Complex Variables and Elliptic Equations
An International Journal

ISSN: 1747-6933 (Print) 1747-6941 (Online) Journal homepage: https://www.tandfonline.com/loi/gcov20

# Boundedness of the anisotropic Riesz potential in anisotropic local Morrey-type spaces

A. Akbulut , V.S. Guliyev & Sh.A. Muradova

**To cite this article:** A. Akbulut , V.S. Guliyev & Sh.A. Muradova (2013) Boundedness of the anisotropic Riesz potential in anisotropic local Morrey-type spaces, Complex Variables and Elliptic Equations, 58:2, 259-280, DOI: 10.1080/17476933.2011.575465

**To link to this article:** https://doi.org/10.1080/17476933.2011.575465

Published online: 01 Jun 2011.

*Complex Variables and Elliptic Equations*, 2013
Vol. 58, No. 2, 259–280, http://dx.doi.org/10.1080/17476933.2011.575465

Taylor & Francis
Taylor & Francis Group

# Boundedness of the anisotropic Riesz potential in anisotropic local Morrey-type spaces†

A. Akbulut[a], V.S. Guliyev[ab]* and Sh.A. Muradova[b]

[a]*Department of Mathematics, Ahi Evran University, Kirşehir 40 200, Turkey;*
[b]*Department of Mathematical Analysis, Institute of Mathematics and Mechanics, Baku, AZ 1145, Azerbaijan*

Communicated by A. Pankov

(*Received 25 November 2010; final version received 9 March 2011*)

The problem of boundedness of the anisotropic Riesz potential in local Morrey-type spaces is reduced to the problem of boundedness of the Hardy operator in weighted $L_p$-spaces on the cone of non-negative non-increasing functions. This allows obtaining sharp sufficient conditions for boundedness for all admissible values of the parameters, which, for a certain range of the parameters wider than known before, coincide with the necessary ones.

**Keywords:** anisotropic Riesz potential; anisotropic local and global Morrey-type spaces; Hardy operator on the cone of monotonic functions

**AMS Subject Classifications:** Primary; 42B20; 42B25; 42B35

## 1. Introduction

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space with the routine norm $|x|$ for each $x \in \mathbb{R}^n$, $S^{n-1}$ denotes the unit sphere on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centred at $x$ of radius $r$ and ${}^{\complement}B(x,r)$ denote the set $\mathbb{R}^n \backslash B(x,r)$. Let $d = (d_1, \ldots, d_n)$, $d_i \geq 1$, $i = 1, \ldots, n$, $|d| = \sum_{i=1}^n d_i$ and $t^d x \equiv \left(t^{d_1} x_1, \ldots, t^{d_n} x_n\right)$. By [1,2], the function $F(x,\rho) = \sum_{i=1}^n x_i^2 \rho^{-2d_i}$, considered for any fixed $x \in \mathbb{R}^n$, is a decreasing one with respect to $\rho > 0$ and the equation $F(x,\rho) = 1$ is uniquely solvable. This unique solution will be denoted by $\rho(x)$. It is a simple matter to check that $\rho(x-y)$ defines a distance between any two points $x, y \in \mathbb{R}^n$. Thus $\mathbb{R}^n$, endowed with the metric $\rho$, defines a homogeneous metric space [1–3]. The balls with respect to $\rho$, centred at $x$ of radius $r$, are just the ellipsoids

$$\mathcal{E}_d(x,r) = \left\{ y \in \mathbb{R}^n : \frac{(y_1 - x_1)^2}{r^{2d_1}} + \cdots + \frac{(y_n - x_n)^2}{r^{2d_n}} < 1 \right\},$$

*Corresponding author. Email: vagif@guliyev.com
†Dedicated to 70th birthday of Prof. V. Burenkov.

with the Lebesgue measure $|\mathcal{E}_d(x,r)| = v_n r^{|d|}$, where $v_n$ is the volume of the unit ball in $\mathbb{R}^n$. Also let ${}^{\complement}\mathcal{E}_d(x,r) = \mathbb{R}^n \backslash \mathcal{E}_d(x,r)$ be the complement of $\mathcal{E}_d(0,r)$. If $d = \mathbf{1} \equiv (1,\dots,1)$, then clearly $\rho(x) = |x|$ and $\mathcal{E}_{\mathbf{1}}(x,r) = B(x,r)$. Note that in the standard parabolic case $d = (1,\dots,1,2)$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \quad x = (x', x_n).$$

For any $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, set

$$\begin{aligned} x_1 &= \rho^{d_1} \cos\varphi_1 \dots \cos\varphi_{n-2} \cos\varphi_{n-1}, \\ x_2 &= \rho^{d_2} \cos\varphi_1 \dots \cos\varphi_{n-2} \sin\varphi_{n-1}, \\ &\dots \\ x_{n-1} &= \rho^{d_{n-1}} \cos\varphi_1 \sin\varphi_2, \\ x_n &= \rho^{d_n} \sin\varphi_1. \end{aligned} \tag{1.1}$$

Thus, $\mathrm{d}x = \rho^{|d|-1} J(\varphi_1,\dots,\varphi_{n-1}) \mathrm{d}\rho\, \mathrm{d}\sigma(x)$, where $\mathrm{d}\sigma$ is the element of the area of $S^{n-1}$ and $\rho^{|d|-1} J(\varphi_1,\dots,\varphi_{n-1})$ is the Jacobian of this transform. In [1,2], it was shown that there exists a constant $M \geq 1$ such that $1 \leq J(\varphi_1,\dots,\varphi_{n-1}) \leq M$ and $J(\varphi_1,\dots,\varphi_{n-1}) \in C^\infty((0,2\pi)^{n-2} \times (0,\pi))$.

If $E$ is a non-empty measurable subset on $\mathbb{R}^n$ and $f$ is a measurable function on $E$, then we put

$$\|f\|_{L_p(E)} := \left( \int_E |f(y)|^p \mathrm{d}y \right)^{\frac{1}{p}}, \quad 0 < p < +\infty,$$

$$\|f\|_{L_\infty(E)} := \sup\{\alpha : |\{y \in E : |f(y)| \geq \alpha\}| > 0\}.$$

Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. The anisotropic Riesz potential $I_\alpha^d$ is defined by

$$I_\alpha^d f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{\rho(x-y)^{|d|-\alpha}} \mathrm{d}y, \quad 0 < \alpha < |d|.$$

If $d = \mathbf{1}$, then $I_\alpha \equiv I_\alpha^{\mathbf{1}}$ is the Riesz potential. The operators $I_\alpha$ and $I_\alpha^d$ play an important role in real and harmonic analysis (see, e.g. [4,5]).

