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BOUNDEDNESS OF THE FRACTIONAL MAXIMAL OPERATOR IN GENERALIZED MORREY SPACE ON THE HEISENBERG GROUP

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In this paper we study the fractional maximal operator M_α, $0 \le \alpha < Q$ on the Heisenberg group $\mathbb{H}_n$ in the generalized Morrey spaces $M_{p,\varphi}(\mathbb{H}_n)$, where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}_n$. We find the conditions on the pair (φ_1, φ_2) which ensures the boundedness of the operator M_α from one generalized Morrey space $M_{p,\varphi_1}(\mathbb{H}_n)$ to another $M_{q,\varphi_2}(\mathbb{H}_n)$, $1 < p < q < \infty$, $1/p - 1/q = \alpha/Q$, and from the space $M_{1,\varphi_1}(\mathbb{H}_n)$ to the weak space $WM_{q,\varphi_2}(\mathbb{H}_n)$, $1 < q < \infty$, $1 - 1/q = \alpha/Q$. We also find conditions on the φ which ensure the Adams type boundedness of M_α from $M_{p,\varphi^{\frac{1}{p}}}(\mathbb{H}_n)$ to $M_{q,\varphi^{\frac{1}{q}}}(\mathbb{H}_n)$ for $1 < p < q < \infty$ and from $M_{1,\varphi}(\mathbb{H}_n)$ to $WM_{q,\varphi^{\frac{1}{q}}}(\mathbb{H}_n)$ for $1 < q < \infty$.

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As applications we establish the boundedness of some Schrödinger type operators on generalized Morrey spaces related to certain nonnegative potentials V belonging to the reverse Hölder class $B_\infty(\mathbb{H}_n)$.

Key words : Heisenberg group; fractional maximal function; generalized Morrey space; Schrödinger operator.

1. Introduction

Heisenberg group appear in quantum physics and many parts of mathematics, including Fourier analysis, several complex variables, geometry and topology. Analysis on the groups is also motivated by their role as the simplest and the most important model in the general theory of vector fields satisfying Hormander's condition. In the present paper we will prove the boundedness of the fractional maximal operator on the Heisenberg group in generalized Morrey spaces.

For $g \in \mathbb{H}_n$ and $r > 0$, we denote by $B(g,r)$ the open ball centered at g of radius r, and by $^{\complement}B(g,r)$ denote its complement. Let $|B(g,r)|$ be the Haar measure of the ball $B(g,r)$.

Given a function f which is integrable on any ball $B(g,r) \subset \mathbb{H}_n$, the fractional maximal function $M_\alpha f$, $0 \leq \alpha < Q$ of f is defined by

$$M_\alpha f(g) = \sup_{r>0} |B(g,r)|^{-1+\frac{\alpha}{Q}} \int_{B(g,r)} |f(h)|dh.$$

The fractional maximal function $M_\alpha f$ coincides for $\alpha = 0$ with the Hardy-Littlewood maximal function $Mf \equiv M_0 f$ (see [8, 26]). The operator M_α play important role in real and harmonic analysis (see, for example [7, 8, 26]).

In this work, we prove the boundedness of the fractional maximal operator M_α, $0 \leq \alpha < Q$ from one generalized Morrey space $M_{p,\varphi_1}(\mathbb{H}_n)$ to $M_{q,\varphi_2}(\mathbb{H}_n)$, $1 < p < q < \infty$, $1/p - 1/q = \alpha/Q$, and from $M_{1,\varphi_1}(\mathbb{H}_n)$ to the weak space $WM_{q,\varphi_2}(\mathbb{H}_n)$, $1 < q < \infty$, $1 - 1/q = \alpha/Q$. We also prove the Adams type boundedness of the operator M_α from $M_{p,\varphi^{\frac{1}{p}}}(\mathbb{H}_n)$ to $M_{q,\varphi^{\frac{1}{q}}}(\mathbb{H}_n)$ for $1 < p <$

$q < \infty$ and from $M_{1,\varphi}(\mathbb{H}_n)$ to $WM_{q,\varphi^{\frac{1}{q}}}(\mathbb{H}_n)$ for $1 < q < \infty$. In all the cases the conditions for the boundedness are given it terms of supremal-type inequalities on (φ_1, φ_2) and φ, which do not assume any assumption on monotonicity of (φ_1, φ_2) and φ in r. Let $L = -\Delta_{\mathbb{H}_n} + V$ be a Schrödinger operator on $\mathbb{H}_n$, where $\Delta_{\mathbb{H}_n}$ is the sub-Laplacian. As applications we establish the boundedness of some Schrödinger type operators $V^\gamma(-\Delta_{\mathbb{H}_n} + V)^{-\beta}$ and $V^\gamma \nabla_{\mathbb{H}_n}(-\Delta_{\mathbb{H}_n} + V)^{-\beta}$ on generalized Morrey spaces related to certain nonnegative potentials V belonging to the reverse Hölder class $B_\infty(\mathbb{H}_n)$.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Notations

We recall some basic facts on the Heisenberg group, which more detailed information can be found in [8, 9, 26] and the references therein. The $(2n+1)$-dimensional Heisenberg group $\mathbb{H}_n$ is the Lie group with underlying manifold $\mathbb{R}^{2n} \times \mathbb{R}$ and multiplication

$$(x,t)(y,s) = \big(x + y, t + s + 2\sum_{j=1}^{n}(x_{n+j}y_j - x_j y_{n+j})\big).$$

