Applied Mathematics and Computation 219 (2013) 8332–8337

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

The incomplete second Appell hypergeometric functions

Ayşegül Çetinkaya

Ahi Evran University, Department of Mathematics, 40100 Kırşehir, Turkey

ARTICLE INFO

Keywords:
Gamma function
Incomplete gamma functions
Pochhammer symbol
Incomplete Pochhammer symbols

ABSTRACT

Srivastava et al. [H.M. Srivastava, M.A. Chaudhry, R.P. Agarwal, The incomplete Pochhammer symbols and their applications to hypergeometric and related functions, Integral Transforms Spec. Funct. 23 (2012) 659–683] introduced the incomplete Pochhammer symbols. In this paper, the incomplete second Appell hypergeometric functions are defined in terms of these symbols. Some properties of these functions are also presented.

1. Introduction

In terms of familiar incomplete gamma functions

$$\gamma(a,x) := \int_0^x t^{a-1}e^{-t}dt \quad (\Re(a) > 0;\ x \geqslant 0), \tag{1}$$

$$\Gamma(a,x) := \int_x^\infty t^{a-1}e^{-t}dt \quad (x \geqslant 0;\ \Re(a) > 0 \text{ when } x = 0), \tag{2}$$

Srivastava et al. [10] introduced the incomplete Pochhammer symbols

$$(\lambda;x)_\nu := \frac{\gamma(\lambda+\nu,x)}{\Gamma(\lambda)} \quad (\lambda,\nu \in \mathbb{C};\ x \geqslant 0), \tag{3}$$

$$[\lambda;x]_\nu := \frac{\Gamma(\lambda+\nu,x)}{\Gamma(\lambda)} \quad (\lambda,\nu \in \mathbb{C};\ x \geqslant 0), \tag{4}$$

which satisfy the decomposition formula

$$(\lambda;x)_\nu + [\lambda;x]_\nu = (\lambda)_\nu \quad (\lambda,\nu \in \mathbb{C};\ x \geqslant 0). \tag{5}$$

Besides, they introduced and investigated the generalized incomplete hypergeometric functions by means of these symbols.

This paper is organized as follows. In Section 2, the incomplete second Appell hypergeometric functions are introduced by means of the incomplete Pochhammer symbols. For these functions, some integral formulas and the transforms formulas are obtained. Also, the connections with the incomplete Gauss hypergeometric functions, the incomplete gamma functions and the complementary error function are determined. In Section 3, the some finite summation formulas are presented.

E-mail address: acetinkaya@ahievran.edu.tr

0096-3003/$ - see front matter © 2012 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.amc.2012.11.050

2. The incomplete second Appell hypergeometric functions

We introduce the incomplete second Appell hypergeometric functions as follows:

$$\gamma_2((a,x),\alpha_1,\alpha_2;\beta_1,\beta_2;x_1,x_2) := \sum_{m,n=0}^{\infty} \frac{(a;x)_{m+n}(\alpha_1)_m(\alpha_2)_n}{(\beta_1)_m(\beta_2)_n} \frac{x_1^m}{m!} \frac{x_2^n}{n!}, \tag{6}$$

$$\Gamma_2((a,x),\alpha_1,\alpha_2;\beta_1,\beta_2;x_1,x_2) := \sum_{m,n=0}^{\infty} \frac{[a;x]_{m+n}(\alpha_1)_m(\alpha_2)_n}{(\beta_1)_m(\beta_2)_n} \frac{x_1^m}{m!} \frac{x_2^n}{n!}. \tag{7}$$

From (5)–(7), we have the decomposition formula

$$\gamma_2((a,x),\alpha_1,\alpha_2;\beta_1,\beta_2;x_1,x_2) + \Gamma_2((a,x),\alpha_1,\alpha_2;\beta_1,\beta_2;x_1,x_2) = F_2(a,\alpha_1,\alpha_2;\beta_1,\beta_2;x_1,x_2), \tag{8}$$

where $F_2(a,\alpha_1,\alpha_2;\beta_1,\beta_2;x_1,x_2)$ is the familiar second Appell hypergeometric function (see [1]).