In the theory of partial differential equations, together with weighted $L_{p,w}$ spaces, Morrey spaces $\mathcal{M}_{p,\lambda}$ play an important role. They were introduced by Morrey in 1938 [6]. These spaces appeared to be quite useful in the study of a number of problems in the theory of partial differential equations, in particular in the study of local behaviour of solutions of parabolic or quasi-elliptic differential equations. The anisotropic Morrey space is defined as follows: for $1 \leq p \leq \infty$, $0 \leq \lambda \leq |d|$, a function $f \in \mathcal{M}_{p,\lambda,d}$ if $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\|f\|_{\mathcal{M}_{p,\lambda,d}} \equiv \|f\|_{\mathcal{M}_{p,\lambda,d}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n,\, r>0} r^{-\lambda/p} \|f\|_{L_p(\mathcal{E}_d(x,r))} < \infty.$$

Note that $\mathcal{M}_{p,\lambda} \equiv \mathcal{M}_{p,\lambda,\mathbf{1}}$. (If $\lambda = 0$, then $\mathcal{M}_{p,0,d} = L_p$; if $\lambda = |d|$, then $\mathcal{M}_{p,|d|,d} = L_\infty$; if $\lambda < 0$ or $\lambda > |d|$, then $\mathcal{M}_{p,\lambda,d} = \Theta$, where $\Theta$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.)

Also, by $W\mathcal{M}_{p,\lambda,d}$ we denote the weak Morrey space of all functions $f \in WL_p^{\rm loc}$ for which

$$\|f\|_{W\mathcal{M}_{p,\lambda,d}} \equiv \|f\|_{W\mathcal{M}_{p,\lambda,d}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\lambda/p} \|f\|_{WL_p(\mathcal{E}_d(x,r))} < \infty,$$

where $WL_p(\mathcal{E}_d(x,r))$ denotes the weak $L_p$-space of measurable functions $f$ for which

$$\begin{aligned}\|f\|_{WL_p(\mathcal{E}_d(x,r))} &\equiv \|f\chi_{\mathcal{E}_d(x,r)}\|_{WL_p(\mathbb{R}^n)} \\ &= \sup_{t>0} t\left|\{y\in\mathcal{E}_d(x,r):\ |f(y)|>t\}\right|^{1/p} \\ &= \sup_{t>0} t^{1/p}\left(f\chi_{\mathcal{E}_d(x,r)}\right)^*(t) < \infty. \qquad (1.2)\end{aligned}$$

Here $g^*$ denotes the non-increasing rearrangement of the function $g$.

The anisotropic result by Hardy–Littlewood–Sobolev states that if $1<p_1<p_2<\infty$, then $I_\alpha^d$ is bounded from $L_{p_1}(\mathbb{R}^n)$ to $L_{p_2}(\mathbb{R}^n)$ if and only if $\alpha = |d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)$ and for $p_1=1<p_2<\infty$, $I_\alpha^d$ is bounded from $L_1(\mathbb{R}^n)$ to $WL_{p_2}(\mathbb{R}^n)$ if and only if $\alpha = |d|\left(1-\frac{1}{p_2}\right)$. Spanne [7] and Adams [8] studied boundedness of the Riesz potential $I_\alpha$ for $0<\alpha<n$ in Morrey spaces $\mathcal{M}_{p,\lambda}$. Later on Chiarenza and Frasca [9] reproved boundedness of the Riesz potential $I_\alpha$ in these spaces. By more general results of Guliyev [10] (see also [11,12]) one can obtain the following generalization of the results in [7–9] to the anisotropic case.

Theorem 1.1 (1) *Let $1<p_1<p_2<\infty$ and $0<\alpha<|d|$. Then $I_\alpha^d$ is bounded from $\mathcal{M}_{p_1,\lambda}$ to $\mathcal{M}_{p_2,\lambda}$ if and only if*

$$\alpha \le |d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right) \quad \textit{and} \quad \lambda = \left(|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)-\alpha\right)\left(\frac{1}{p_1}-\frac{1}{p_2}\right)^{-1}.$$

(2) *Let $1<p_2<\infty$ and $0<\alpha<|d|$. Then $I_\alpha^d$ is bounded from $\mathcal{M}_{1,\lambda}$ to $W\mathcal{M}_{p_2,\lambda}$ if and only if*

$$\alpha \le |d|\left(1-\frac{1}{p_2}\right) \quad \textit{and} \quad \lambda = \left(|d|\left(1-\frac{1}{p_2}\right)-\alpha\right)\left(1-\frac{1}{p_2}\right)^{-1}.$$

If $\alpha = |d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)$, then $\lambda=0$ and the statement of Theorem 1.1 reduces to the aforementioned result by Hardy–Littlewood–Sobolev.

If in the place of the power function $r^{-\lambda/p}$ in the definition of $\mathcal{M}_{p,\lambda,d}$ we consider any positive measurable weight function $w$ defined on $(0,\infty)$, then it becomes the Morrey-type space $\mathcal{M}_{p,w,d}$. Guliyev [10] and Fan et al. [13] (see also [11,12,14,15]) generalized Theorem 1.1 and obtained sufficient conditions on weights $w_1$ and $w_2$ ensuring boundedness of the anisotropic Riesz potential $I_\alpha^d$ for the limiting case $\alpha = |d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)$ from $\mathcal{M}_{p_1,w_1,d}$ to $\mathcal{M}_{p_2,w_2,d}$.

The following statement, containing the results in [13] was proved in [10] (see also [11,12,14,15]).

THEOREM 1.2 *Let* $1 \le p_1 \le p_2 < \infty$ *and* $\alpha = |d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$. *Moreover, let* $w_1$, $w_2$ *be positive measurable functions satisfying the following condition*:

$$\sup_{t>0} w_2(t) t^{\frac{|d|}{p_2}} \int_t^\infty \frac{s^{-\frac{|d|}{p_2}-1}}{w_1(s)} \mathrm{d}s < \infty. \tag{1.3}$$

*Then for* $p_1 > 1$ $I_\alpha^d$ *is bounded from* $\mathcal{M}_{p_1,w_1,d}$ *to* $\mathcal{M}_{p_2,w_2,d}$ *and for* $p_1 = 1$ $I_\alpha^d$ *is bounded from* $\mathcal{M}_{1,w_1,d}$ *to* $W\mathcal{M}_{p_2,w,d}$.