The inverse element of $g = (x,t)$ is $g^{-1} = (-x, -t)$ and we write the identity of $\mathbb{H}_n$ as $e = (0,0)$. The Heisenberg group is a connected, simply connected nilpotent Lie group. We define one-parameter dilations on $\mathbb{H}_n$, for $r > 0$, by $\delta_r(x,t) = (rx, r^2t)$. These dilations are group automorphisms and the Jacobian determinant is r^Q, where $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}_n$. A homogeneous norm on $\mathbb{H}_n$ is given by $|g| = |(x,t)| = (|x|^2 + |t|)^{1/2}$. This norm satisfies the triangle inequality and leads to a left-invariant distant $d(g,h) = |g^{-1}h|$. With this norm, we define the Heisenberg ball centered at $g = (x,t)$ with radius r by $B(g,r) = \{h \in \mathbb{H}_n : |g^{-1}h| < r\}$, and we denote by $^{\complement}B(g,r) = \mathbb{H}_n \setminus B(g,r)$ its complement. The volume of the ball $B(g,r)$ is $C_n r^Q$, where C_n

is the volume of the unit ball B_1:

$$C_n = |B(e,1)| = \frac{2\pi^{n+\frac{1}{2}}\Gamma\left(\frac{n}{2}\right)}{(n+1)\Gamma(n)\Gamma\left(\frac{n+1}{2}\right)}.$$

Using coordinates $g = (x,t)$ for points in $\mathbb{H}_n$, the left-invariant vector fields $X_1, \dots, X_{2n}, X_{2n+1}$ on $\mathbb{H}_n$ equal to $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{2n}}, \frac{\partial}{\partial t}$ at the origin are given by

$$X_j = \frac{\partial}{\partial x_j} + 2x_{n+j}\frac{\partial}{\partial t}, \;\; X_{n+j} = \frac{\partial}{\partial x_{n+j}} - 2x_j\frac{\partial}{\partial t}, \;\; j = 1, \dots, n, \;\; X_{2n+1} = \frac{\partial}{\partial t},$$

respectively. These $2n+1$ vector fields form a basis for the Lie algebra of $\mathbb{H}_n$ with commutation relations

$$[X_j, X_{n+j}] = -4X_{2n+1}$$

for $j = 1, \dots, n$, and all other commutators equal to 0. The sub-Laplacian $\Delta_{\mathbb{H}_n}$ and the gradient $\nabla_{\mathbb{H}_n}$ are defined respectively by

$$\Delta_{\mathbb{H}_n} = \sum_{j=1}^{2n} X_j^2 \quad \text{and} \quad \nabla_{\mathbb{H}_n} = (X_1, \dots, X_{2n}).$$

The sub-Laplacian operator (which is hypoelliptic by Hormanderfs theorem [16]) is well known to play the same fundamental role on $\mathbb{H}_n$ as the ordinary does on $\mathbb{R}^n$.

In the study of local properties of solutions to of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $L_{p,\lambda}(\mathbb{H}_n)$ play an important role, see [10, 17]. They were introduced by Morrey in 1938 [22]. The Morrey space in a Heisenberg group is defined as follows: for $1 \le p \le \infty$, $0 \le \lambda \le Q$, a function $f \in L_{p,\lambda}(\mathbb{H}_n)$ if $f \in L_p^{loc}(\mathbb{H}_n)$ and

$$\|f\|_{L_{p,\lambda}} := \sup_{u\in\mathbb{H}_n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(g,r))} < \infty,$$

(If $\lambda = 0$, then $L_{p,0}(\mathbb{H}_n) = L_p(\mathbb{H}_n)$; if $\lambda = Q$, then $L_{p,Q}(\mathbb{H}_n) = L_\infty(\mathbb{H}_n)$; if $\lambda < 0$ or $\lambda > Q$, then $L_{p,\lambda}(\mathbb{H}_n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{H}_n$.)

We also denote by $WL_{p,\lambda}(\mathbb{H}_n)$ the weak Morrey space of all functions $f \in WL_p^{loc}(\mathbb{H}_n)$ for which

$$\|f\|_{WL_{p,\lambda}} = \sup_{u\in\mathbb{H}_n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(g,r))} < \infty,$$

where $WL_p(B(g,r))$ denotes the weak L_p-space of measurable functions f for which

$$\|f\|_{WL_p(B(g,r))} = \sup_{\tau>0} \tau \left|\{h \in B(g,r) : \, |f(h)| > \tau\}\right|^{1/p}.$$

Note that

$$\begin{aligned} WL_p(\mathbb{H}_n) &= WL_{p,0}(\mathbb{H}_n), \;\; L_{p,\lambda}(\mathbb{H}_n) \subset WL_{p,\lambda}(\mathbb{H}_n) \\ &\text{and} \;\; \|f\|_{WL_{p,\lambda}} \le \|f\|_{L_{p,\lambda}}. \end{aligned}$$

Everywhere in the sequel the functions $\varphi(g,r),\ \varphi_1(g,r)$ and $\varphi_2(g,r)$ used in the body of the paper, are non-negative measurable function on $\mathbb{H}_n \times (0,\infty)$.

