a) If N and N' are submodules of M then $M/N \cong M/N'$ if and only if $N \cong N'$.

(b) M is finitely generated if and only if M is noetherian.

Proof. (1)⇒(2). Given $K \subseteq M$, let $K \cong M/N$ where $N \subseteq M$ by (1). Then $M/K \cong N$ because M is morphic, and (2) follows.

(2)⇒(1). Given $K \subseteq M$, let $M/K \cong N \subseteq M$ by (2). Then $M/N \cong K$ because M is morphic, so (1) is proved.

(a). If $M/N \cong M/N'$, let $M/N \cong K \subseteq M$ by (2). Then $N \cong M/K \cong N'$ because M is morphic. Conversely, suppose $N \cong N'$ and, by (2), let $M/N \cong K \subseteq M$ and $M/N' \cong K' \subseteq M$. Since M is morphic we have $M/K \cong N \cong N' \cong M/K'$, so $K \cong K'$ as above. Hence $M/N \cong M/N'$, as required.

(b). If M is finitely generated and $K \subseteq M$, then $K \cong M/N$ for some $N \subseteq M$ by (1). Hence K is finitely generated, and so M is noetherian. □

A left morphic ring R satisfies conditions (1) and (2) in Theorem 15 if and only if every left ideal is principal, and these rings are called left *P-morphic* in [11]. Accordingly, we call a module **P-morphic** if it is morphic and satisfies the equivalent conditions (1) and (2) in Theorem 15.

Every simple module is P-morphic; a semisimple module is P-morphic if and only if it is morphic because conditions (1) and (2) in Theorem 15 are automatic. The $\mathbb{Z}$-module $\mathbb{Z}_n$ is P-morphic because it is morphic by Example and, if $K \subseteq \mathbb{Z}_n$ is a subgroup, $|K| = m$, then $\mathbb{Z}_n$ has a subgroup N with $|N| = n/m$ so $K \cong \mathbb{Z}_n/N$. On the other hand, the module M in Example 10 is a uniserial, morphic module of length 2 that is not P-morphic.

2. Matrix Rings

It is an open problem to determine when $M_n(R)$ is left morphic. With an eye on this question, we examine the relationship between R-modules and modules over the matrix ring $M_n(R)$. The free module R^n plays a basic role. We assume that $n = 2$ for convenience.

Write $S = M_2(R)$. If $K \subseteq S$ is a left ideal then K has the form $K = \begin{bmatrix} X \\ X \end{bmatrix}$ where ${}_RX \subseteq {}_RR^2$ is given by $X = \{\bar{x} \in R^2 \mid \begin{bmatrix} \bar{x} \\ 0 \end{bmatrix} \in K\}$. In fact, $X \mapsto \begin{bmatrix} X \\ X \end{bmatrix}$ is a lattice isomorphism from the left R-submodules of R^2 to the left ideals of S. Given ${}_RM$, $\begin{bmatrix} M \\ M \end{bmatrix}$ is a left S-module via matrix multiplication. If $\alpha : {}_RM \to {}_RN$ the map $\begin{bmatrix} \alpha \\ \alpha \end{bmatrix} : \begin{bmatrix} M \\ M \end{bmatrix} \to \begin{bmatrix} N \\ N \end{bmatrix}$ is S-linear where $\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = \begin{bmatrix} m_1\alpha \\ m_2\alpha \end{bmatrix}$. Moreover, every S-linear map $\lambda : \begin{bmatrix} M \\ M \end{bmatrix} \to \begin{bmatrix} N \\ N \end{bmatrix}$ has this form. Indeed, if $m \in M$ then $\begin{bmatrix} m \\ 0 \end{bmatrix} \lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} m \\ 0 \end{bmatrix} \lambda \right) = \begin{bmatrix} m\alpha \\ 0 \end{bmatrix}$ for some element $m\alpha \in N$, and this defines an R-morphism $\alpha : {}_RM \to {}_RN$. Now observe that $\begin{bmatrix} 0 \\ m \end{bmatrix} \lambda = \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} m \\ 0 \end{bmatrix} \right) \lambda = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} m\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ m\alpha \end{bmatrix}$ for all $m \in M$, and it follows that $\lambda = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix}$. Clearly $\begin{bmatrix} \alpha \\ \alpha \end{bmatrix}$ is one-to-one or onto if and only if the same is true of α.

Lemma 16. *Let* $K = \begin{bmatrix} X \\ X \end{bmatrix}$ *and* $N = \begin{bmatrix} Y \\ Y \end{bmatrix}$ *denote left ideals of* $S = M_2(R)$. *Then:*

(1) $S/K \cong \begin{bmatrix} R^2/X \\ R^2/X \end{bmatrix}$.

(2) $S/K \cong N$ *as* S*-modules if and only if* $R^2/X \cong Y$ *as* R*-modules.*

Proof. The map $\theta : S \to \begin{bmatrix} R^2/X \\ R^2/X \end{bmatrix}$ given by $\begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix} \mapsto \begin{bmatrix} \bar{p} + X \\ \bar{q} + X \end{bmatrix}$ is an S-morphism with kernel $\begin{bmatrix} X \\ X \end{bmatrix}$. This proves (1). Hence $S/K \cong N$ if and only if $\begin{bmatrix} R^2/X \\ R^2/X \end{bmatrix} \cong \begin{bmatrix} Y \\ Y \end{bmatrix}$; if and only if $R^2/X \cong Y$. This proves (2). □

Lemma 17. *Let* R *be a ring and write* $S = M_n(R)$.