Earlier, in [13] a weaker version of Theorem 1.2 was proved: it was assumed that $w_1 = w_2 = w$ and that $w$ is a positive non-increasing function satisfying the pointwise doubling condition, namely that for some $c > 0$

$$c^{-1} w(r) \le w(t) \le c w(r)$$

for all $t, r > 0$ such that $0 < r \le t \le 2r$.

In [10,11,14–27] boundedness of maximal operator, fractional maximal operator, Riesz potential and singular integral operators from one local Morrey-type space $LM_{p_1\theta_1,w_1}$ to another one $LM_{p_2\theta_2,w_2}$ have been investigated and, in particular, in [24,25] for a certain range of the parameters necessary and sufficient conditions for the operator $I_\alpha$ to be bounded from $LM_{p_1\theta_1,w_1}$ to $LM_{p_2\theta_2,w_2}$ were obtained. (The definition and basic properties of these spaces are given in Section 2. In particular it is noted there that local Morrey-type spaces are non-trivial only if $w_1$, $w_2$ belong to classes $\Omega_{\theta_1}$, $\Omega_{\theta_2}$, respectively, defined in that section.)

THEOREM 1.3 (1) *If* $1 < p_1 < p_2 < \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $\alpha = n(1/p_1 - 1/p_2)$, $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$, *then the Burenkov–Guliyevs condition*

$$\left\| w_2(r) \left( \frac{r}{t+r} \right)^{n/p_2} \right\|_{L_{\theta_2}(0,\infty)} \le c \|w_1\|_{L_{\theta_1}(t,\infty)} \tag{1.4}$$

*for all* $t > 0$, *where* $c > 0$ *is independent of* $t$, *is necessary and sufficient for the boundedness of* $I_\alpha$ *from* $LM_{p_1\theta_1,w_1}$ *to* $LM_{p_2\theta_2,w_2}$.

(2) *If* $1 \le p_1 < p_2 < \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $\alpha = n(1/p_1 - 1/p_2)$, $w_1 \in \Omega_1$ *and* $w_2 \in \Omega_{\theta_2}$, *then the Burenkov–Guliyevs condition* (1.4) *is necessary and sufficient for the boundedness of* $I_\alpha$ *from* $LM_{p_1\theta_1,w_1}$ *to* $WLM_{p_2\theta_2,w_2}$.

Condition (1.4) for the first time was introduced in [20,21] for the case of the maximal operator and in [22,23] for the case of the fractional maximal operator. It appeared to be rather 'stable': for $\theta_1 \le \theta_2$ it serves as necessary and sufficient condition not only for the maximal and the fractional maximal operators, but also, under the appropriate assumptions on the parameters, for the Riesz potential [24,25] and genuine singular integral operators [26,27].

Theorem 1.3 in the case $\theta_1 \le p_1$ was proved in [24,25] and in the case $\theta_1 > p_1$ in [19]. In [24,25] the proof was based on a certain estimate for $L_p$-norms of $I_\alpha f$ over balls $B(x,r)$, which allowed to reduce the problem of boundedness of $I_\alpha$ in local Morrey-type spaces to the problem of boundedness of the Hardy operator on the cone of non-negative non-decreasing functions. In [19], the problem of boundedness of $I_\alpha$ from $LM_{p_1\theta_1,w_1}$ to $LM_{p_2\theta_2,w_2}$ was reduced to the problem of boundedness of the so-called Hardy operator on the cone of non-negative non-decreasing functions. Also for the case $p_1 = 1$, $0 < p_2 < \infty$, and $n(1 - \frac{1}{p_2})_+ < \alpha < n$ necessary and sufficient conditions ensuring boundedness of $I_\alpha$ from $LM_{1\theta_1,w_1}$ to $LM_{p_2\theta_2,w_2}$ were obtained in [19] for all $0 < \theta_1, \theta_2 \le \infty$ and $w_1 \in \Omega_{\theta_1}$, $w_2 \in \Omega_{\theta_2}$.

## 2. Definitions and basic properties of Morrey-type spaces

*Definition 2.1* Let $0 < p, \theta \le \infty$ and let $w$ be a non-negative measurable function on $(0,\infty)$. We denote by $LM_{p\theta,w,d}$, $GM_{p\theta,w,d}$, the anisotropic local Morrey-type spaces, the global Morrey-type spaces, respectively, the spaces of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorms

$$\|f\|_{LM_{p\theta,w,d}} \equiv \|f\|_{LM_{p\theta,w,d}(\mathbb{R}^n)} = \left\| w(r) \|f\|_{L_p(\mathcal{E}_d(0,r))} \right\|_{L_\theta(0,\infty)},$$

$$\|f\|_{GM_{p\theta,w,d}} = \sup_{x \in \mathbb{R}^n} \|f(x + \cdot)\|_{LM_{p\theta,w,d}},$$

respectively.

*Definition 2.2* Let $0 < p, \theta \le \infty$ and let $w$ be a non-negative measurable function on $(0,\infty)$. Denote by $WLM_{p\theta,w,d}$ $WGM_{p\theta,w,d}$, the anisotropic local weak Morrey-type spaces, the anisotropic global weak Morrey-type spaces, respectively, the spaces of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorms

$$\|f\|_{WLM_{p\theta,w,d}} \equiv \|f\|_{WLM_{p\theta,w,d}(\mathbb{R}^n)} = \left\| w(r) \|f\|_{WL_p(\mathcal{E}_d(0,r))} \right\|_{L_\theta(0,\infty)},$$

$$\|f\|_{WGM_{p\theta,w,d}} = \sup_{x \in \mathbb{R}^n} \|f(x + \cdot)\|_{WLM_{p\theta,w,d}},$$

respectively.