We find it convenient to define the generalized Morrey spaces in the form as follows.

Definition 2.1 — Let $1 \le p < \infty$. The generalized Morrey space $M_{p,\varphi}(\mathbb{H}_n)$ is defined of all functions $f \in L_p^{loc}(\mathbb{H}_n)$ by the finite norm

$$\|f\|_{M_{p,\varphi}} = \sup_{u\in\mathbb{H}_n, r>0} \varphi(g,r)^{-1} \, |B(g,r)|^{-\frac{1}{p}} \, \|f\|_{L_p(B(g,r))}.$$

According to this definition, we recover the space $L_{p,\lambda}(\mathbb{H}_n)$ under the choice $\varphi(g,r) = r^{\frac{\lambda-Q}{p}}$:

$$L_{p,\lambda}(\mathbb{H}_n) = M_{p,\varphi}(\mathbb{H}_n)\Big|_{\varphi(g,r)=r^{\frac{\lambda-Q}{p}}}.$$

In [11, 12, 23, 24] there were obtained sufficient conditions on weights φ_1 and φ_2 for the boundedness of the maximal operator M and the singular integral operators T from $M_{p,\varphi_1}(\mathbb{H}_n)$ to $M_{p,\varphi_2}(\mathbb{H}_n)$. In [23, 24] the following condition was imposed on $w(g,r)$:

$$c^{-1}\varphi(g,r) \le \varphi(g,\tau) \le c\,\varphi(g,r) \tag{2.1}$$

whenever $r \leq \tau \leq 2r$, where $c(\geq 1)$ does not depend on t, r and $u \in \mathbb{H}_n$, jointly with the condition:

$$\int_r^\infty \varphi(g,\tau)^p \frac{d\tau}{\tau} \leq C\, \varphi(g,r)^p. \tag{2.2}$$

for the maximal or singular operators and the condition

$$\int_r^\infty \tau^{\alpha p}\varphi(g,\tau)^p \frac{d\tau}{\tau} \leq C\, r^{\alpha p}\varphi(g,r)^p. \tag{2.3}$$

for potential and fractional maximal operators, where $C(>0)$ does not depend on r and $u \in \mathbb{H}_n$.

In [24] the following statements were proved.

Theorem **2.1** — *Let* $1 \leq p < \infty, 0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$ *and* $\varphi(g,\tau)$ *satisfy conditions (2.1) and (2.3). Then for* $p > 1$ *the operator* M_α *is bounded from* $M_{p,\varphi}(\mathbb{H}_n)$ *to* $M_{q,\varphi}(\mathbb{H}_n)$ *and for* $p = 1$ M_α *is bounded from* $M_{1,\varphi}(\mathbb{H}_n)$ *to* $WM_{q,\varphi}(\mathbb{H}_n)$.

The following statements, containing results obtained in [24] was proved in [11] (see also [12, 13, 14, 15]).

Theorem **2.2** — *Let* $1 \leq p < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_\tau^\infty r^\alpha \varphi_1(g,r) \frac{dr}{r} \leq C\, \varphi_2(x,\tau), \tag{2.4}$$

where C *does not depend on* g *and* τ. *Then the operator* M_α *is bounded from* $M_{p,\varphi_1}(\mathbb{H}_n)$ *to* $M_{q,\varphi_2}(\mathbb{H}_n)$ *for* $p > 1$ *and from* $M_{1,\varphi_1}(\mathbb{H}_n)$ *to* $WM_{q,\varphi_2}(\mathbb{H}_n)$ *for* $p = 1$.

3. Boundedness of the Fractional Maximal Operator in the Spaces $M_{p,\varphi}(\mathbb{H}_n)$

3.1 *Spanne type result*

We denote by $L_{\infty,v}(0,\infty)$ the space of all functions $g(t)$, $t > 0$ with finite norm

$$\|g\|_{L_{\infty,v}(0,\infty)} = \operatorname*{ess\,sup}_{t>0} v(t)g(t)$$

and $L_\infty(0,\infty) \equiv L_{\infty,1}(0,\infty)$. Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset consisting of all nonnegative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and

$$\mathbb{A} = \left\{ \varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0 \right\}.$$

Let u be a continuous and non-negative function on $(0,\infty)$. We define the supremal operator $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(t,\infty)}, \;\; t \in (0,\infty).$$

The following theorem was proved in [3].

***Theorem* 3.3** — *Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L_\infty(t,\infty)} < \infty$ for any $t > 0$ and let u be a continuous non-negative function on* $(0,\infty)$

Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(0,\infty)$ to $L_{\infty,v_2}(0,\infty)$ on the cone $\mathbb{A}$ if and only if

$$\left\| v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)} \right) \right\|_{L_\infty(0,\infty)} < \infty. \tag{3.1}$$

Sufficient conditions on φ for the boundedness of M and M_α in generalized Morrey spaces $\mathcal{M}_{p,\varphi}(\mathbb{H}_n)$ have been obtained in [2, 3, 13, 14, 15, 24].

The following lemma is true.