(1) Every left ideal of S *is isomorphic to an image of* S *if and only if every submodule of* ${}_RR^n$ *is isomorphic to an image of* ${}_RR^n$.

(2) Every image of ${}_SS$ *is isomorphic to a left ideal of* S *if and only if every image of* ${}_RR^n$ *is isomorphic to a submodule of* ${}_RR^n$.

Proof. We prove the result for $n = 2$; the general case is analogous. Assume that left ideals of S are images of S. If $X \subseteq R^n$ and we write $K = \begin{bmatrix} X \\ X \end{bmatrix}$ then $K \cong S/N$ for some $N = \begin{bmatrix} Y \\ Y \end{bmatrix} \subseteq S$, so $X \cong R^n/Y$. This proves half of (1); the converse is similar as is the proof of (2). □

Theorem 18. *Let* R *be a ring.*

(1) ${}_RR^n$ *is morphic if and only if* $M_n(R)$ *is left morphic.*

(2) ${}_RR^n$ *is P-morphic if and only if* $M_n(R)$ *is left P-morphic.*

Proof. We prove it for $n = 2$; the general case is analogous.

(1). If ${}_RR^2$ is morphic, let $K = \begin{bmatrix} X \\ X \end{bmatrix}$ and $N = \begin{bmatrix} Y \\ Y \end{bmatrix}$ be left ideals of $S = M_2(R)$ such that $K \cong S/N$. Then $X \cong R^2/Y$ by Lemma 16, so $Y \cong R^2/X$ by hypothesis, whence $N \cong S/K$. Hence S is left morphic, and the converse is similar.

(2). If ${}_RR^2$ is P-morphic then $S = M_2(R)$ is left morphic by (1). If ${}_RR^2$ is P-morphic, let $K = \begin{bmatrix} X \\ X \end{bmatrix}$ be a left ideal of S, where $X \subseteq R^2$. Then $X \cong R^2/Y$ for some $Y \subseteq R^2$ by hypothesis. If we write $N = \begin{bmatrix} Y \\ Y \end{bmatrix}$ then $S/N \cong K$, proving that S is left P-morphic. The converse is similar. □

The property of being left morphic (or being left P-morphic) does not pass to matrix rings. In fact, Example 7 of [11] exhibits a left and right artinian, left P-morphic ring R such that $M_2(R)$ is not left morphic. Accordingly, the following classes of rings are of interest. A ring R is called **strongly left morphic** (respectively **strongly left P-morphic**) if every matrix ring $M_n(R)$ is left morphic (respectively left P-morphic).

Direct products of strongly left morphic rings, and finite direct products of strongly left P-morphic rings, are again of the same type by [10, Example 2] and [11, Example 3], respectively, using the canonical isomorphism $M_n(\Pi_i R_i) \cong \Pi_i M_n(R_i)$.

Every unit regular ring is strongly left morphic (unit regularity is a Morita invariant by [8, Corollary 3]). However:

Question. *If a ring R is strongly left morphic and $J(R) = 0$, is R unit regular?*

Call a ring R left *special* if R is local, left morphic and $J = J(R)$ is nilpotent (see [10, Theorem 9]). Then the left and right special rings are all strongly P-morphic by [11, Theorem 15]. Note that Example 7 of [11] is actually a left special ring R for which $M_2(R)$ is not left morphic.

Theorem 19. *The following are equivalent for a ring R :*

(1) R is strongly left morphic (respectively strongly left P-morphic).

(2) ${}_RR^n$ is morphic (respectively P-morphic) for each $n \geq 1$.

(3) Every finitely generated projective left R-module is morphic (respectively P-morphic).

Proof. (1)⇔(2) by Theorem 18 and (2)⇔(3) because direct summands of morphic or P-morphic modules are again of the same type (Theorem 23 below). □

Theorem 20. *Let R denote a ring.*

(1) If R is strongly left morphic the same is true of eRe for any idempotent $e \in R$.

(2) Being strongly left morphic is a Morita invariant.

Proof. Write $S = M_n(R)$.

(1). If R is strongly left morphic and $e^2 = e \in R$, write $\bar{e} = eI \in S$ where I is the identity matrix. Then $M_n(eRe) = \bar{e}S\bar{e}$ is left morphic by [10, Theorem 15] because S is left morphic. Hence eRe is is strongly left morphic.

(2). If R is strongly left morphic then $M_m(S) \cong M_{mn}(R)$ is left morphic for all $m \geq 1$ by hypothesis. Hence S is strongly left morphic, and we are done by (1). □

We do not know if part (1) of Theorem 20 holds for strongly left P-morphic rings because we do not know the answer to:

Question. *If R is left P-morphic and $ReR = R$ where $e^2 = e$, is eRe left P-morphic?*

Note that Re is a P-morphic module by Theorem 23.

A ring R is said to be *stably finite* if $M_n(R)$ is directly finite for every $n \geq 1$. Hence every strongly left morphic ring is stably finite by Corollary 2 because $M_n(R)$ is morphic.

