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The solution of the time-fractional diffusion equation by the generalized differential transform method

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ABSTRACT

In the present paper, a general recurrence relation for determining the solutions of the time-fractional diffusion equation is obtained with the generalized differential transform method. The obtained relation will help us to solve time-fractional diffusion equations with various external forces and initial conditions. Four illustrative examples are given.

1. Introduction

Fractional diffusion equations represent extensions of basic equations of mathematical physics. Analytical methods used to solve these equations have very restricted applications and the numerical techniques commonly used give rise to rounding of errors. These kinds of equations have been intensively studied since the nineties [1–3].

Recently, Saha Ray and Bera have used the Adomian decomposition method (ADM) to find the solution of a time-fractional diffusion equation of order $\beta = 1/2$ in [4]. Das has used the variational iteration method (VIM) for the same equation with the initial conditions 1, x and x^2 in [5], and for time-fractional diffusion equation of order β, $(0 < \beta \leq 1)$ with the initial conditions x^n, $n \in \mathbb{N}$ in [6].

There are several definitions of a fractional derivative. In this paper, we deal only with the Caputo fractional derivative. The Caputo fractional derivative of order β is defined as

$$D_{t_0}^{\beta} f(t) = J_{t_0}^{m-\beta}\left[\frac{d^m}{dt^m} f(t)\right], \quad m-1 < \beta \leq m$$

where m is a positive integer [7,8]. Here $J_{t_0}^{\mu}$ is the Riemann–Liouville integral operator of order $\mu \geqslant 0$ defined by

$$J_{t_0}^{\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_{t_0}^{t} (t-\tau)^{\mu-1} f(\tau) d\tau, \quad t > t_0,$$

$$J_{t_0}^{0} f(t) = f(t).$$

In this work, we consider the following time-fractional diffusion equation

$$D_t^{\beta} u = \lambda \frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}\left(F(x) u(x,t)\right), \quad 0 < \beta \leq 1, \ x, t > 0 \tag{1}$$

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with initial condition $u(x, 0) = f(x)$. Here λ is a positive constant, $F(x)$ is the external force, $u(x, t)$ represents the probability density function of finding a particle at the point x in the time t and $D_t^{\beta} u(x, t) = J_0^{1-\beta}\left[\frac{\partial}{\partial t} u(x, t)\right]$.

The main object of this work is to give a general recurrence relation for obtaining the solutions of (1) with the generalized differential transform method (GDTM). This is the first study that this type of problem is solved for a given general external force $F(x)$ and initial condition $f(x)$.

2. Generalized differential transform

Consider a function of two variables $u(x, t)$, and suppose that it can be represented as a product of two single-variable functions, i.e., $u(x, t) = f(x)g(t)$. Based on the properties of generalized one-dimensional differential transform, the function $u(x, t)$ can be represented as

$$\begin{aligned} u(x, t) &= \sum_{k=0}^{\infty} F_{\alpha}(k)(x - x_0)^{k\alpha} \sum_{h=0}^{\infty} G_{\beta}(h)(t - t_0)^{h\beta} \\ &= \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U_{\alpha,\beta}(k, h)(x - x_0)^{k\alpha}(t - t_0)^{h\beta} \end{aligned} \tag{2}$$

where $0 < \alpha, \beta \le 1$. The generalized two-dimensional differential transform of the function $u(x, t)$ is given by

$$U_{\alpha,\beta}(k, h) = \frac{1}{\Gamma(\alpha k + 1)\Gamma(\beta h + 1)}\left[(D_{x_0}^{\alpha})^k (D_{t_0}^{\beta})^h u(x, t)\right]_{(x_0,t_0)} \tag{3}$$

where $(D_{x_0}^{\alpha})^k = D_{x_0}^{\alpha} D_{x_0}^{\alpha} \cdots D_{x_0}^{\alpha}$, k-times. Besides, Eq. (2) is also called as the generalized inverse differential transform of $U_{\alpha,\beta}(k, h)$. In case of $\alpha = \beta = 1$, the generalized two-dimensional differential transform (3) reduces to the classical two-dimensional differential transform.