Lemma 3.1 — Let $1 \le p < \infty$, $0 \le \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$. Then for $p > 1$ and any ball $B = B(g,r)$ in $\mathbb{H}_n$ the inequality

$$\|M_\alpha f\|_{L_q(B(g,r))} \lesssim \|f\|_{L_p(B(g,2r))} + r^{\frac{Q}{q}} \sup_{\tau > 2r} \tau^{-Q+\alpha} \|f\|_{L_1(B(g,\tau))} \tag{3.2}$$

holds for all $f \in L_p^{\rm loc}(\mathbb{H}_n)$.

Moreover for $p = 1$ the inequality

$$\|M_\alpha f\|_{WL_q(B(g,r))} \lesssim \|f\|_{L_1(B(g,2r))} + r^{\frac{Q}{q}} \sup_{\tau > 2r} \tau^{-Q+\alpha} \|f\|_{L_1(B(g,\tau))} \tag{3.3}$$

holds for all $f \in L_1^{\rm loc}(\mathbb{H}_n)$.

PROOF : Let $1 < p < q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$. For arbitrary ball $B = B(g, r)$ let $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$, $f_2 = f\chi_{\complement_{(2B)}}$ and $2B = B(g, 2r)$.

$$\|M_\alpha f\|_{L_q(B)} \leq \|M_\alpha f_1\|_{L_q(B)} + \|M_\alpha f_2\|_{L_q(B)}.$$

By the continuity of the operator $M_\alpha : L_p(\mathbb{H}_n) \to L_q(\mathbb{H}_n)$ we have

$$\|M_\alpha f_1\|_{L_q(B)} \lesssim \|f\|_{L_p(2B)}.$$

Let h be an arbitrary point from B. If $B(h, \tau) \cap {}^{\complement}(2B) \neq \emptyset$, then $\tau > r$. Indeed, if $w \in B(h, \tau) \cap {}^{\complement}(2B)$, then $\tau > |h^{-1}w| \geq |g^{-1}w| - |g^{-1}h| > 2r - r = r$.

On the other hand, $B(h, \tau) \cap {}^{\complement}(2B) \subset B(g, 2\tau)$. Indeed, $w \in B(h, \tau) \cap {}^{\complement}(2B)$, then we get $|g^{-1}w| \leq |h^{-1}w| + |g^{-1}h| < \tau + r < 2\tau$.

Hence

$$M_\alpha f_2(h) = \sup_{\tau > 0} \frac{1}{|B(h, \tau)|^{1-\alpha/Q}} \int_{B(h,\tau) \cap {}^{\complement}(2B)} |f(w)| dw$$

$$\leq 2^{Q-\alpha} \sup_{\tau > r} \frac{1}{|B(g, 2\tau)|^{1-\alpha/Q}} \int_{B(g,2\tau)} |f(w)| dw$$

$$= 2^{Q-\alpha} \sup_{\tau > 2r} \frac{1}{|B(g, \tau)|^{1-\alpha/Q}} \int_{B(g,\tau)} |f(w)| dw.$$

Therefore, for all $h \in B$ we have

$$M_\alpha f_2(h) \leq 2^{Q-\alpha} \sup_{\tau > 2r} \frac{1}{|B(g, \tau)|^{1-\alpha/Q}} \int_{B(g,\tau)} |f(w)| dw. \tag{3.4}$$

Thus

$$\|M_\alpha f\|_{L_q(B)} \lesssim \|f\|_{L_p(2B)} + |B|^{\frac{1}{q}} \left(\sup_{\tau > 2r} \frac{1}{|B(g, \tau)|^{1-\alpha/Q}} \int_{B(g,\tau)} |f(w)| dw \right).$$

Let $p = 1$. It is obvious that for any ball $B = B(g, r)$

$$\|M_\alpha f\|_{WL_q(B)} \leq \|M_\alpha f_1\|_{WL_q(B)} + \|M_\alpha f_2\|_{WL_q(B)}.$$

By the continuity of the operator $M_\alpha : L_1(\mathbb{H}_n) \to WL_q(\mathbb{H}_n)$ we have

$$\|M_\alpha f_1\|_{WL_q(B)} \lesssim \|f\|_{L_1(2B)}.$$

Then by (3.4) we get the inequality (3.3). □

Lemma 3.2 — Let $1 \leq p < \infty$, $0 \leq \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$. Then for $p > 1$ and any ball $B = B(g, r)$ in $\mathbb{H}_n$, the inequality

$$\|M_\alpha f\|_{L_q(B(g,r))} \lesssim r^{\frac{Q}{q}} \sup_{\tau > 2r} \tau^{-\frac{Q}{q}} \|f\|_{L_p(B(g,\tau))} \tag{3.5}$$

holds for all $f \in L_p^{\text{loc}}(\mathbb{H}_n)$.

Moreover for $p = 1$ the inequality

$$\|M_\alpha f\|_{WL_q(B(g,r))} \lesssim r^{\frac{Q}{q}} \sup_{\tau > 2r} \tau^{-\frac{Q}{q}} \|f\|_{L_1(B(g,\tau))} \tag{3.6}$$

holds for all $f \in L_1^{\text{loc}}(\mathbb{H}_n)$.