Theorem 21. *A ring R is strongly left morphic and semiperfect if and only if R is a finite product of matrix rings over local, strongly left morphic rings.*

Proof. This follows from [10, Theorem 29] and the fact that $M_n(R)$ is strongly left morphic if R is local and strongly left morphic. □

Question. *If $M_2(R)$ is left morphic, is R strongly left morphic?*

A ring R is called right *FP-injective* if every R-morphism from a finitely generated submodule of a free right R-module F to R extends to F. Every strongly left morphic ring R is right FP-injective by [14, Theorem 1] because every left morphic ring is right P-injective by [10, Theorem 24]. (Here a ring R is called right *P-injective* [12] if every R-morphism $aR \to R$, $a \in R$, extends to R, equivalently if aR is a right annihilator for each $a \in R$.)

Example 22. *There exists a commutative, local, FP-injective ring R with $J^3 = 0$ and J^2 simple and essential in R, but which is not morphic.*

Proof. Let $R = F[x_1, x_2, \cdots]$ where F is a field and the x_i are commuting indeterminants satisfying the relations

$$x_i^3 = 0 \text{ for all } i, \quad x_i x_j = 0 \text{ for all } i \neq j, \quad \text{and} \quad x_i^2 = x_j^2 \text{ for all } i \text{ and } j.$$

Write $m = x_1^2 = x_2^2 = \cdots$, so that $m^2 = 0 = x_i m$ for all i. Then R is commutative and $J = span_F\{m, x_1, x_2, \cdots\}$, so R is local, $R/J \cong F$, $J^3 = 0$ and $J^2 = Rm = Fm$ is simple and essential in R (see also [2, Example 6]). Moreover, R is P-injective; in fact $\mathtt{lr}(A) = A$ for every ideal A. This follows from the

Claim. If we denote $X = span_F\{x_1, x_2, \cdots\}$, the maps $A \mapsto A \cap X$ and $U \mapsto Fm \oplus U$ are mutually inverse lattice isomorphisms between the lattice of all ideals $A \neq 0, R$ of R and the lattice of all F-subspaces U of X.

Proof. Observe first that $Fm \oplus U$ is an ideal for each U because $(Fm \oplus U)R \subseteq FmR + UR \subseteq Fm + (UF + mF) \subseteq Fm + U$. The composites of the given maps are $U \mapsto Fm \oplus U \mapsto (Fm \oplus U) \cap X = U$ because $U \subseteq X$, and $A \mapsto A \cap X \mapsto Fm \oplus (A \cap X) = A$ by the modular law because $Fm \subseteq A \subseteq J = Fm \oplus X$. Hence the two maps are mutually inverse; they clearly preserve inclusions. This proves the Claim. Note that $J = Fm \oplus X$.

The Claim also shows that R is not morphic since otherwise R is special and has only finitely many ideals. R is FP-injective by [13, Example 5.45]. □

3. Direct Sums

The classes of morphic and P-morphic modules are not closed under taking direct sums. In fact The $\mathbb{Z}$-modules $\mathbb{Z}_2$ and $\mathbb{Z}_4$ are both P-morphic but $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ is not even morphic by Example 4. However we do have

Theorem 23. *Every direct summand of a morphic or P-morphic module is again of the same type.*

Proof. Let $M = P \oplus Q$ be morphic, and suppose that $P/K \cong N$ where $K \subseteq P$ and $N \subseteq P$. Then $M/K \cong (P/K) \oplus Q \cong N \oplus Q$, so (as M is morphic) $K \cong M/(N \oplus Q) \cong P/N$. Hence P is morphic by Theorem 5.

Now assume that $M = P \oplus Q$ is P-morphic. Let $N \subseteq P$ be a submodule; we must show that N is an image of P. As M is P-morphic, $M/(N \oplus 0)$ is isomorphic to a submodule of M by Theorem 15, say $M/(N \oplus 0) \cong H \subseteq M$. Since M is morphic, this gives $M/H \cong N \oplus 0 \cong N$, so it suffices to show that M/H is an image of P. Observe that $H \cong M/(N \oplus 0) \cong (P/N) \oplus Q$, so write $H = H_1 \oplus Q_1$ where $Q_1 \cong Q$. Now $(M/Q_1)/(H/Q_1) \cong M/H \cong N$ so N is an image of M/Q_1. Hence it suffices to show that $M/Q_1 \cong P$. But M is P-morphic so the fact that $Q \cong Q_1$ gives $M/Q \cong M/Q_1$ by Theorem 15, so $P \cong M/Q_1$, as required. □

Lemma 24. *Let $M = K \oplus N$ be a morphic module. If $\lambda : K \to N$ is R-linear then*

$$K \oplus (N/K\lambda) \cong ker(\lambda) \oplus N.$$

In particular:

(1) If λ is monic then $N \cong K \oplus (N/K\lambda)$.

(2) If λ is epic then $K \cong ker(\lambda) \oplus N$.

Hence if K is isomorphic to either a submodule or an image of N then K is isomorphic to a direct summand of N.