Suppose that $U_{\alpha,\beta}(k, h)$, $V_{\alpha,\beta}(k, h)$ and $W_{\alpha,\beta}(k, h)$ are the differential transformations of the functions $u(x, t)$, $v(x, t)$ and $w(x, t)$, respectively. Based on Eqs. (2) and (3), we have the following results [9–13].

1. If $u(x, t) = v(x, t) \pm w(x, t)$, then $U_{\alpha,\beta}(k, h) = V_{\alpha,\beta}(k, h) \pm W_{\alpha,\beta}(k, h)$.
2. If $u(x, t) = \lambda v(x, t)$, $\lambda \in \mathbb{R}$, then $U_{\alpha,\beta}(k, h) = \lambda V_{\alpha,\beta}(k, h)$.
3. If $u(x, t) = v(x, t)w(x, t)$, then
$$U_{\alpha,\beta}(k, h) = \sum_{r=0}^{k} \sum_{s=0}^{h} V_{\alpha,\beta}(r, h - s)W_{\alpha,\beta}(k - r, s).$$
4. If $u(x, t) = (x - x_0)^{n\alpha}(t - t_0)^{m\beta}$, then $U_{\alpha,\beta}(k, h) = \delta(k - n)\delta(h - m)$.
5. If $u(x, t) = f(x)g(t)$ and the function $f(x) = x^{\lambda}h(x)$, where $\lambda > -1$, $h(x)$ has the generalized Taylor series expansion $h(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^{\alpha n}$, and
(a) $\beta < \lambda + 1$ and α arbitrary or
(b) $\beta \ge \lambda + 1$, α arbitrary and $a_n = 0$ for $n = 0, 1, \ldots, m - 1$, where $m - 1 < \beta \le m$,
then the generalized differential transform (3) becomes
$$U_{\alpha,\beta}(k, h) = \frac{1}{\Gamma(\alpha k + 1)\Gamma(\beta h + 1)}\left[D_{x_0}^{\alpha k}(D_{t_0}^{\beta})^h u(x, t)\right]_{(x_0,t_0)}.$$
6. If $u(x, t) = D_{x_0}^{\gamma} v(x, t)$, $m - 1 < \gamma \le m$, and $v(x, t) = f(x)g(t)$, then
$$U_{\alpha,\beta}(k, h) = \frac{\Gamma(\alpha k + \gamma + 1)}{\Gamma(\alpha k + 1)} V_{\alpha,\beta}(k + \gamma/\alpha, h).$$
7. If $u(x, t) = D_{t_0}^{\gamma} v(x, t)$, $m - 1 < \gamma \le m$, and $v(x, t) = f(x)g(t)$, then
$$U_{\alpha,\beta}(k, h) = \frac{\Gamma(\beta h + \gamma + 1)}{\Gamma(\beta h + 1)} V_{\alpha,\beta}(k, h + \gamma/\beta).$$

3. The solution of the general problem

In this section we present a general recurrence relation for the generalized differential transform (GDT) of Eq. (1) by the following theorem. With the help of the given general recurrence relation, the solutions of time-fractional diffusion equations will be easily calculate with GDTM.

Theorem 1. *If the function $F(x)$ has the Maclaurin series expansion $F(x) = \sum_{n=0}^{\infty} a_n x^n$ with a radius of convergence $R > 0$ where $a_n = \frac{F^{(n)}(0)}{n!}$ for $n = 0, 1, 2, \ldots$, then the GDT of Eq. (1) is*

$$U_{1,\beta}(k, h+1) = \frac{\Gamma(\beta h + 1)}{\Gamma(\beta(h+1)+1)}(k+1)\left[\lambda(k+2)U_{1,\beta}(k+2, h) - \sum_{i=0}^{k+1} a_{k+1-i}U_{1,\beta}(i, h)\right]. \tag{4}$$