PROOF : Let $1 < p < \infty$, $0 \leq \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$. Denote

$$\mathcal{M}_1 := |B|^{\frac{1}{q}} \left(\sup_{\tau > 2r} \frac{1}{|B(g,\tau)|^{1-\alpha/Q}} \int_{B(g,\tau)} |f(w)| dw \right),$$
$$\mathcal{M}_2 := \|f\|_{L_p(2B)}.$$

Applying Hölder's inequality, we get

$$\mathcal{M}_1 \lesssim |B|^{\frac{1}{q}} \left(\sup_{\tau > 2r} \frac{1}{|B(g,\tau)|^{\frac{1}{q}}} \left(\int_{B(g,\tau)} |f(w)|^p dw \right)^{\frac{1}{p}} \right).$$

On the other hand,

$$|B|^{\frac{1}{q}} \left(\sup_{\tau>2r} \frac{1}{|B(g,\tau)|^{\frac{1}{q}}} \left(\int_{B(g,\tau)} |f(w)|^p dw \right)^{\frac{1}{p}} \right)$$

$$\gtrsim |B|^{\frac{1}{q}} \left(\sup_{\tau>2r} \frac{1}{|B(g,\tau)|^{\frac{1}{q}}} \right) \|f\|_{L_p(2B)} \approx \mathcal{M}_{\in}.$$

Since by Lemma 3.1

$$\|M_\alpha f\|_{L_q(B)} \le \mathcal{M}_1 + \mathcal{M}_2,$$

we arrive at (3.5).

Let $p=1$. The inequality (3.6) directly follows from (3.3). □

***Theorem* 3.4** — *Let $1 \le p < \infty$, $0 \le \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, and (φ_1, φ_2) satisfies the condition*

$$\sup_{r<t<\infty} t^{\alpha-\frac{Q}{p}} \operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(g,s)\, s^{\frac{Q}{p}} \le C\, \varphi_2(g,r), \tag{3.7}$$

where C does not depend on g and r. Then for $p>1$, M_α is bounded from $M_{p,\varphi_1}(\mathbb{H}_n)$ to $M_{q,\varphi_2}(\mathbb{H}_n)$ and for $p=1$, M_α is bounded from $M_{1,\varphi_1}(\mathbb{H}_n)$ to $WM_{q,\varphi_2}(\mathbb{H}_n)$.

PROOF : By Theorem 3.3 and Lemma 3.2 we get

$$\|M_\alpha f\|_{M_{q,\varphi_2}} \lesssim \sup_{g\in\mathbb{H}_n, r>0} \varphi_2(g,r)^{-1} \sup_{\tau>r} \tau^{-\frac{Q}{q}} \|f\|_{L_p(B(g,\tau))}$$

$$\lesssim \varphi_1(g,r)^{-1}\, r^{-\frac{Q}{p}}\, \|f\|_{L_p(B(g,r))} = \|f\|_{M_{p,\varphi_1}},$$

if $p \in (1,\infty)$ and

$$\|M_\alpha f\|_{WM_{q,\varphi_2}} \lesssim \sup_{g\in\mathbb{H}_n, r>0} \varphi_2(g,r)^{-1} \sup_{\tau>r} \tau^{-\frac{Q}{q}} \|f\|_{L_1(B(g,\tau))}$$

$$\lesssim \sup_{g\in\mathbb{H}_n, r>0} \varphi_1(g,r)^{-1}\, r^{-Q}\, \|f\|_{L_1(B(g,r))} = \|f\|_{M_{1,\varphi_1}},$$

if $p=1$. □

In the case $\alpha = 0$ and $p = q$ from Theorem 3.4 we get the following corollary, which proven in [2] on $\mathbb{R}^n$.

Corollary 3.1 — Let $1 \le p < \infty$ and (φ_1, φ_2) satisfies the condition

$$\sup_{r<t<\infty} t^{-\frac{Q}{p}} \operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(g,s)\, s^{\frac{Q}{p}} \le C\, \varphi_2(g,r), \tag{3.8}$$

where C does not depend on g and r. Then for $p > 1$, M is bounded from $M_{p,\varphi_1}(\mathbb{H}_n)$ to $M_{p,\varphi_2}(\mathbb{H}_n)$ and for $p = 1$, M is bounded from $M_{1,\varphi_1}(\mathbb{H}_n)$ to $WM_{1,\varphi_2}(\mathbb{H}_n)$.

Corollary 3.2 — Let $p \in [1, \infty)$ and let $\varphi : (0,\infty) \to (0,\infty)$ be an decreasing function. Assume that the mapping $r \mapsto \varphi(r)\, r^{\frac{Q}{p}}$ is almost increasing (there exists a constant c such that for $s < r$ we have $\varphi(s)\, s^{\frac{Q}{p}} \le c\varphi(r)\, r^{\frac{Q}{p}}$). Then there exists a constant $C > 0$ such that

$$\|Mf\|_{\mathcal{M}_{p,\varphi}} \le C\|f\|_{\mathcal{M}_{p,\varphi}} \quad \text{if} \quad p > 1,$$

and

$$\|Mf\|_{W\mathcal{M}_{1,\varphi}} \le C\|f\|_{\mathcal{M}_{1,\varphi}}.$$

3.2 *Adams type result*

The following is a result of Adams type for the fractional maximal operator (see [1]).