Proof. Given λ define $\bar{\lambda} : M \to M$ by $(k+n)\bar{\lambda} = k\lambda$ for all $k+n \in M$. Then $M\bar{\lambda} = K\lambda$ and $ker(\bar{\lambda}) = ker(\lambda) \oplus N$. Since M is morphic we have $M/M\bar{\lambda} \cong ker(\bar{\lambda})$, that is $K \oplus (N/K\lambda) \cong ker(\lambda) \oplus N$. Now (1) and (2) are immediate. As to the last sentence: If K is isomorphic to a submodule of N then K is isomorphic to a direct summand of N by (1); If K is isomorphic to an image of N then K is isomorphic to a direct summand of N by (2) with K and N interchanged. □

We can refine Lemma 24 as follows: If $M = K \oplus N$ is morphic and $X \subseteq K$, then

$$K/X \cong N \quad \text{if and only if} \quad K \cong X \oplus N.$$

In fact, $K/X \cong M/(X \oplus N)$, so $K/X \cong N \Leftrightarrow M/(X \oplus N) \cong N \Leftrightarrow M/N \cong X \oplus N \Leftrightarrow K \cong X \oplus N$.

We conclude with a consequence of Theorems 23 and 19 characterizing the morphic, finitely generated abelian groups. The following will be needed.

Lemma 25. *If ${}_RM$ and ${}_RN$ are morphic modules for which $hom_R(M,N) = 0 = hom_R(N,M)$, then $M \oplus N$ is morphic.*

Proof. If $\lambda \in end(M \oplus N)$ then $\lambda = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ in matrix form where $\alpha \in end(M)$ and $\beta \in end(N)$. If $\alpha' \in end(M)$ and $\beta' \in end(N)$ satisfy $M\alpha = ker(\alpha')$, $ker(\alpha) = M\alpha'$, $M\beta = ker(\beta')$, $ker(\beta) = M\beta'$ then $im(\lambda) = ker(\lambda')$ and $ker(\lambda) = im(\lambda')$ where $\lambda' = \begin{bmatrix} \alpha' & 0 \\ 0 & \beta' \end{bmatrix}$.
□

Applying the fundamental theorem of finitely generated abelian groups, we obtain

Theorem 26. *A finitely generated abelian group is morphic if and only if it is finite and each p-primary component has the form $(\mathbb{Z}_{p^k})^n$ for some $n \geq 0$ and $k \geq 0$.*

Proof. If ${}_{\mathbb{Z}}M$ is finitely generated and morphic, then M is a direct sum of cyclic groups, and contains no infinite summand by Theorem 23 because $\mathbb{Z}$ is not morphic. Hence M is finite and so is the direct sum of its primary components. If M_p is the p-primary component of M then $M_p \cong \mathbb{Z}_{p^{m_1}} \oplus \mathbb{Z}_{p^{m_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{m_n}}$ where $m_1 \geq m_2 \geq \cdots \geq m_s$. But if $m_i > m_{i+1}$ then $\mathbb{Z}_{p^{m_i}} \oplus \mathbb{Z}_{p^{m_{i+1}}}$ is not morphic by Lemma 24 because $\mathbb{Z}_{p^{m_{i+1}}}$ is indecomposable and $\mathbb{Z}_{p^{m_i}}$ embeds in it. This contradicts Theorem 23, and so shows that $M \cong (\mathbb{Z}_{p^{m_1}})^n$.

Conversely, by Lemma 25 it suffices to show that $(\mathbb{Z}_{p^k})^n$ is morphic for each $n \geq 0$ and $k \geq 0$. The ring $S = \mathbb{Z}_{p^k}$ is (left and right) special so $M_n(S)$ is left morphic by [10, Theorem 17]. Hence S^n is morphic as a left S-module by Theorem 19, and hence as a left $\mathbb{Z}$-module. □

We remark in passing that, for integers $n \geq 1$ and $m \geq 1$, $\mathbb{Z}_m \oplus \mathbb{Z}_n$ is morphic if and only if $m = da$ and $n = db$ where $gcd(d, a) = 1$, $gcd(d, b) = 1$, and $gcd(a, b) = 1$

4. Endomorphism Rings

The original motivation for studying morphic modules stems from Erlich's theorem characterizing when an endomorphism α is unit regular. This is included in the next lemma (along with Azumaya's characterization of when α is regular) and reveals the connection with morphic endomorphisms. We sketch brief proofs for completeness.

Lemma 27. *Let α be an endomorphism of ${}_RM$.*

(1) ***Azumaya*** *[1]. α is regular if and only if both $M\alpha$ and $ker(\alpha)$ are direct summands of M.*

(2) ***Ehrlich*** *[5]. α is unit regular if and only if it is both regular and morphic.*

Proof. (1) If $M = M\alpha \oplus K = ker(\alpha) \oplus N$ then $M\alpha = N\alpha$ so $M = N\alpha \oplus K$. Then $\alpha\beta\alpha = \alpha$ where $\beta : M \to M$ is (well-) defined by $(n\alpha + k)\beta = n$, $n \in N$, $k \in K$. The converse is routine.