Proof. Applying GDT to both sides of the Eq. (1), we get

$$\frac{\Gamma(\beta(h+1)+1)}{\Gamma(\beta h+1)}U_{1,\beta}(k, h+1) = \lambda(k+1)(k+2)U_{1,\beta}(k+2, h) - (k+1)\sum_{r=0}^{k+1}\sum_{s=0}^{h}\sum_{n=0}^{\infty} a_n\delta(r-n)\delta(h-s)U_{1,\beta}(k-r+1, s).$$

To make $\delta(r-n) \neq 0$ and $\delta(h-s) \neq 0$, we take $s = h$ and $n = r$. Thus,

$$\frac{\Gamma(\beta(h+1)+1)}{\Gamma(\beta h+1)}U_{1,\beta}(k, h+1) = \lambda(k+1)(k+2)U_{1,\beta}(k+2, h) - (k+1)\sum_{r=0}^{k+1} a_r U_{1,\beta}(k-r+1, h).$$

Consequently, taking $r = k - i + 1$, we obtain (4). □

Furthermore, the reader easily see that

$$U_{1,\beta}(k, 0) = \frac{f^{(k)}(0)}{k!}, \quad k = 0, 1, 2, \ldots \tag{5}$$

is the GDT of the initial condition $u(x, 0) = f(x)$.

4. Numerical examples

In this section, we have selected four examples which will ultimately show the simplicity and effectiveness of the proposed general recurrence relation (4). The first three examples also solved with VIM and ADM before.

Example 2. Taking $F(x) = -x$, $\lambda = 1$ in (1) and choosing $f(x) = 1$, we get the following initial value problem:

$$\begin{cases} D_t^{\beta} u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial}{\partial x}(xu), \\ u(x, 0) = 1. \end{cases} \tag{6}$$

Since $F(x) = -x$, we find $a_1 = -1$ and $a_n = 0$ for $n \neq 1$. Also, substituting $f(x) = 1$ in (5), we have

$$U_{1,\beta}(0, 0) = 1, \qquad U_{1,\beta}(k, 0) = 0 \quad \text{for } k \neq 0.$$

Therefore, using the general recurrence relation (4), we get

$$\begin{aligned}
U_{1,\beta}(0, 1) &= \frac{1}{\Gamma(\beta+1)}, \qquad U_{1,\beta}(k, 1) = 0 \quad \text{for } k \neq 0, \\
U_{1,\beta}(0, 2) &= \frac{1}{\Gamma(2\beta+1)}, \qquad U_{1,\beta}(k, 2) = 0 \quad \text{for } k \neq 0, \\
U_{1,\beta}(0, 3) &= \frac{1}{\Gamma(3\beta+1)}, \qquad U_{1,\beta}(k, 3) = 0 \quad \text{for } k \neq 0, \\
U_{1,\beta}(0, 4) &= \frac{1}{\Gamma(4\beta+1)}, \qquad U_{1,\beta}(k, 4) = 0 \quad \text{for } k \neq 0, \\
&\vdots \\
U_{1,\beta}(0, h) &= \frac{1}{\Gamma(h\beta+1)}, \qquad U_{1,\beta}(k, h) = 0 \quad \text{for } k \neq 0.
\end{aligned}$$

Thus, the solution of (6) is given by

$$u(x, t) = \sum_{h=0}^{\infty} \frac{\left(t^{\beta}\right)^h}{\Gamma(h\beta+1)} = E_{\beta}(t^{\beta})$$

where $E_{\beta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n\beta+1)}$, $\beta > 0$ is the Mittag-Leffler function in one parameter.

For the special case $\beta = 1/2$, the above result is in complete agreement with [5].