Note that $GM_{p\theta,w,\mathbf{1}} = GM_{p\theta,w}$, $LM_{p\theta,w,\mathbf{1}} = LM_{p\theta,w}$ and

$$\|f\|_{LM_{p\infty,1,d}} = \|f\|_{GM_{p\infty,1,d}} = \|f\|_{L_p}.$$

Also $WGM_{p\theta,w,\mathbf{1}} = WGM_{p\theta,w}$, $WLM_{p\theta,w,\mathbf{1}} = WLM_{p\theta,w}$ and

$$\|f\|_{WLM_{p\infty,1,d}} = \|f\|_{WGM_{p\infty,1,d}} = \|f\|_{WL_p}.$$

Furthermore, $GM_{p\infty,r^{-\lambda/p},d} \equiv \mathcal{M}_{p,\lambda,d}$, $WGM_{p\infty,r^{-\lambda/p},d} \equiv W\mathcal{M}_{p,\lambda,d}$, $0 \le \lambda \le |d|$.

Lemma 2.3 [16] *Let $0 < p, \theta \le \infty$ and let $w$ be a non-negative measurable function on $(0,\infty)$.*

(1) *If for all $t > 0$*

$$\|w(r)\|_{L_\theta(t,\infty)} = \infty, \tag{2.1}$$

*then $LM_{p\theta,w,d} = GM_{p\theta,w,d} = \Theta$, where $\Theta$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.*

(2) *If for all $t > 0$*

$$\|w(r) r^{|d|/p}\|_{L_\theta(0,t)} = \infty, \tag{2.2}$$

*then for all functions $f \in LM_{p\theta,w,d}$, continuous at 0, $f(0) = 0$, and for $0 < p < \infty$ $GM_{p\theta,w,d} = \Theta$.*

*Definition 2.4* Let $0<p,\theta\le\infty$. We denote by $\Omega_\theta$ the set of all functions $w$ which are non-negative, measurable on $(0,\infty)$, not equivalent to 0 and such that for some $t>0$

$$\|w\|_{L_\theta(t,\infty)}<\infty.$$

Moreover, we denote by $\Omega_{p,\theta,d}$ the set of all functions $w$ which are non-negative, measurable on $(0,\infty)$, not equivalent to 0 and such that for some $t_1,\ t_2>0$

$$\|w(r)\|_{L_\theta(t_1,\infty)}<\infty,\qquad \|w(r)r^{|d|/p}\|_{L_\theta(0,t_2)}<\infty.$$

In [16] (see also [21]), it was proved that if $\|w\|_{L_\theta(t,\infty)}=\infty$ for all $t>0$, then $GM_{p\theta,w,d}=LM_{p\theta,w,d}=\Theta$ and if $\|w(r)r^{|d|/p}\|_{L_\theta(0,t_2)}=\infty$ for all $t>0$, then $GM_{p\theta,w,d}=\Theta$. For this reason when considering spaces $LM_{p\theta,w,d}$ we always assume that $w\in\Omega_\theta$ and when considering spaces $GM_{p\theta,w,d}$ we always assume that $w\in\Omega_{p,\theta,d}$.

Lemma 2.5 [16] *Let $0<p<\infty$, $r>0$. Then for $\beta>-|d|/p$*

$$\left\|\rho(x)^\beta\right\|_{L_p(\mathcal{E}_d(0,r))}=(|d|+\beta p)^{-1/p}\,C_0\,r^{|d|/p+\beta},$$

*and for $\beta<-|d|/p$*

$$\left\|\rho(x)^\beta\right\|_{L_p({}^{\complement}\mathcal{E}_d(0,r))}=\big||d|+\beta p\big|^{-1/p}\,C_0\,r^{|d|/p+\beta},$$

*where ${}^{\complement}\mathcal{E}_d(0,r)$ is the complement of $\mathcal{E}_d(0,r)$, and*

$$\begin{aligned}C_0&=\left(\int_{S^{n-1}}\mathrm{d}\sigma(x')\right)^{1/p}\\&=\left(\int_0^\pi\int_0^\pi\cdots\int_0^{2\pi}J(\varphi_1,\dots,\varphi_{n-1})\mathrm{d}\varphi_1\mathrm{d}\varphi_2\cdots\mathrm{d}\varphi_{n-1}\right)^{1/p}<\infty.\end{aligned}$$

Corollary 2.6 [16] *Let $0<p,\ \theta,\ t<\infty$ and $w\in\Omega_\theta$. Then*

(1) $\rho(x)^\beta\in LM_{p\theta,w,d}\Longleftrightarrow\beta>-|d|/p$ *and* $\|w(r)r^{|d|/p+\beta}\|_{L_\theta(0,\infty)}<\infty$;

(2) $\rho(x)^\beta\chi_{\mathcal{E}_d(0,t)}\in LM_{p\theta,w,d}\Longleftrightarrow\beta>-|d|/p$ *and*
$\|w(r)r^{|d|/p+\beta}\|_{L_\theta(0,t)}<\infty,\ \|w(r)\|_{L_\theta(t,\infty)}<\infty$;

(3a) $\rho(x)^\beta\chi_{{}^{\complement}\mathcal{E}_d(0,t)}\in LM_{p\theta,w,d}$ *for* $\beta>-|d|/p$
$\Longleftrightarrow\|(r^{|d|/p+\beta}-t^{|d|/p+\beta})w(r)\|_{L_\theta(t,\infty)}<\infty$;

(3b) $\rho(x)^\beta\chi_{{}^{\complement}\mathcal{E}_d(0,t)}\in LM_{p\theta,w,d}$ *for* $\beta=-|d|/p\Longleftrightarrow\|w(r)(\ln\frac{r}{t})^{1/p}\|_{L_\theta(t,\infty)}<\infty$;

(3c) $\rho(x)^\beta\chi_{{}^{\complement}\mathcal{E}_d(0,t)}\in LM_{p\theta,w,d}$ for $\beta<-|d|/p$
$\Longleftrightarrow\|(t^{|d|/p+\beta}-r^{|d|/p+\beta})w(r)\|_{L_\theta(t,\infty)}<\infty$.