(2) If α is unit regular it is morphic by Example 3. Conversely, if $M = M\alpha \oplus K = N \oplus ker(\alpha)$ then $K \cong M/M\alpha \cong ker(\alpha)$ because α is morphic, say via $\gamma : K \to ker(\alpha)$. We have $M = N\alpha \oplus K$ as in (1), and we define $\sigma : M \to M$ by $(n\alpha + k)\sigma = n + k\gamma$, $n \in N$, $k \in K$. Then σ is well defined, $M\sigma = N + K\gamma = N + ker(\alpha) = M$, and $ker(\sigma) = 0$ because γ is monic, so σ is a unit in $end(M)$. Finally, $\alpha\sigma\alpha = \alpha$ because they agree on $ker(\alpha)$ and N. □

Example 28. *If $end(M)$ is unit regular then M is morphic.*

Zelmanowitz [15] calls a module ${}_RM$ *regular* if for any $m \in M$ there exists $\lambda \in M^* = hom_R(M, R)$ such that $(m\lambda)m = m$. In this case, if we write $e = m\lambda$, then $e^2 = e$, $\lambda : Rm \to Re$ is an isomorphism (so Rm is projective), and $M = Rm \oplus W$ where $w = \{w \in M \mid (w\lambda)m = 0\}$. Zelmanowitz proves [15, Theorem 1.6] that every finitely generated submodule of a regular module M is a projective direct summand of M. Our interest lies in a larger class of modules wherein every principal submodule Rm, $m \in M$, is a direct summand of M (equivalently [15, Corollary 1.3] if every finitely generated submodule is a summand).

Proposition 29. *Let M be a finitely generated module in which every principal submodule is a direct summand. Then M is morphic if and only if $end(M)$ is unit regular. In particular, every finite dimensional, regular module M is morphic.*

Proof. If M is morphic and $\alpha \in end(M)$, then $M\alpha$ is a summand by [15, Corollary 1.3]. Since M is morphic, we have $ker(\alpha) \cong M/M\alpha$ so $ker(\alpha)$ is finitely generated, and so is a summand. Hence α is regular in $end(M)$, and so is unit regular by Lemma 27. The converse also follows from Lemma 27. If M is finite dimensional then M is finitely generated (the proof of [15, Theorem 1.8] goes through), so the last statement follows by Example 28 because $end(M)$ is semisimple artinian by [15, Theorem 4.8]. □

Corollary 30. *Let M be a finitely generated module over a commutative ring. Then M is regular and morphic if and only if M is projective and $end(M)$ is unit regular.*

Proof. If M is regular and finitely generated then M is projective by [15, Corollary 1.7] and $end(M)$ is unit regular by Proposition 29. Conversely, if M is projective and $end(M)$ is unit regular then M is regular (by [15, Theorem 3.8]) and morphic (by Proposition 29).□

A module ${}_RM$ will be called **image-projective** if, whenever $M\gamma \subseteq M\alpha$ where $\alpha, \gamma \in E = end(M)$, then $\gamma \in E\alpha$, that is if the map δ exists in the diagram when α and γ are given. If we insist on this whenever $M\alpha$ is replaced by an arbitrary module (equivalently any quotient of M) the module M is called quasi-projective.

$$\begin{array}{ccccc} & & M & & \\ & \overset{\delta}{\swarrow} & \downarrow \gamma & & \\ M & \overset{\alpha}{\to} & M\alpha & \to 0 \end{array}$$

Hence every quasi-projective module ${}_RM$ is image-projective. In a different direction, [12, Theorem 1.5] shows that ${}_RM$ is image-projective if $E = end(M)$ is right P-injective, and that the converse holds if M cogenerates $M/M\beta$ for every $\beta \in E$.

A module M is said to *generate* a submodule $K \subseteq M$ if $K = \Sigma\{M\lambda \mid \lambda \in E, M\lambda \subseteq K\}$, and we say that M **generates its kernels** if M generates $ker(\beta)$ for each $\beta \in E$.

Lemma 31. *Let ${}_RM$ be a module and write $E = end({}_RM)$.*
(1) If E is left morphic then M is image-projective.
(2) If M is morphic and image-projective, then E is left morphic.
(3) If M is morphic then it generates its kernels.
(4) If E is left morphic and M generates its kernels, then M is morphic.

Proof. (1). Assume that E is left morphic. If $M\gamma \subseteq M\alpha$ then $\mathtt{r}_E(\alpha) \subseteq \mathtt{r}_E(\gamma)$, so $E\gamma \subseteq E\alpha$ because E is right P-injective by hypothesis using [10, Theorem 24].

(2). If M is morphic and image-projective, and given $\alpha \in E$, choose (by hypothesis) $\beta \in E$ such that $M\alpha = ker(\beta)$ and $M\beta = ker(\alpha)$. Then $E\alpha \subseteq \mathtt{l}_E(\beta)$ because $\alpha\beta = 0$. Conversely, if $\gamma \in \mathtt{l}_E(\beta)$ then $M\gamma \subseteq ker(\beta) = M\alpha$, so $\gamma \in E\alpha$ because M is image-projective. Thus $E\alpha = \mathtt{l}_E(\beta)$, and $E\beta = \mathtt{l}_E(\alpha)$ follows in the same way. Hence E is left morphic by [10, Lemma 1].

(3). Let $\beta \in E$. Since M is morphic, choose $\lambda \in E$ such that $M\lambda = ker(\beta)$. This proves that M generates its kernels.