Example 3. Taking $F(x) = -x$, $\lambda = 1$ in (1) and choosing $f(x) = x$, we get the following initial value problem:

$$\begin{cases} D_t^\beta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial}{\partial x}(xu), \\ u(x, 0) = x. \end{cases} \tag{7}$$

Since $F(x) = -x$, we find $a_1 = -1$ and $a_n = 0$ for $n \neq 1$. Also, substituting $f(x) = x$ in (5), we have

$$U_{1,\beta}(1, 0) = 1, \qquad U_{1,\beta}(k, 0) = 0 \quad \text{for } k \neq 1.$$

Therefore, using the general recurrence relation (4), we get

$$\begin{aligned}
&U_{1,\beta}(1, 1) = \frac{2}{\Gamma(\beta + 1)}, \qquad U_{1,\beta}(k, 1) = 0 \quad \text{for } k \neq 1,\\
&U_{1,\beta}(1, 2) = \frac{2^2}{\Gamma(2\beta + 1)}, \qquad U_{1,\beta}(k, 2) = 0 \quad \text{for } k \neq 1,\\
&U_{1,\beta}(1, 3) = \frac{2^3}{\Gamma(3\beta + 1)}, \qquad U_{1,\beta}(k, 3) = 0 \quad \text{for } k \neq 1,\\
&U_{1,\beta}(1, 4) = \frac{2^4}{\Gamma(4\beta + 1)}, \qquad U_{1,\beta}(k, 4) = 0 \quad \text{for } k \neq 1,\\
&\vdots\\
&U_{1,\beta}(1, h) = \frac{2^h}{\Gamma(h\beta + 1)}, \qquad U_{1,\beta}(k, h) = 0 \quad \text{for } k \neq 1.
\end{aligned}$$

Thus, the solution of (7) is given by

$$u(x, t) = x \sum_{h=0}^{\infty} \frac{\left(2t^\beta\right)^h}{\Gamma(h\beta + 1)} = xE_\beta(2t^\beta).$$

The same solution has been obtained by Das [6]. Also the above result is in complete agreement with [4,5] for $\beta = 1/2$.

Example 4. Taking $F(x) = -x$, $\lambda = 1$ in (1) and choosing $f(x) = x^2$, we get the following initial value problem:

$$\begin{cases} D_t^\beta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial}{\partial x}(xu), \\ u(x, 0) = x^2. \end{cases} \tag{8}$$

Since $F(x) = -x$, we find $a_1 = -1$ and $a_n = 0$ for $n \neq 1$. Also, substituting $f(x) = x^2$ in (5), we have

$$U_{1,\beta}(2, 0) = 1, \qquad U_{1,\beta}(k, 0) = 0 \quad \text{for } k \neq 2.$$

Therefore, using the general recurrence relation (4), we get

$$\begin{aligned}
&U_{1,\beta}(0, 1) = \frac{2}{\Gamma(\beta + 1)}, \qquad U_{1,\beta}(2, 1) = \frac{3}{\Gamma(\beta + 1)}, \qquad U_{1,\beta}(k, 1) = 0 \quad \text{for } k \neq 0, 2,\\
&U_{1,\beta}(0, 2) = \frac{8}{\Gamma(2\beta + 1)}, \qquad U_{1,\beta}(2, 2) = \frac{3^2}{\Gamma(2\beta + 1)}, \qquad U_{1,\beta}(k, 2) = 0 \quad \text{for } k \neq 0, 2,\\
&U_{1,\beta}(0, 3) = \frac{26}{\Gamma(3\beta + 1)}, \qquad U_{1,\beta}(2, 3) = \frac{3^3}{\Gamma(3\beta + 1)}, \qquad U_{1,\beta}(k, 3) = 0 \quad \text{for } k \neq 0, 2,\\
&\vdots\\
&U_{1,\beta}(0, h) = \frac{3^h - 1}{\Gamma(h\beta + 1)}, \qquad U_{1,\beta}(2, h) = \frac{3^h}{\Gamma(h\beta + 1)}, \qquad U_{1,\beta}(k, h) = 0 \quad \text{for } k \neq 0, 2.
\end{aligned}$$