To determine the properties and characteristics of $\gamma_2((a,x),\alpha_1,\alpha_2;\beta_1,\beta_2;x_1,x_2)$, it is sufficient to examine the properties of $\Gamma_2((a,x),\alpha_1,\alpha_2;\beta_1,\beta_2;x_1,x_2)$ according to the decomposition formula (8).

Theorem 1. Let the function defined by

$$u = u(x_1,x_2) := \gamma_2((a,x),\alpha_1,\alpha_2;\beta_1,\beta_2;x_1,x_2) + \Gamma_2((a,x),\alpha_1,\alpha_2;\beta_1,\beta_2;x_1,x_2).$$

Then this function satisfies the following system of partial differential equations:

$$\begin{aligned} &x_1(1-x_1)\frac{\partial^2 u}{\partial x_1^2} - x_1x_2\frac{\partial^2 u}{\partial x_1 \partial x_2} + [\beta_1 - (a+\alpha_1+1)x_1]\frac{\partial u}{\partial x_1} - \alpha_1 x_2 \frac{\partial u}{\partial x_2} - a\alpha_1 u = 0, \\ &x_2(1-x_2)\frac{\partial^2 u}{\partial x_2^2} - x_1x_2\frac{\partial^2 u}{\partial x_1 \partial x_2} + [\beta_2 - (a+\alpha_2+1)x_2]\frac{\partial u}{\partial x_2} - \alpha_2 x_1 \frac{\partial u}{\partial x_1} - a\alpha_2 u = 0. \end{aligned} \tag{9}$$

Proof. The proof is obvious from (8) and the fact that $F_2(a,\alpha_1,\alpha_2;\beta_1,\beta_2;x_1,x_2)$ satisfies the system of partial differential equations given by (9) (see [1, p. 76]). □

Theorem 2. (Integral formula)

$$\Gamma_2((a,x),\alpha_1,\alpha_2;\beta_1,\beta_2;x_1,x_2) = \frac{1}{\Gamma(a)} \int_x^{\infty} t^{a-1} e^{-t} {}_1F_1(\alpha_1;\beta_1;x_1 t) {}_1F_1(\alpha_2;\beta_2;x_2 t) dt \tag{10}$$

$$(x \geqslant 0;\ \Re(a) > 0 \text{ when } x = 0).$$

Proof. Replacing the incomplete Pochhammer symbol $[a;x]_{m+n}$ in the definition (7) by its integral representation which obtained from (2) and (4), we get the desired result. □

Theorem 3. (Transformation formulas)

$$\Gamma_2((a,x),\alpha_1,\alpha_2;\beta_1,\beta_2;x_1,x_2) = (1-x_1)^{-a}\Gamma_2\left((a,x(1-x_1)),\beta_1-\alpha_1,\alpha_2;\beta_1,\beta_2;-\frac{x_1}{1-x_1},\frac{x_2}{1-x_1}\right), \tag{11}$$

$$\Gamma_2((a,x),\alpha_1,\alpha_2;\beta_1,\beta_2;x_1,x_2) = (1-x_2)^{-a}\Gamma_2\left((a,x(1-x_2)),\alpha_1,\beta_2-\alpha_2;\beta_1,\beta_2;\frac{x_1}{1-x_2},-\frac{x_2}{1-x_2}\right), \tag{12}$$

$$\begin{aligned} \Gamma_2((a,x),\alpha_1,\alpha_2;\beta_1,\beta_2;x_1,x_2) &= (1-x_1-x_2)^{-a} \\ &\cdot \Gamma_2\left((a,x(1-x_1-x_2)),\beta_1-\alpha_1,\beta_2-\alpha_2;\beta_1,\beta_2;-\frac{x_1}{1-x_1-x_2},-\frac{x_2}{1-x_1-x_2}\right). \end{aligned} \tag{13}$$