Theorem 3.5 — *Let $1 \le p < q < \infty$, $0 < \alpha < \frac{Q}{p}$ and let $\varphi(g,\tau)$ satisfy the condition*

$$\sup_{r<t<\infty} t^{-Q} \operatorname*{ess\,inf}_{t<s<\infty} \varphi(g,s)\, s^{Q} \le C\, \varphi(g,r) \tag{3.9}$$

and

$$\sup_{r<\tau<\infty} t^{\alpha}\varphi(g,\tau)^{\frac{1}{p}} \le C r^{-\frac{\alpha p}{q-p}}, \tag{3.10}$$

where C does not depend on $g \in \mathbb{H}_n$ and $r > 0$.

Then the operator M_α is bounded from $M_{p,\varphi^{\frac{1}{p}}}(\mathbb{H}_n)$ to $M_{q,\varphi^{\frac{1}{q}}}(\mathbb{H}_n)$ for $p > 1$ and from $M_{1,\varphi}(\mathbb{H}_n)$ to $WM_{q,\varphi^{\frac{1}{q}}}(\mathbb{H}_n)$.

PROOF : Let $1 \le p < q < \infty$, $0 < \alpha < \frac{Q}{p}$ and $f \in M_{p,\varphi^{\frac{1}{p}}}(\mathbb{H}_n)$. Write $f = f_1 + f_2$, where $B = B(g,r)$, $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{(2B)}}$.

For $M_\alpha f_2(h)$ for all $h \in B$ from (3.4) we have

$$M_\alpha(f_2)(h) \le 2^{Q-\alpha} \sup_{\tau>2r} \frac{1}{|B(g,\tau)|^{1-\alpha/Q}} \int_{B(g,\tau)} |f(w)|dw$$
$$\lesssim \sup_{\tau>2r} \tau^{-\frac{Q}{q}} \|f\|_{L_p(B(g,\tau))} \tag{3.11}$$

Then from conditions (3.10) and (3.11) we get

$$\begin{aligned} M_\alpha f(h) &\lesssim r^\alpha\, Mf(h) + \sup_{\tau>2r} t^{\alpha-\frac{Q}{p}} \|f\|_{L_p(B(g,\tau))} \\ &\le r^\alpha\, Mf(h) + \|f\|_{M_{p,\varphi^{\frac{1}{p}}}} \sup_{\tau>2r} \tau^\alpha \varphi(g,\tau)^{\frac{1}{p}} \\ &\lesssim r^\alpha\, Mf(h) + r^{-\frac{\alpha p}{q-p}} \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}. \end{aligned}$$

Hence choose $r = \left(\frac{\|f\|_{M_{p,\varphi^{1/p}}}}{Mf(h)}\right)^{\frac{q-p}{\alpha q}}$ for every $h \in B$, we have

$$|M_\alpha f(h)| \lesssim (Mf(h))^{\frac{p}{q}} \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}}.$$

Hence the statement of the theorem follows in view of the boundedness of the maximal operator M in $M_{p,\varphi^{\frac{1}{p}}}(\mathbb{H}_n)$ provided by Corollary 3.1 in virtue of condition (3.9).

$$\begin{aligned} \|M_\alpha f\|_{M_{q,\varphi^{\frac{1}{q}}}} &= \sup_{g\in\mathbb{H}_n,\,\tau>0} \varphi(g,\tau)^{-\frac{1}{q}} \tau^{-\frac{Q}{q}} \|M_\alpha f\|_{L_q(B(g,\tau))} \\ &\lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \sup_{g\in\mathbb{H}_n,\,\tau>0} \varphi(g,\tau)^{-\frac{1}{q}} \tau^{-\frac{Q}{q}} \|Mf\|_{L_p(B(g,\tau))}^{\frac{p}{q}} \\ &= \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \left(\sup_{g\in\mathbb{H}_n,\,\tau>0} \varphi(g,\tau)^{-\frac{1}{p}} \tau^{-\frac{Q}{p}} \|Mf\|_{L_p(B(g,\tau))}\right)^{\frac{p}{q}} \\ &= \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}^{1-\frac{p}{q}} \|Mf\|_{M_{p,\varphi^{\frac{1}{p}}}}^{\frac{p}{q}} \\ &\lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}, \end{aligned}$$

if $1 < p < q < \infty$ and

$$\begin{aligned}
\|M_\alpha f\|_{WM_{q,\varphi^{\frac{1}{q}}}} &= \sup_{g\in\mathbb{H}_n,\, \tau>0} \varphi(g,\tau)^{-\frac{1}{q}} t^{-\frac{Q}{q}} \|M_\alpha f\|_{WL_q(B(g,\tau))} \\
&\lesssim \|f\|_{M_{1,\varphi}}^{1-\frac{1}{q}} \sup_{g\in\mathbb{H}_n,\, \tau>0} \varphi(g,\tau)^{-\frac{1}{q}} \tau^{-\frac{Q}{q}} \|Mf\|_{WL_1(B(g,\tau))}^{\frac{1}{q}} \\
&= \|f\|_{M_{1,\varphi}}^{1-\frac{1}{q}} \left(\sup_{g\in\mathbb{H}_n,\, \tau>0} \varphi(g,\tau)^{-1} \tau^{-Q} \|Mf\|_{WL_1(B(g,\tau))} \right)^{\frac{1}{q}} \\
&= \|f\|_{M_{1,\varphi}}^{1-\frac{1}{q}} \|Mf\|_{WM_{1,\varphi}}^{\frac{1}{q}} \\
&\lesssim \|f\|_{M_{1,\varphi}},
\end{aligned}$$

if $1 < q < \infty$. □

In the case $\varphi(g,\tau) = \tau^{\lambda - Q}$, $0 < \lambda < Q$ from Theorem 3.5 we get the following Adams type result [1] for the fractional maximal operator.