*If, in addition, $w$ is continuous on $(0,\infty)$ then conditions* (3a)–(3c) *take simpler form, namely*

(3a′) $\rho(x)^\beta\chi_{{}^{\complement}\mathcal{E}_d(0,t)}\in LM_{p\theta,w,d}$ *for* $\beta>-|d|/p$
$\Longleftrightarrow\|(r^{|d|/p+\beta}-t^{|d|/p+\beta})w(r)\|_{L_\theta(1,t)}<\infty$;

(3b′) $\rho(x)^{\beta}\chi_{\complement\mathcal{E}_d(0,t)} \in LM_{p\theta,w,d}$ *for* $\beta = -|d|/p \Longleftrightarrow \|w(r)(\ln r)^{1/p}\|_{L_\theta(1,t)} < \infty$;

(3c′) $\rho(x)^{\beta}\chi_{\complement\mathcal{E}_d(0,t)} \in LM_{p\theta,w,d}$ *for* $\beta < -|d|/p$.

Lemma 2.7 *Let* $1<p_1\le\infty$, $0<p_2\le\infty$, $0<\alpha<|d|$, $0<\theta_1,\theta_2\le\infty$, $w_1\in\Omega_{\theta_1}$ *and* $w_2\in\Omega_{\theta_2}$. *Then the conditions*

$$p_1<\infty \quad and \quad \alpha<\frac{|d|}{p_1}$$

*are necessary for the boundedness of* $I^d_\alpha$ *from* $LM_{p_1\theta_1,w_1,d}$ *to* $LM_{p_2\theta_2,w_2,d}$.

*Proof* Assume that $\alpha>\frac{|d|}{p_1}$ and $I^d_\alpha$ is bounded from $LM_{p_1\theta_1,w_1,d}$ to $LM_{p_2\theta_2,w_2,d}$. Let $f(x)=\rho(x)^{-\beta}$ if $\rho(x)\ge 1$ where $\frac{|d|}{p_1}<\beta<\alpha$, and $f(x)=0$ if $\rho(x)<1$. Then by Lemma 2.5 we have $f\in LM_{p_1\theta_1,w_1,d}$ since

$$\|f\|_{LM_{p_1\theta_1,w_1,d}} \le \|w\|_{L_{\theta_1}(1,\infty)}\|\rho(x)^{-\beta}\|_{L_{p_1}(\complement\mathcal{E}_d(0,1))}<\infty.$$

On the other hand for all $x\in\mathbb{R}^n$

$$I^d_\alpha f(x)=\int_{\complement\mathcal{E}_d(0,1)}\frac{\rho(y)^{-\beta}}{\rho(x-y)^{|d|-\alpha}}\,\mathrm{d}y=\infty.$$

Assume that $\alpha=\frac{|d|}{p_1}$ and $I^d_{\frac{|d|}{p_1}}$ is bounded from $LM_{p_1\theta_1,w_1,d}$ to $LM_{p_2\theta_2,w_2,d}$. Let $f(x)=\rho(x)^{-\frac{|d|}{p_1}}(\log\rho(x))^{-\gamma}$ if $\rho(x)\ge 2$ where $\frac{1}{p_1}<\gamma\le 1$, and $f(x)=0$ if $\rho(x)<2$. Then $f\in LM_{p_1\theta_1,w_1,d}$ since for $\gamma>\frac{1}{p_1}$

$$\|f\|_{LM_{p_1\theta_1,w_1,d}} \le \|w\|_{L_{\theta_1}(2,\infty)}\left\|\rho(x)^{-\frac{|d|}{p_1}}(\log\rho(x))^{-\gamma}\right\|_{L_{p_1}(\complement\mathcal{E}_d(0,2))}<\infty.$$

On the other hand, since $\rho(x-y)\le 2\rho(y)$ for $\rho(y)\ge\rho(x)$, by passing to generalized spherical coordinates (1.1) we have that for all $x\in\mathbb{R}^n$

$$\begin{aligned} I^d_{\frac{|d|}{p_1}} f(x) &\ge \int_{\rho(y)\ge\max\{2,\rho(x)\}}\rho(x-y)^{-\frac{|d|}{p_1'}}\rho(y)^{-\frac{|d|}{p_1}}(\log\rho(y))^{-\gamma}\mathrm{d}y\\ &\ge 2^{-\frac{|d|}{p_1'}}\int_{\rho(y)\ge\max\{2,\rho(x)\}}\rho(y)^{-|d|}(\log\rho(y))^{-\gamma}\mathrm{d}y=\infty, \end{aligned}$$

because $\gamma\le 1$. ∎

Throughout this article $a\lesssim b$ ($b\gtrsim a$) means that $a\le\lambda b$, where $\lambda>0$ depends on unessential parameters. If $b\lesssim a\lesssim b$, then we write $a\approx b$.

## 3. $L_p$-estimates of the anisotropic Riesz potential over ellipsoids

We consider the following 'partial' anisotropic Riesz potentials

$$\underline{I}^d_{\alpha,r}f(x)\equiv I^d_\alpha(f\chi_{\mathcal{E}_d(x,r)})(x)=\int_{\mathcal{E}_d(x,r)}\frac{|f(y)|}{\rho(x-y)^{|d|-\alpha}}\mathrm{d}y,$$

$$\overline{I}^d_{\alpha,r}f(x)\equiv I^d_\alpha(f\chi_{\complement\mathcal{E}_d(x,r)})(x)=\int_{\complement\mathcal{E}_d(x,r)}\frac{|f(y)|}{\rho(x-y)^{|d|-\alpha}}\mathrm{d}y.$$

Lemma 3.1 *Let* $0<p<\infty$, $0<\alpha<|d|$ *and* $f\in L_1^{\rm loc}(\mathbb{R}^n)$. *Then for any ball* $\mathcal{E}_d(x,r)$ *in* $\mathbb{R}^n$

$$\|I_\alpha^d(|f|)\|_{WL_p(\mathcal{E}_d(x,r))}\gtrsim r^{\frac{n}{p}}\,\overline{I}_{\alpha,r}^{\,d}(|f|)(x).$$

*Proof* If $y\in\mathcal{E}_d(x,r)$ and $z\in{}^{\complement}\mathcal{E}_d(x,r)$, then $\rho(y-z)\le 2\rho(x-y)$ and

$$\begin{aligned} I_\alpha^d(|f|)(y) &\ge \int_{{}^{\complement}\mathcal{E}_d(x,r)}\frac{|f(z)|}{\rho(y-z)^{|d|-\alpha}}\,\mathrm{d}z\\ &\ge 2^{\alpha-|d|}\int_{{}^{\complement}\mathcal{E}_d(x,r)}\frac{|f(z)|}{\rho(x-z)^{|d|-\alpha}}\,\mathrm{d}z = 2^{\alpha-|d|}\,\overline{I}_{\alpha,r}^{\,d}(|f|)(x). \end{aligned}$$