Proof. We will prove only (11) since the others can be proved similarly. Using Kummer's first formula [6, p. 125]

$${}_1F_1(\alpha;\beta;x) = e^x {}_1F_1(\beta-\alpha;\beta;-x)$$

in (10), we have

$$\Gamma_2((a,x),\alpha_1,\alpha_2;\beta_1,\beta_2;x_1,x_2) = \frac{1}{\Gamma(a)} \int_x^{\infty} t^{a-1} e^{-(1-x_1)t} {}_1F_1(\beta_1-\alpha_1;\beta_1;-x_1 t) {}_1F_1(\alpha_2;\beta_2;x_2 t) dt. \tag{14}$$

Substituting $\tau = (1 - x_1)t$ in (14), we obtain the formula (11). □

Theorem 4. (Connections with the incomplete Gauss hypergeometric function ${}_2\Gamma_1$)

$$\Gamma_2((a,x),\alpha_1,\alpha_2;\beta_1,\beta_2;0,x_2) = {}_2\Gamma_1((a,x),\alpha_2;\beta_2;x_2), \tag{15}$$

$$\Gamma_2((a,x),\alpha_1,\alpha_2;\beta_1,\beta_2;x_1,0) = {}_2\Gamma_1((a,x),\alpha_1;\beta_1;x_1), \tag{16}$$

$$\Gamma_2((a,x),0,\alpha_2;\beta_1,\beta_2;x_1,x_2) = {}_2\Gamma_1((a,x),\alpha_2;\beta_2;x_2), \tag{17}$$

$$\Gamma_2((a,x),\alpha_1,0;\beta_1,\beta_2;x_1,x_2) = {}_2\Gamma_1((a,x),\alpha_1;\beta_1;x_1), \tag{18}$$

$$\Gamma_2((a,x),\alpha_1,\alpha_2;\alpha_1,\beta_2;x_1,x_2) = (1-x_1)^{-a}{}_2\Gamma_1((a,x(1-x_1)),\alpha_2;\beta_2;\frac{x_2}{1-x_1}), \tag{19}$$

$$\Gamma_2((a,x),\alpha_1,\alpha_2;\beta_1,\alpha_2;x_1,x_2) = (1-x_2)^{-a}{}_2\Gamma_1((a,x(1-x_2)),\alpha_1;\beta_1;\frac{x_1}{1-x_2}), \tag{20}$$

where ${}_2\Gamma_1((a,x),b;c;z)$ is the incomplete Gauss hypergeometric function defined by [10, p. 664]

$${}_2\Gamma_1((a,x),b;c;z) = \sum_{n=0}^{\infty} \frac{[a;x]_n (b)_n}{(c)_n} \frac{z^n}{n!}.$$

Proof. The proof of (15)–(18) is a direct consequence of the definition (7). The relation (19) is obtained setting $\beta_1 = \alpha_1$ in (11) and then using (17). Similarly, the relation (20) is derived setting $\beta_2 = \alpha_2$ in (12) and then using (18). □

Theorem 5. (Connection with the incomplete gamma function $\gamma(a,x)$ defined by (1))

$$\Gamma_2((a,x),\alpha_1,\alpha_2;\alpha_1+1,\alpha_2+1;-x_1,-x_2) = \frac{\alpha_1\alpha_2 x_1^{-\alpha_1} x_2^{-\alpha_2}}{\Gamma(a)} \int_x^{\infty} t^{a-\alpha_1-\alpha_2-1} e^{-t} \gamma(\alpha_1,x_1 t)\gamma(\alpha_2,x_2 t)dt \tag{21}$$