Corollary 3.3 — Let $0 < \alpha < Q$, $1 \le p < \frac{Q}{\alpha}$, $0 < \lambda < Q - \alpha p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q-\lambda}$. Then for $p > 1$, the operator M_α is bounded from $L_{p,\lambda}(\mathbb{H}_n)$ to $L_{q,\lambda}(\mathbb{H}_n)$ and for $p = 1$, M_α is bounded from $L_{1,\lambda}(\mathbb{H}_n)$ to $WL_{q,\lambda}(\mathbb{H}_n)$.

4. THE GENERALIZED MORREY ESTIMATES FOR THE OPERATORS $V^\gamma\big(-\Delta_{\mathbb{H}_n} + V\big)^{-\beta}$ AND $V^\gamma \nabla_{\mathbb{H}_n}\big(-\Delta_{\mathbb{H}_n} + V\big)^{-\beta}$

In this section we consider the Schrödinger operator $-\Delta_{\mathbb{H}_n} + V$ on $\mathbb{H}_n$, where the nonnegative potential V belongs to the reverse Hölder class $B_\infty(\mathbb{H}_n)$. The generalized Morrey $M_{p,\varphi}(\mathbb{H}_n)$ estimates for the operators $V^\gamma(-\Delta_{\mathbb{H}_n} + V)^{-\beta}$ and $V^\gamma \nabla_{\mathbb{H}_n}(-\Delta_{\mathbb{H}_n} + V)^{-\beta}$ are obtained.

The investigation of Schrödinger operators on the Euclidean space $\mathbb{R}^n$ with nonnegative potentials which belong to the reverse Hölder class has attracted attention of a number of authors (cf. [5, 25, 29]). Shen [25] studied the Schrödinger operator $-\Delta + V$, assuming the nonnegative potential V belongs to the reverse Hölder class $B_q(\mathbb{R}^n)$ for $q \ge n/2$ and he proved the L_p boundedness of the operators $(-\Delta + V)^{i\gamma}$, $\nabla^2(-\Delta + V)^{-1}$, $\nabla(-\Delta + V)^{-\frac{1}{2}}$ and $\nabla(-\Delta + V)^{-1}$. Kurata and Sugano generalized Shens results to uniformly elliptic operators in [18]. Sugano [27] also extended some results of Shen to the operator $V^{\gamma}(-\Delta+V)^{-\beta}$, $0\le\gamma\le\beta\le 1$ and $V^{\gamma}\nabla_{\mathbb{H}_n}(-\Delta_{\mathbb{H}_n}+V)^{-\beta}$, $0\le\gamma\le\frac{1}{2}\le\beta\le 1$ and $\beta-\gamma\ge\frac{1}{2}$. Later, Lu [21] and Li [19] investigated the Schrödinger operators in a more general setting.

The main purpose of this section is investigate the generalized Morrey M_{p,φ_1}-M_{q,φ_2} boundedness of the operators

$$\begin{aligned} T_1 &= V^{\gamma}(-\Delta_{\mathbb{H}_n}+V)^{-\beta},\ \ 0\le\gamma\le\beta\le 1, \\ T_2 &= V^{\gamma}\nabla_{\mathbb{H}_n}(-\Delta_{\mathbb{H}_n}+V)^{-\beta},\ \ 0\le\gamma\le\frac{1}{2}\le\beta\le 1,\ \beta-\gamma\ge\frac{1}{2}. \end{aligned}$$

Note that the operators $V(-\Delta_{\mathbb{H}_n}+V)^{-1}$ and $V^{\frac{1}{2}}\nabla_{\mathbb{H}_n}(-\Delta_{\mathbb{H}_n}+V)^{-1}$ in [19] are the special case of T_1 and T_2, respectively.

It is worth pointing out that we need to establish pointwise estimates for T_1, T_2 and their adjoint operators by using the estimates of fundamental solution for the Schrödinger operator on $\mathbb{H}_n$ in [19]. And we prove the generalized Morrey estimates by using $M_{p,\varphi_1}-M_{q,\varphi_2}$ boundedness of the fractional maximal operators.

Let $V\ge 0$. We say $V\in B_{\infty}(\mathbb{H}_n)$, if there exists a constant $C>0$ such that

$$\|V\|_{L_\infty(B)}\le\frac{C}{|B|}\int_B V(g)dg$$

holds for every ball B in $\mathbb{H}_n$ (see [19]).