Thus, the solution of (8) is given by

$$\begin{aligned} u(x,t) &= x^2 + \frac{(2+3x^2)}{\Gamma(\beta+1)}t^{\beta} + \frac{(8+9x^2)}{\Gamma(2\beta+1)}t^{2\beta} + \frac{(26+27x^2)}{\Gamma(3\beta+1)}t^{3\beta} + \cdots \\ &= \sum_{h=0}^{\infty} \frac{3^h(1+x^2)-1}{\Gamma(h\beta+1)}t^{h\beta} \\ &= E_{\beta}(pt^{\beta}) \end{aligned}$$

where $p^h = 3^h(1+x^2) - 1$.

The same solution has been obtained by Das [6]. Also this solution is in complete agreement with [5] for $\beta = 1/2$.

Example 5. Taking $F(x) = e^{-x}$, $\lambda = 1$ in (1) and choosing $f(x) = e^x$, we get the following initial value problem:

$$\begin{cases} D_t^{\beta} u = \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial}{\partial x}(e^{-x}u), \\ u(x,0) = e^x. \end{cases} \tag{9}$$

Since $F(x) = e^{-x}$, we find $a_n = \frac{(-1)^n}{n!}$, $n = 0, 1, 2 \ldots$. Also, substituting $f(x) = e^x$ in (5), we have

$$U_{1,\beta}(k,0) = \frac{1}{k!}, \quad k = 0, 1, 2, \ldots.$$

Therefore, using the general recurrence relation (4), we get

$$U_{1,\beta}(k,h) = \frac{1}{k!\Gamma(h\beta+1)}, \quad k, h = 0, 1, 2, \ldots.$$

The first few components of $U_{1,\beta}(k,h)$ can be seen in Table 1. Thus, the solution of (9) is given by

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!\Gamma(h\beta+1)} x^k t^{h\beta} \\ &= e^x E_{\beta}(t^{\beta}). \end{aligned}$$

Table 1
Some values of the components $U_{1,\beta}(k,h)$ obtained from (4) for Eq. (9).

$U_{1,\beta}(k,h)$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$h=0$	1	1	$\frac{1}{2!}$	$\frac{1}{3!}$	$\frac{1}{4!}$
$h=1$	$\frac{1}{\Gamma(\beta+1)}$	$\frac{1}{\Gamma(\beta+1)}$	$\frac{1}{2!\Gamma(\beta+1)}$	$\frac{1}{3!\Gamma(\beta+1)}$	$\frac{1}{4!\Gamma(\beta+1)}$
$h=2$	$\frac{1}{\Gamma(2\beta+1)}$	$\frac{1}{\Gamma(2\beta+1)}$	$\frac{1}{2!\Gamma(2\beta+1)}$	$\frac{1}{3!\Gamma(2\beta+1)}$	$\frac{1}{4!\Gamma(2\beta+1)}$
$h=3$	$\frac{1}{\Gamma(3\beta+1)}$	$\frac{1}{\Gamma(3\beta+1)}$	$\frac{1}{2!\Gamma(3\beta+1)}$	$\frac{1}{3!\Gamma(3\beta+1)}$	$\frac{1}{4!\Gamma(3\beta+1)}$
$h=4$	$\frac{1}{\Gamma(4\beta+1)}$	$\frac{1}{\Gamma(4\beta+1)}$	$\frac{1}{2!\Gamma(4\beta+1)}$	$\frac{1}{3!\Gamma(4\beta+1)}$	$\frac{1}{4!\Gamma(4\beta+1)}$

5. Conclusion

In this paper, we present a general recurrence relation for (1) by using GDTM. The recurrence relation presented in this study is applied to four time-fractional diffusion equations that exist in the literature except the last one. The results evaluated are in good agreement with the already existing ones. Shortly, the general recurrence relation works successfully in handling time-fractional diffusion equations with a minimum size of calculations.

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