$(x \geqslant 0;\ \Re(a) > 0 \text{ when } x = 0).$

Proof. Setting $\beta_1 = \alpha_1 + 1, \beta_2 = \alpha_2 + 1$ and replacing x_1, x_2 by $-x_1$, $-x_2$ in the integral formula (10) and using the relation [5, p. 726]

$${}_1F_1(\alpha;\alpha+1;-x) = \alpha x^{-\alpha}\gamma(\alpha,x),$$

we obtain the relation (21). □

Theorem 6. (Connection with the incomplete gamma function $\Gamma(a,x)$ defined by (2))

$$\Gamma_2((a,x),\alpha_1,\alpha_2;\alpha_1,\alpha_2;x_1,x_2) = (1-x_1-x_2)^{-a}\frac{\Gamma(a,x(1-x_1-x_2))}{\Gamma(a)}. \tag{22}$$

Proof. Setting $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2$ in (13), we get the result. □

Theorem 7. (Connection with the complementary error function $\mathrm{erfc}(z)$)

$$\Gamma_2\left(\left(\frac{1}{2},x\right),\alpha_1,\alpha_2;\alpha_1,\alpha_2;0,1-z\right) = \Gamma_2\left(\left(\frac{1}{2},x\right),\alpha_1,\alpha_2;\alpha_1,\alpha_2;1-z,0\right) = \frac{1}{\sqrt{z}}\mathrm{erfc}(\sqrt{xz}) \tag{23}$$

where $\mathrm{erfc}(z)$ is the complementary error function defined by [5, p.726]

$$\mathrm{erfc}(\sqrt{z}) = \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2},z\right).$$

Proof. Putting $x_1 = 0, x_2 = 1 - z$ or $x_1 = 1 - z, x_2 = 0$ and taking $a = \frac{1}{2}$ in (22), we have the relation (23). □

Theorem 8. (Second integral formula)

$$\Gamma_2((a,x),\alpha_1,\alpha_2;\beta_1,\beta_2;x_1,x_2)=\frac{1}{B(\alpha_1,\beta_1-\alpha_1)B(\alpha_2,\beta_2-\alpha_2)} \cdot\int_0^1\int_0^1 t^{\alpha_1-1}\tau^{\alpha_2-1}(1-t)^{\beta_1-\alpha_1-1}(1-\tau)^{\beta_2-\alpha_2-1}(1-x_1t-x_2\tau)^{-a}\frac{\Gamma(a,x(1-x_1t-x_2\tau))}{\Gamma(a)}dt\,d\tau \quad (\Re(\beta_j)>\Re(\alpha_j)>0,\ j=1,2) \tag{24}$$

Proof. Taking into account the elementary identity

$$\frac{(\alpha_1)_m}{(\beta_1)_m}=\frac{B(\alpha_1+m,\beta_1-\alpha_1)}{B(\alpha_1,\beta_1-\alpha_1)}=\frac{1}{B(\alpha_1,\beta_1-\alpha_1)}\int_0^1 t^{\alpha_1+m-1}(1-t)^{\beta_1-\alpha_1-1}dt \quad (\Re(\beta_1)>\Re(\alpha_1)>0;\ m\in\mathbb{N}_0)$$

in the definition (7), we have

$$\Gamma_2((a,x),\alpha_1,\alpha_2;\beta_1,\beta_2;x_1,x_2)=\frac{1}{B(\alpha_1,\beta_1-\alpha_1)B(\alpha_2,\beta_2-\alpha_2)} \cdot\int_0^1\int_0^1 t^{\alpha_1-1}\tau^{\alpha_2-1}(1-t)^{\beta_1-\alpha_1-1}(1-\tau)^{\beta_2-\alpha_2-1}\Gamma_2((a,x),\alpha_1,\alpha_2;\alpha_1,\alpha_2;x_1t,x_2\tau)dt\,d\tau,$$

which yields (24) in accordance with (22). □

Remark 1. For $x=0$, the incomplete second Appell hypergeometric function Γ_2 reduces to the second Appell hypergeometric function F_2. Therefore, taking $x=0$, the obtained results in this section for Γ_2 reduce to the well-known results for F_2 (see, [1,2,5,7,9]).