By the functional calculus, we may write, for all $0<\beta<1$,

$$(-\Delta_{\mathbb{H}_n}+V)^{-\beta}=\frac{1}{\pi}\int_0^{\infty}\lambda^{-\beta}\ (-\Delta_{\mathbb{H}_n}+V+\lambda)^{-1}\ d\lambda.$$

Let $f\in C_0^{\infty}(\mathbb{H}_n)$. From $(-\Delta_{\mathbb{H}_n}+V+\lambda)^{-1}f(g)=\int_{\mathbb{H}_n}\Gamma(g,h,\lambda)f(h)dh$, it follows that

$$\mathcal{T}_1 f(g)=\int_{\mathbb{H}_n}K_1(g,h)V(g)^{\gamma}f(h)dh,$$

where

$$K_1(g,h)=\begin{cases}\frac{1}{\pi}\int_0^{\infty}\lambda^{-\beta}\,\Gamma(g,h,\lambda)\,d\lambda & \text{for } 0<\beta<1\\ \Gamma(g,h,0) & \text{for } \beta=1.\end{cases}$$

The following two pointwise estimates for T_1 and T_2 which proven in [29], Lemma 3.2 with the potential $V \in B_\infty(\mathbb{H}_n)$.

Theorem B — *Suppose $V \in B_\infty(\mathbb{H}_n)$ and $0 \le \gamma \le \beta \le 1$. Then there exists a constant $C > 0$ such that*

$$|T_1 f(g)| \le C M_\alpha f(g), \quad f \in C_0^\infty(\mathbb{H}_n),$$

where $\alpha = 2(\beta - \gamma)$.

Theorem C — *Suppose $V \in B_\infty(\mathbb{H}_n)$, $0 \le \gamma \le \frac{1}{2} \le \beta \le 1$ and $\beta - \gamma \ge \frac{1}{2}$. Then there exists a constant $C > 0$ such that*

$$|T_2 f(g)| \le C M_\alpha f(g), \quad f \in C_0^\infty(\mathbb{H}_n),$$

where $\alpha = 2(\beta - \gamma) - 1$.

The above theorems will yield the generalized Morrey estimates for T_1 and T_2.

Corollary 4.4 — Assume that $V \in B_\infty(\mathbb{H}_n)$, and $0 \le \gamma \le \beta \le 1$. Let $1 \le p \le q < \infty$, $2(\beta - \gamma) = Q\left(\frac{1}{p} - \frac{1}{q}\right)$ and the condition (3.7) be satisfied for $\alpha = 2(\beta - \gamma)$. Then for any $f \in C_0^\infty(\mathbb{H}_n)$

$$\|\mathcal{T}_1 f\|_{M_{q,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}}, \qquad \text{for } p > 1$$

and

$$\|\mathcal{T}_1 f\|_{WM_{q,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}} \qquad \text{for } p = 1$$

Corollary 4.5 — Assume that $V \in B_\infty(\mathbb{H}_n)$, $0 \le \gamma \le \frac{1}{2} \le \beta \le 1$ and $\beta - \gamma \ge \frac{1}{2}$. Let $1 \le p \le q < \infty$, $2(\beta - \gamma) - 1 = Q\left(\frac{1}{p} - \frac{1}{q}\right)$ and the condition (3.7) be satisfied for $\alpha = 2(\beta - \gamma) - 1$. Then for any $f \in C_0^\infty(\mathbb{H}_n)$

$$\|\mathcal{T}_2 f\|_{M_{q,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}}, \qquad \text{for } p > 1$$

and

$$\|\mathcal{T}_2 f\|_{WM_{q,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}} \qquad \text{for } p = 1$$

Corollary 4.6 — Assume that $V \in B_\infty(\mathbb{H}_n)$ and $0 \le \gamma \le \beta \le 1$. Let $1 \le p < q < \infty$, $2(\beta-\gamma) = Q\left(\frac{1}{p} - \frac{1}{q}\right)$ and the conditions (3,9), (3.10) be satisfied for $\alpha = 2(\beta-\gamma)$.

Then for any $f \in C_0^\infty(\mathbb{H}_n)$

$$\|\mathcal{T}_1 f\|_{M_{q,\varphi^{\frac{1}{q}}}} \lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}, \qquad \text{for } p > 1$$

and

$$\|\mathcal{T}_1 f\|_{WM_{q,\varphi^{\frac{1}{q}}}} \lesssim \|f\|_{M_{1,\varphi}} \qquad \text{for } p = 1.$$

Corollary 4.7 — Assume that $V \in B_\infty(\mathbb{H}_n)$, $0 \le \gamma \le \frac{1}{2} \le \beta \le 1$ and $\beta - \gamma \ge \frac{1}{2}$. Let $1 \le p < q < \infty$, $2(\beta-\gamma) - 1 = Q\left(\frac{1}{p} - \frac{1}{q}\right)$ and the conditions (3.9), (3.10) be satisfied for $\alpha = 2(\beta-\gamma) - 1$.

Then for any $f \in C_0^\infty(\mathbb{H}_n)$

$$\|\mathcal{T}_2 f\|_{M_{q,\varphi^{\frac{1}{q}}}} \lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}}}}, \qquad \text{for } p > 1$$

and

$$\|\mathcal{T}_2 f\|_{WM_{q,\varphi^{\frac{1}{q}}}} \lesssim \|f\|_{M_{1,\varphi}} \qquad \text{for } p = 1.$$

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