(4). If E is left morphic and M generates its kernels, and given $\alpha \in E$, choose $\beta \in E$ such that $E\alpha = \mathtt{l}_E(\beta)$ and $E\beta = \mathtt{l}_E(\alpha)$. Then $M\alpha \subseteq ker(\beta)$ because $\alpha\beta = 0$. By hypothesis we have $ker(\beta) = \Sigma\{M\lambda \mid \lambda \in E \text{ and } M\lambda \subseteq ker(\beta)\}$. But $M\lambda \subseteq ker(\beta)$ implies $\lambda \in \mathtt{l}_E(\beta) = E\alpha$, say $\lambda = \gamma\alpha$, $\gamma \in E$. Hence $M\lambda = M\gamma\alpha \subseteq M\alpha$, whence $ker(\beta) \subseteq M\alpha$. This shows that $M\alpha = ker(\beta)$, and $M\beta = ker(\alpha)$ is proved in the same way. □

Combining these we get a characterization of the image-projective, morphic modules.

Theorem 32. *The following are equivalent for a module M :*
(1) M is morphic and image-projective.
(2) $end(M)$ is left morphic and M generates its kernels.

Proof. (1)⇒(2) by (2) and (3) of Lemma 31, and (2)⇒(1) by (4) and (1) of Lemma 31. □

Let R be a ring. Since R^n is image-projective and generates its submodules, taking $M = R^n$ in Theorem 32 provides another proof of the fact (in Theorem 18) that $M_n(R)$ is left morphic if and only if R^n is morphic as a left R-module.

If every principal submodule of a module ${}_RM$ is a direct summand, it is routine to check that M generates all its submodules. Hence Theorem 32 gives

Corollary 33. *Assume that $Rm \subseteq^{\oplus} M$ for all $m \in {}_RM$ (for example if M is regular). Then M is morphic and image-projective if and only if $end(M)$ is left morphic.*

Corollary 34. *Let ${}_RM$ be a module and assume that $E = end(M)$ is regular. Then M is morphic and image-projective if and only if E is unit regular.*

Proof. If M is morphic and image-projective then E is morphic by Theorem 32. Since E is regular, it is unit regular by Lemma 27. Conversely, if E is unit regular then M is morphic by Example 3. But E is left morphic by Lemma 27, so M is image-projective by Lemma 31. □

We can say more for a semisimple module.

Theorem 35. *The following are equivalent for a semisimple module M :*
(1) M is morphic.
(2) $end(M)$ is unit regular.
(3) Each homogeneous component of M is artinian.
In this case $end(M)$ is a direct product of matrix rings over division rings.

Proof. Write $E = end(M)$ and $E_i = end(H_i)$ where $\{H_i \mid i \in I\}$ are the homogeneous components of M. Then $E \cong \Pi_i E_i$.

(1)⇒(2). Assume that M is morphic. Since M is image-projective (every semisimple module is quasi-projective), E is left morphic by Lemma 31. Now (2) follows by Lemma 27 because E is regular (M is semisimple).

(2)⇒(3). Note that $end(H_i)$ is left morphic for each i by (2) and Lemma 27. If $H_i = K \oplus K \oplus \cdots$ where K is simple, then $(k_1, k_2, \cdots) \mapsto (0, k_1, k_2, \cdots)$ is monic in $end(H_i)$ and not epic, contrary to Corollary 2. This proves (3).

(3)⇒(1). Given (3) the last statement follows because $E \cong \Pi_i E_i$. In particular E is unit regular, and hence left morphic. But M generates its kernels because $end(M)$ is regular, so M is morphic by Lemma 31. □

Proposition 36. *A ring R is semisimple artinian if and only if every finitely generated (respectively every 2-generated) left module is morphic.*

Proof. If R is semisimple artinian then every module is semisimple, so Theorem 35 applies. Conversely, let $X \subseteq R$ be a left ideal and let $\lambda : R \to R/X$ be the coset map. Since $M = R \oplus (R/X)$ is morphic by hypothesis, Lemma 24 shows that $R \cong X \oplus (R/X)$. In particular, R/X is projective so X is a direct summand of R. It follows that R is semisimple artinian. □

Note that Corollary 9 shows that if R is left uniserial of finite length then every principal left R-module is morphic, but the converse is false (consider $\mathbb{Z}_n$).

Question. *Which rings have every principal left module morphic*?

Call a module M **kernel-direct** if $ker(\alpha) \subseteq^{\oplus} M$ for every $\alpha \in end(M)$, and call M **image-direct** if $im(\alpha) \subseteq^{\oplus} M$ for each $\alpha \in end(M)$. Modules with regular endomorphism ring (and hence all semisimple modules) enjoy both properties. Note that, by Lemma 1, a morphic module is kernel direct if and only if it is image direct. We can give a partial converse for Example 28.

Theorem 37. *The following are equivalent for a module M :*
(1) $end(M)$ is unit regular.
(2) M is morphic and kernel-direct.
(3) M is morphic and image-direct.

Proof. (1)⇒(2). Given (1), M is morphic by Example 28 and kernel direct by Lemma 27.
(2)⇒(3). Given $\alpha \in end(M)$, then $M\alpha = ker(\beta)$ for some $\beta \in end(M)$ by Lemma 1.
(3)⇒(1). Given $\alpha \in end(M)$, $M\alpha$ is a summand of M by (3), as is $ker(\alpha)$ (since $ker(\alpha) = M\beta$ for some $\beta \in end(M)$). Hence $end(M)$ is regular by (1) of Lemma 27, so it is unit regular by (2) of the same lemma. □

If R is a ring then ${}_RR$ is image direct if and only if R is regular, so Theorem 37 shows again that the unit regular rings are just the regular, left morphic rings. On the other hand, ${}_RR$ is kernel-direct if and only if $\mathtt{l}(a) \subseteq^{\oplus} {}_RR$ for all $a \in R$, that is if and only if every principal left ideal Ra is projective. These are called left *PP rings*, and Theorem 37 gives

Corollary 38. *A ring R is unit regular if and only if it is a left morphic, left PP ring.*

Corollary 39. *The following are equivalent for a finite dimensional module M :*
(1) M is morphic and kernel-direct.
(2) M is morphic and image-direct.
(3) $end(M)$ is semisimple artinian.