3. Some finite summation formulas

Theorem 9. The following finite summation formula holds true:

$$\sum_{k=0}^{n}\Gamma_2((a,x),-k,-n+k;1,1;x_1,x_2)=(n+1)\,{}_2\Gamma_1((a,x),-n;2;x_1+x_2) \quad (x\geqslant 0;\ \Re(a)>0 \text{ when } x=0). \tag{25}$$

Proof. Using the integral formula (10) and the well-known relations for the Laguerre polynomials (see [3, p. 189, Eq. (14)] and [3, p. 192, Eq. (41)]), we get

$$\begin{aligned}\sum_{k=0}^{n}\Gamma_2((a,x),-k,-n+k;1,1;x_1,x_2)&=\frac{1}{\Gamma(a)}\sum_{k=0}^{n}\int_x^\infty t^{a-1}e^{-t}\,{}_1F_1(-k;1;x_1t)\,{}_1F_1(-n+k;1;x_2t)dt\\&=\frac{1}{\Gamma(a)}\int_x^\infty t^{a-1}e^{-t}\left(\sum_{k=0}^{n}L_k^{(0)}(x_1t)L_{n-k}^{(0)}(x_2t)\right)dt\\&=\frac{1}{\Gamma(a)}\int_x^\infty t^{a-1}e^{-t}L_n^{(1)}((x_1+x_2)t)dt\\&=\frac{(n+1)}{\Gamma(a)}\int_x^\infty t^{a-1}e^{-t}\,{}_1F_1(-n;2;(x_1+x_2)t)dt.\end{aligned} \tag{26}$$

Finally, taking into consideration the integral representation [10, p. 665]

$${}_2\Gamma_1((a,x),b;c;z)=\frac{1}{\Gamma(a)}\int_x^\infty t^{a-1}e^{-t}{}_1F_1(b;c;zt)dt \quad (x\geqslant 0;\ \Re(a)>0 \text{ when } x=0)$$

in (26), the proof is completed. □

Theorem 10. The following finite summation formula holds true:

$$\sum_{k=0}^{n}\Gamma_2\left((a,x),-k,-n+k;1,1;\frac{1}{2},\frac{1}{2}\right)=(n+1)\left\{\frac{\Gamma(2-a+n)}{\Gamma(2-a)\Gamma(2+n)}-\frac{x^a}{\Gamma(a+1)}\,{}_2F_2(2+n,a;2,a+1;-x)\right\}. \tag{27}$$

Proof. Setting $x_1=x_2=\frac{1}{2}$ in (25) and applying the formula [10, p. 667]

$$_2\Gamma_1((a,x),b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-\frac{x^a}{\Gamma(a+1)}\,{}_2F_2(c-b,a;c,a+1;-x),$$

we get the formula (27). □

Theorem 11. The following integral formula holds true:

$$\Gamma_2((a,x),-m,-n;\beta_1+1,\beta_2+1;x_1,x_2)=\frac{m!n!}{(\beta_1+1)_m(\beta_2+1)_n}\frac{1}{\Gamma(a)}\int_x^\infty t^{a-1}e^{-t}L_m^{(\beta_1)}(x_1t)L_n^{(\beta_2)}(x_2t)dt \tag{28}$$

$(x \geqslant 0;\ \Re(a)>0 \text{ when } x=0)$.

Proof. Using the integral formula (10) and the relation given in [3, p. 189, Eq. (14)], we have

$$\begin{aligned}\Gamma_2((a,x),-m,-n;\beta_1+1,\beta_2+1;x_1,x_2)&=\frac{1}{\Gamma(a)}\int_x^\infty t^{a-1}e^{-t}\,{}_1F_1(-m;\beta_1+1;x_1t)\,{}_1F_1(-n;\beta_2+1;x_2t)dt\\&=\frac{m!n!}{(\beta_1+1)_m(\beta_2+1)_n}\frac{1}{\Gamma(a)}\int_x^\infty t^{a-1}e^{-t}L_m^{(\beta_1)}(x_1t)L_n^{(\beta_2)}(x_2t)dt,\end{aligned}$$