Hence[1]

$$\|I_\alpha^d(|f|)\|_{WL_p(\mathcal{E}_d(x,r))}\ge (v_n r^{|d|})^{\frac{1}{p}}\,2^{\alpha-|d|}\,\overline{I}_{\alpha,r}^{\,d}(|f|)(x),$$

where $v_n$ is the volume of the unit ball in $\mathbb{R}^n$. ■

Lemma 3.2 *Let* $0<p<\infty$, $0<\alpha<|d|$ *and* $f\in L_1^{\rm loc}(\mathbb{R}^n)$. *Then for any ball* $\mathcal{E}_d(x,r)$ *in* $\mathbb{R}^n$

$$\|I_\alpha^d(|f|)\|_{L_p(\mathcal{E}_d(x,r))}\approx \|I_\alpha^d(|f|\chi_{\mathcal{E}_d(x,2r)})\|_{L_p(\mathcal{E}_d(x,r))}+r^{\frac{n}{p}}\,\overline{I}_{\alpha,2r}^{\,d}(|f|)(x) \tag{3.1}$$

*and*

$$\|I_\alpha^d(|f|)\|_{WL_p(\mathcal{E}_d(x,r))}\approx \|I_\alpha^d(|f|\chi_{\mathcal{E}_d(x,2r)})\|_{WL_p(\mathcal{E}_d(x,r))}+r^{\frac{n}{p}}\,\overline{I}_{\alpha,2r}^{\,d}(|f|)(x). \tag{3.2}$$

*Proof* Clearly

$$\|I_\alpha^d(|f|)\|_{L_p(\mathcal{E}_d(x,r))}\lesssim \|I_\alpha^d(|f|\chi_{\mathcal{E}_d(x,2r)})\|_{L_p(\mathcal{E}_d(x,r))}+\|I_\alpha^d(|f|\chi_{{}^{\complement}\mathcal{E}_d(x,2r)})\|_{L_p(\mathcal{E}_d(x,r))}$$

and

$$\|I_\alpha^d(|f|)\|_{WL_p(\mathcal{E}_d(x,r))}\lesssim \|I_\alpha^d(|f|\chi_{\mathcal{E}_d(x,2r)})\|_{WL_p(\mathcal{E}_d(x,r))}+\|I_\alpha^d(|f|\chi_{{}^{\complement}\mathcal{E}_d(x,2r)})\|_{WL_p(\mathcal{E}_d(x,r))}.$$

If $y\in\mathcal{E}_d(x,r)$, $z\in{}^{\complement}\mathcal{E}_d(x,2r)$, then $\rho(x-z)/2\le\rho(y-z)\le 3\rho(x-z)/2$. Therefore,

$$\begin{aligned} \left\|I_\alpha^d\left(|f|\chi_{{}^{\complement}\mathcal{E}_d(x,2r)}\right)\right\|_{WL_p(\mathcal{E}_d(x,r))} &\le \left\|I_\alpha^d\left(|f|\chi_{{}^{\complement}\mathcal{E}_d(x,2r)}\right)\right\|_{L_p(\mathcal{E}_d(x,r))}\\ &=\left(\int_{\mathcal{E}_d(x,r)}\left(\int_{{}^{\complement}\mathcal{E}_d(x,2r)}\frac{f(z)}{\rho(y-z)^{|d|-\alpha}}\,\mathrm{d}z\right)^p\mathrm{d}y\right)^{\frac{1}{p}}\\ &\approx r^{\frac{|d|}{p}}\int_{{}^{\complement}\mathcal{E}_d(x,2r)}\frac{|f(z)|}{\rho(x-z)^{|d|-\alpha}}\,\mathrm{d}z\\ &= r^{\frac{|d|}{p}}\,\overline{I}_{\alpha,2r}^{\,d}(|f|)(x), \end{aligned}$$

and the right-hand side inequalities in (3.1) and (3.2) follow.

The left-hand side inequalities in (3.1) and (3.2) follow by Lemma 3.1 and obvious inequalities

$$\|I_\alpha^d(|f|)\|_{L_p(\mathcal{E}_d(x,r))}\ge \|I_\alpha^d(|f|\chi_{\mathcal{E}_d(x,2r)})\|_{L_p(\mathcal{E}_d(x,r))},$$

and

$$\|I^d_\alpha(|f|)\|_{WL_p(\mathcal{E}_d(x,r))} \geq \|I^d_\alpha(|f|\chi_{\mathcal{E}_d(x,2r)})\|_{WL_p(\mathcal{E}_d(x,r))}. \qquad \blacksquare$$

Lemma 3.3 *Let* $1 \leq p_1 < p_2 < \infty$ *and* $0 < \alpha < |d|$. *The inequality*

$$\|I^d_\alpha(f\chi_{\mathcal{E}_d(x,2r)})\|_{L_{p_2}(\mathcal{E}_d(x,r))} \lesssim r^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|f\|_{L_{p_1}(\mathcal{E}_d(x,2r))} \tag{3.3}$$

*holds for any ball* $\mathcal{E}_d(x,r) \subset \mathbb{R}^n$ *and for all* $f \in L^{\text{loc}}_{p_1}(\mathbb{R}^n)$ *if and only if in the case* $p_1 > 1$

$$\alpha \geq |d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right) \tag{3.4}$$

*and in the case* $p_1 = 1$

$$\alpha > |d|\left(1-\frac{1}{p_2}\right).$$

*Moreover for* $1 < p_2 < \infty$ *and* $\alpha = |d|\left(1-\frac{1}{p_2}\right)$ *the inequality*

$$\|I^d_\alpha(f\chi_{\mathcal{E}_d(x,2r)})\|_{WL_{p_2}(\mathcal{E}_d(x,r))} \lesssim \|f\|_{L_1(\mathcal{E}_d(x,2r))} \tag{3.5}$$

*holds for any ball* $\mathcal{E}_d(x,r) \subset \mathbb{R}^n$ *and for all* $f \in L^{\text{loc}}_1(\mathbb{R}^n)$.