Proof. We have seen (1)⇔(2), and these conditions imply that $E = end(M)$ is semilocal by Corollary 2 and the Camps-Dicks theorem [4]. Now (3) follows because E is unit regular by Theorem 37. Conversely, given (3), M is morphic by Example 28, and hence kernel-direct by Lemma 27. □

Lemma 40. *Every kernel-direct module is image-projective.*

Proof. Let $M\beta \subseteq M\alpha$ where α and β are in $end(M)$. Write $M = ker(\alpha) \oplus N$. Hence $M\alpha = N\alpha$ so $\sigma : M\alpha \to N$ is well defined by $(n\alpha)\sigma = n$, $n \in N$. Then $M \xrightarrow{\beta} M\beta \subseteq M\alpha = N\alpha \xrightarrow{\sigma} N \subseteq M$ so $\lambda = \beta\sigma$ is in $end(M)$ and $\lambda\alpha = \beta$. This shows that M is image-projective. □

Since kernel-direct modules generate their kernels, Theorem 32 gives

Corollary 41. *If M is kernel-direct then M is morphic if and only if $end(M)$ is left morphic.*

We conclude this section with a look at when $end({}_RM)$ is right morphic. We call a module ${}_RM$ **image-injective** if R-linear maps $M\beta \to M$ extend to M for each $\beta \in end({}_RM)$, and we say that M **cogenerates its cokernels** if it cogenerates $M/M\beta$ for each $\beta \in end({}_RM)$. Note that ${}_RR$ is image-injective if and only if R is left P-injective, and ${}_RR$ cogenerates its cokernels if and only if R is right P-injective. With this, we can obtain "dual" versions of Lemma 31 and Theorem 32.

Lemma 42. *Let $_RM$ be a module and write $E = end(_RM)$.*
(1) If E is right morphic then M is image-injective.
(2) If M is morphic and image-injective, then E is right morphic.
(3) If M is morphic then it cogenerates its cokernels.
(4) If E is right morphic and M cogenerates its cokernels, then M is morphic.

Proof. (1). Let $\gamma : M\beta \to M$, $\beta \in E$. Then $\lambda = \beta\gamma \in E$ and $\mathtt{l}_E(\beta) \subseteq \mathtt{l}_E(\lambda)$, so $\lambda \in \beta E$ because E is left P-injective, say $\lambda = \beta\alpha$. Hence $(m\beta)\alpha = m\lambda = (m\beta)\gamma$, so α extends γ.

(2). Let $\alpha \in E$. Since M is morphic, choose $\beta \in E$ such that $M\beta = ker(\alpha)$ and $M\alpha = ker(\beta)$. Then $\beta E \subseteq \mathtt{r}_E(\alpha)$, and we claim that this is equality. If $\lambda \in \mathtt{r}_E(\alpha)$ then $M\alpha \subseteq ker(\lambda)$, that is $ker(\beta) \subseteq ker(\lambda)$. Hence $\gamma : M\beta \to M$ is well defined by $(m\beta)\gamma = m\lambda$. By hypothesis, let $\delta \in E$ extend γ. Then $\lambda = \beta\delta \in \beta E$, and we have shown that $\beta E = \mathtt{r}_E(\alpha)$. The proof that $\alpha E = \mathtt{r}_E(\beta)$ is similar.

(3). Given $\beta \in E$, let $\alpha \in E$ satisfy $M\beta = ker(\alpha)$. Then $\gamma : M/M\beta \to M$ is well defined by $(m + M\beta)\gamma = m\alpha$. Moreover, if $m_1 + M\beta \neq 0$ then $m_1 \notin M\beta = ker(\alpha)$, so $(m_1 + M\beta)\gamma = m_1\alpha \neq 0$. This proves (3).

(4). Let $\alpha \in E$. Since E is right morphic, there exists $\beta \in E$ such that $\beta E = \mathtt{r}_E(\alpha)$ and $\alpha E = \mathtt{r}_E(\beta)$. Then $M\beta \subseteq ker(\alpha)$ and we claim this is equality. If $k \in ker(\alpha)$ but $k \notin M\beta$ then, by hypothesis, choose $\gamma : M/M\beta \to M$ such that $(k + M\beta)\gamma \neq 0$. If $\delta \in E$ is defined by $m\delta = (m + M\beta)\gamma$ then $\beta\delta = 0$ so $\delta \in \mathtt{r}_E(\beta) = \alpha E$, say $\delta = \alpha\lambda$ where $\lambda \in E$. But then $0 \neq k\delta = k\alpha\lambda = 0\lambda = 0$; and it follows that $M\beta = ker(\alpha)$. Similarly, $M\alpha = ker(\beta)$. □

Theorem 43. *The following are equivalent for a module $_RM$.*
(1) M is morphic and image-injective.
(2) $end(M)$ is right morphic and M cogenerates its cokernels.