which completes the proof. □

Theorem 12. The following finite summation formula holds true:

$$\begin{aligned}&\sum_{n=0}^{m}\binom{\beta+n}{n}\Gamma_2((a,x),-n,-n;\beta+1,\beta+1;x_1,x_2)\\&\quad=\frac{(\beta+1)_{m+1}}{m\,!(a-1)}(x_1-x_2)^{-1}\Gamma_2((a-1,x),-m,-m-1;\beta+1,\beta+1;x_1,x_2)+x_1\rightleftharpoons x_2,\end{aligned} \tag{29}$$

where $x \geqslant 0;\ \Re(a)>1$ when $x=0$, and $x_1\rightleftharpoons x_2$ indicates the presence of a second term that originates from the first by interchanging x_1 and x_2.

Proof. Using (28), we have

$$\begin{aligned}&\sum_{n=0}^{m}\binom{\beta+n}{n}\Gamma_2((a,x),-n,-n;\beta+1,\beta+1;x_1,x_2)\\&\quad=\frac{1}{\Gamma(a)}\sum_{n=0}^{m}\frac{n!}{(\beta+1)_n}\int_x^\infty t^{a-1}e^{-t}L_n^{(\beta)}(x_1t)L_n^{(\beta)}(x_2t)dt\quad(x\geqslant 0;\ \Re(a)>0 \text{ when } x=0).\end{aligned}$$

By interchanging the order of summation and integration and applying the formula [3, p. 188, Eq. (9)]

$$\sum_{n=0}^{m}\frac{n!}{(\beta+1)_n}L_n^{(\beta)}(x_1)L_n^{(\beta)}(x_2)=\frac{(m+1)!}{(\beta+1)_m}(x_1-x_2)^{-1}\{L_m^{(\beta)}(x_1)L_{m+1}^{(\beta)}(x_2)-L_{m+1}^{(\beta)}(x_1)L_m^{(\beta)}(x_2)\}$$

and then taking into consideration (28), we obtain the formula (29). □

Remark 2. Taking $x=0$, the obtained results in this section for the incomplete second Appell hypergeometric function Γ_2 reduce to the well-known results for the second Appell hypergeometric function F_2 (see [4,8]).

Acknowledgement

The author is very grateful to the anonymous referees for many valuable comments and suggestions which helped to improve the draft.

References

[1] W.N. Bailey, Generalized Hypergeometric Series, Cambridge Tracts in Mathematics and Mathematical Physics, vol. 32, Cambridge University Press, Cambridge, 1935; Reprinted by Stechert-Hafner Service Agency, New York and London, 1964.
[2] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Higher Transcendental Functions, vol. I, McGraw-Hill Book Company, New York, 1953.
[3] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Higher Transcendental Functions, vol. II, McGraw-Hill Book Company, New York, 1953.
[4] P.A. Padmanabham, H.M. Srivastava, Summation formulas associated with the Lauricella function $F_A^{(r)}$, Appl. Math. Lett. 13 (2000) 65–70.
[5] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, Integrals and Series, vol. II, Gordon and Breach Science Publishers, New York, 1992.
[6] E.D. Rainville, Special Functions, Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.

[7] L.J. Slater, Confluent Hypergeometric Functions, Cambridge University Press, Cambridge, 1960.
[8] H.M. Srivastava, On a summation formula for the Appell function F_2, Proc. Camb. Phil. Soc. 63 (1967) 1087–1089.
[9] H.M. Srivastava, P.W. Karlsson, Multiple Gaussian Hypergeometric Series, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, 1985.
[10] H.M. Srivastava, M.A. Chaudhry, R.P. Agarwal, The incomplete Pochhammer symbols and their applications to hypergeometric and related functions, Integral Transforms Spec. Funct. 23 (2012) 659–683.