*Proof* Recall the well-known inequalities for the anisotropic Riesz potential [5]. If $1 < p < q < \infty$, then

$$\|I^d_{|d|\left(\frac{1}{p}-\frac{1}{q}\right)}f\|_{L_q(\mathbb{R}^n)} \lesssim \|f\|_{L_p(\mathbb{R}^n)}. \tag{3.6}$$

Also if $1 < q < \infty$, then

$$\|I^d_{|d|\left(1-\frac{1}{q}\right)}f\|_{WL_q(\mathbb{R}^n)} \lesssim \|f\|_{L_1(\mathbb{R}^n)}. \tag{3.7}$$

If $1 < p_1 < p_2 < \infty$, inequality (3.4) holds and $z \in \mathcal{E}_d(x,r)$, then

$$I^d_\alpha\left(|f|\chi_{\mathcal{E}_d(x,2r)}\right)(z) \lesssim r^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} I^d_{|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\left(|f|\chi_{\mathcal{E}_d(x,2r)}\right)(z),$$

and by (3.6)

$$\|I^d_\alpha\left(|f|\chi_{\mathcal{E}_d(x,2r)}\right)\|_{L_{p_2}(\mathcal{E}_d(x,r))} \lesssim r^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|f\|_{L_{p_1}(\mathcal{E}_d(x,2r))}.$$

If $1 < p_2 < \infty$ and inequality (3.5) holds then by (3.7)

$$\begin{aligned}
\|I^d_\alpha\left(|f|\chi_{\mathcal{E}_d(x,2r)}\right)\|_{L_{p_2}(\mathcal{E}_d(x,r))} &\leq \|\left(I^d_\alpha\left(|f|\chi_{\mathcal{E}_d(x,2r)}\right)\right)^*\|_{L_{p_2}(0,|\mathcal{E}_d(x,r)|)} \\
&\leq \sup_{0<t\leq|\mathcal{E}_d(x,r)|} t^{1-\frac{\alpha}{|d|}}\left(I^d_\alpha\left(|f|\chi_{\mathcal{E}_d(x,2r)}\right)\right)^*(t)\|t^{\frac{\alpha}{|d|}-1}\|_{L_{p_2}(0,|\mathcal{E}_d(x,r)|)} \\
&\approx r^{\alpha-|d|\left(1-\frac{1}{p_2}\right)}\left\|I^d_\alpha\left(|f|\chi_{\mathcal{E}_d(x,2r)}\right)\right\|_{WL_{\frac{|d|}{|d|-\alpha}}(\mathcal{E}_d(x,r))} \\
&\lesssim r^{\alpha-|d|\left(1-\frac{1}{p_2}\right)}\|f\|_{L_1(\mathcal{E}_d(x,2r))}.
\end{aligned}$$

If $p_1 \geq 1$ and $\alpha < |d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, then inequality (3.3) cannot hold for all $f \in L_{p_1}^{\rm loc}(\mathbb{R}^n)$. Indeed if $f \in L_{p_1}(\mathbb{R}^n)$ and $f \nsim 0$ then by passing in (3.3) to the limit as $r \to +\infty$ we arrive at a contradiction.

Assume that $p_1 = 1$, $1 < p_2 < \infty$, $\alpha = |d|\left(1 - \frac{1}{p_2}\right)$ and $f \in L_1(\mathbb{R}^n)$. Then by passing to the limit in (3.3) as $r \to +\infty$ we get

$$\|I_\alpha^d f\|_{L_{p_2}(\mathbb{R}^n)} \lesssim \|f\|_{L_1(\mathbb{R}^n)},$$

which, according to known results [5], is not possible. ∎

COROLLARY 3.4 *Let*

$$1 < p_1 \leq \infty,\ 0 < p_2 \leq \infty \ \textit{or}\ p_1 = 1, \quad 0 < p_2 < \infty, \quad \textit{and}\ |d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ < \alpha < |d|, \tag{3.8}$$

*or*

$$1 < p_1 < p_2 < \infty \quad \textit{and} \quad \alpha = |d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right). \tag{3.9}$$

*Then the inequality*

$$\|I_\alpha^d (f\chi_{\mathcal{E}_d(x,2r)})\|_{L_{p_2}(\mathcal{E}_d(x,r))} \lesssim r^{\alpha - |d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right)} \|f\|_{L_{p_1}(\mathcal{E}_d(x,2r))}$$

*holds for any ball* $\mathcal{E}_d(x,r) \subset \mathbb{R}^n$ *and for all* $f \in L_{p_1}^{\rm loc}(\mathbb{R}^n)$.

*Moreover for* $1 < p_2 < \infty$ *and* $\alpha = |d|\left(1 - \frac{1}{p_2}\right)$, *then the inequality*

$$\|I_\alpha^d (f\chi_{\mathcal{E}_d(x,2r)})\|_{WL_{p_2}(\mathcal{E}_d(x,r))} \lesssim \|f\|_{L_1(\mathcal{E}_d(x,2r))}$$

*holds for any ball* $\mathcal{E}_d(x,r) \subset \mathbb{R}^n$ *and for all* $f \in L_1^{\rm loc}(\mathbb{R}^n)$.

*Proof* If $p_2 > p_1$, the statement follows by Lemma 3.3.

If $p_2 = p_1$, then by applying Minkowski's inequality for integrals we have

$$\begin{aligned}
\left\|I_\alpha^d\left(f\chi_{\mathcal{E}_d(x,2r)}\right)\right\|_{L_{p_1}(\mathcal{E}_d(x,r))} &\leq \left\| \int_{\mathcal{E}_d(x,2r)} \frac{\left|\left(f\chi_{\mathcal{E}_d(x,2r)}\right)(y)\right|}{\rho(\cdot - y)^{|d|-\alpha}} \mathrm{d}y \right\|_{L_{p_1}(\mathcal{E}_d(x,r))} \\
&\leq \left\| \int_{\mathcal{E}_d(0,,3r)} \frac{\left|\left(f\chi_{\mathcal{E}_d(x,2r)}\right)(\cdot - u)\right|}{\rho(u)^{|d|-\alpha}} \mathrm{d}u \right\|_{L_{p_1}(\mathbb{R}^n)} \\
&\leq \int_{\mathcal{E}_d(0,,3r)} \frac{\mathrm{d}u}{\rho(u)^{|d|-\alpha}} \ \left\|f\chi_{\mathcal{E}_d(x,2r)}\right\|_{L_{p_1}(\mathbb{R}^n)} \\
&\lesssim r^\alpha \|f\|_{L_{p_1}(\mathcal{E}_d(x,2r))}.
\end{aligned}$$