If R is left and right P-injective and we take $M = {_RR}$ then this shows (again) that R is left morphic if and only if R is right morphic. Note finally that the "dual" of Lemma 40 (every kernel-direct module is image-projective) is true: Every image-direct module is clearly image-injective.

5. Internal Cancellation

A module $_RM$ is said to have **internal cancellation** (**IC**) if it satisfies the following condition:

$$\text{If } M = N \oplus K = N_1 \oplus K_1 \text{ and } N \cong N_1, \text{ then } K \cong K_1.$$

Each indecomposable module M has IC, and we have

Proposition 44. *Every direct summand of an IC module has IC.*

Proof. If $M = N \oplus K$ has IC, let $N = N_1 \oplus N_2 = N' \oplus N''$ where $N_1 \cong N'$. Then $M = (N_1 \oplus K) \oplus N_2 = (N' \oplus K) \oplus N''$ where $N_1 \oplus K \cong N' \oplus K$. Hence $N_2 \cong N''$ by hypothesis, as required. □

We say that a ring R has **left internal cancellation** (**left IC**) if $_RR$ has IC. This holds if and only if $Re \cong Rf$, $e^2 = e$, $f^2 = f$, implies that $R(1-e) \cong R(1-f)$. In this case, we have $f = u^{-1}eu$ for some unit $u \in R$.

If $\pi^2 = \pi$ and $\tau^2 = \tau$ in $E = end(_RM)$, it is routine to verify that $M\pi \cong M\tau$ as R-modules if and only if $E\pi \cong E\tau$ as left E-ideals. It follows that $_RM$ has IC if and only if $E = end(_RM)$ has left IC. Hence Proposition 44 gives

Corollary 45. *If R has left IC then eRe has left IC for every idempotent $e \in R$.*

Goodearl [7, Theorem 4.1] shows that for a module M with $end(M)$ regular, internal cancellation is equivalent to $end(M)$ being unit regular. In fact

Theorem 46. *A module ${}_RM$ has IC if and only if every regular element in $end({}_RM)$ is morphic.*

Proof. If M has IC let α be regular in $end(M)$, so that $M = M\alpha \oplus K = ker(\alpha) \oplus N$. Then $M\alpha \cong M/ker(\alpha) \cong N$ so $K \cong ker(\alpha)$. Hence $M/M\alpha \cong K \cong ker(\alpha)$, proving (2). Conversely, if $M = N \oplus K = N_1 \oplus K_1$ and $\gamma : N \to N_1$ is an isomorphism, define $\alpha : M \to M$ by $(n + k)\alpha = n\gamma$, $n \in N$, $k \in K$. Then $M\alpha = N\gamma = N_1$ and $ker(\alpha) = K$ are both summands of R. Thus α is regular in $end({}_RM)$ (using Lemma 27) and so our hypothesis gives $M/M\alpha \cong ker(\alpha)$. Hence $K_1 \cong M/N_1 = M/M\alpha \cong ker(\alpha) = K$, as required. □

Corollary 47. *Every morphic module has IC.*

The converse to of Corollary 47 is false: Every local ring has left (and right) IC, but need not be left morphic. In fact $\mathbb{Z}_{(p)}$ is a counterexample that is a local integral domain, and Example 22 shows that the counterexample can actually be chosen to be commutative and P-injective. For an artinian example, the $\mathbb{Z}$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ can be verified to have IC but is not morphic by Example 4.

Corollary 48. *Given ${}_RM$, $end(M)$ is unit regular if and only if M has IC and $end(M)$ is regular.*

Proof. If $end(M)$ is unit regular then M has IC by Theorem 46. Conversely, if M has IC and $end(M)$ is regular then $end(M)$ is morphic by Theorem 46, and so is unit regular by Lemma 27. □

The next result extends Theorem 35.

Corollary 49. *A semisimple module is morphic if and only if it has IC.*

Proof. If M has IC, let N and K be submodules with $M/K \cong N$. Since M is semisimple let $M = K \oplus K' = N \oplus N'$. Then $N \cong M/K \cong K'$ so, because M has IC, $K \cong N' \cong M/N$. Hence M is morphic. The converse is by Corollary 48. □

A ring R is said to have *stable range* 1 if $aR + bR = R$ implies that $a + bt$ is a unit in R for some t. Evans [6] showed that if $end(M)$ has stable range 1 then M is *cancellable* in the sense that $M \oplus A \cong M \oplus B$ implies $A \cong B$. Camillo and Yu [3, Theorem 3] show that an exchange ring R has stable range 1 if and only if every regular element of R is unit regular (extending the same result of Kaplansky in the regular case).

Theorem 50. *A morphic module is cancellable if it is either injective or has the finite exchange property.*

Proof. Let M be a morphic module. If M is injective then $end(M)$ is right morphic by Theorem 43, and so is directly finite by Corollary 2. This means that M is directly finite ([7, Lemma 5.1]) and so is cancellable by Mohamed and Müller [9, Proposition 1.29]. Now suppose that M has the finite exchange property. Since M has IC by Corollary 47, it is cancellable by Mohamed and Müller [9, Proposition 1.23]. □

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