If $p_2 < p_1$, then by applying Hölder's inequality and this inequality we get

$$\begin{aligned}
&\|I_\alpha^d (f\chi_{\mathcal{E}_d(x,2r)})\|_{L_{p_2}(\mathcal{E}_d(x,r))} \\
&\quad \lesssim r^{\frac{|d|}{p_2} - \frac{|d|}{p_1}} \|I_\alpha^d (f\chi_{\mathcal{E}_d(x,2r)})\|_{L_{p_1}(\mathcal{E}_d(x,r))} \\
&\quad \lesssim r^{\alpha - |d|\left(\frac{1}{p_1} - \frac{1}{p_2}\right)} \|f\|_{L_{p_1}(\mathcal{E}_d(x,2r))}.
\end{aligned}$$

∎

Lemma 3.2 and Corollary 3.4 imply the following statement.

Lemma 3.5 *Let condition* (3.8) *or condition* (3.9) *be satisfied. Then the inequality*

$$\|I^d_\alpha f\|_{L_{p_2}(\mathcal{E}_d(x,r))} \lesssim r^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|f\|_{L_{p_1}(\mathcal{E}_d(x,2r))} + r^{\frac{|d|}{p_2}}\overline{I}^{\,d}_{\alpha,2r}(|f|)(x) \tag{3.10}$$

*holds for any ball* $\mathcal{E}_d(x,r)\subset\mathbb{R}^n$ *and for all* $f\in L^{\mathrm{loc}}_{p_1}(\mathbb{R}^n)$.

*Moreover, for* $1<p_2<\infty$ *and* $\alpha=|d|\left(1-\frac{1}{p_2}\right)$ *the inequality*

$$\|I^d_\alpha f\|_{WL_{p_2}(\mathcal{E}_d(x,r))} \lesssim \|f\|_{L_1(\mathcal{E}_d(x,2r))} + r^{\frac{|d|}{p_2}}\overline{I}^{\,d}_{\alpha,2r}(|f|)(x) \tag{3.11}$$

*holds for any ball* $\mathcal{E}_d(x,r)\subset\mathbb{R}^n$ *and for all* $f\in L^{\mathrm{loc}}_{1}(\mathbb{R}^n)$.

Lemma 3.6 *Let the condition* (3.8) *or condition* (3.9) *be satisfied. Then the inequality*

$$\|I^d_\alpha f\|_{L_{p_2}(\mathcal{E}_d(x,r))} \lesssim r^{\frac{|d|}{p_2}}\int_r^\infty \|f\|_{L_{p_1}(\mathcal{E}_d(x,t))}\frac{\mathrm{d}t}{t^{\frac{|d|}{p_1}-\alpha+1}} \tag{3.12}$$

*holds for any ball* $\mathcal{E}_d(x,r)\subset\mathbb{R}^n$ *and for all* $f\in L^{\mathrm{loc}}_{p_1}(\mathbb{R}^n)$.

*Proof* Note that if $\alpha\ge\frac{|d|}{p_1}$ and $f$ is not equivalent to 0 on $\mathbb{R}^n$, then the right-hand side of (3.12) is infinite, and in this case inequality (3.12) is trivial.

Let $\alpha<\frac{|d|}{p_1}$. By Lemma 6 in [23] and Hölder's inequality

$$\begin{aligned}
r^{\frac{|d|}{p_2}}\overline{I}^{\,d}_{\alpha,2r}(|f|)(x) &= r^{\frac{|d|}{p_2}}\int_{\complement\mathcal{E}_d(x,2r)}\frac{|f(y)|}{\rho(x-y)^{|d|-\alpha}}\mathrm{d}y\\
&= (|d|-\alpha)\,r^{\frac{|d|}{p_2}}\int_{2r}^\infty\left(\int_{2r\le\rho(x-y)\le t}|f(y)|\mathrm{d}y\right)\frac{\mathrm{d}t}{t^{|d|-\alpha+1}}\\
&\le (|d|-\alpha)\,r^{\frac{|d|}{p_2}}\int_{2r}^\infty\|f\|_{L_1(\mathcal{E}_d(x,t))}\frac{\mathrm{d}t}{t^{|d|-\alpha+1}}\\
&\lesssim r^{\frac{|d|}{p_2}}\int_{2r}^\infty\|f\|_{L_{p_1}(\mathcal{E}_d(x,t))}\frac{\mathrm{d}t}{t^{\frac{|d|}{p_1}-\alpha+1}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&r^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|f\|_{L_{p_1}(\mathcal{E}_d(x,2r))}\\
&\quad=\left(\frac{|d|}{p_1}-\alpha\right)2^{\alpha-\frac{|d|}{p_1}}\,r^{\frac{|d|}{p_2}}\|f\|_{L_{p_1}(\mathcal{E}_d(x,2r))}\int_{2r}^\infty\frac{\mathrm{d}t}{t^{\frac{|d|}{p_1}-\alpha+1}}\\
&\quad\lesssim r^{\frac{|d|}{p_2}}\int_{2r}^\infty\|f\|_{L_{p_1}(\mathcal{E}_d(x,t))}\frac{\mathrm{d}t}{t^{\frac{|d|}{p_1}-\alpha+1}}.
\end{aligned}$$

Hence the statement of the lemma follows by inequalities (3.10) and (3.11). ■

*Remark 3.7* Note that inequality (37) in [24]

$$\|I^d_\alpha f\|_{L_{p_2}(\mathcal{E}_d(x,r))} \lesssim r^{\frac{|d|}{p_2}-\delta}\left(\int_r^\infty\left(\int_{\mathcal{E}_d(x,t)}|f(y)|^{p_1}\mathrm{d}y\right)\frac{\mathrm{d}t}{t^{|d|-(\alpha+\delta)p_1+1}}\right)^{\frac{1}{p_1}}$$

follows from the inequality (3.12) by applying Hölder's inequality.

Indeed for any $\delta > 0$ by (3.12)

$$\begin{aligned}\|I_\alpha^d f\|_{L_{p_2}(\mathcal{E}_d(x,r))} &\lesssim r^{\frac{|d|}{p_2}} \int_r^\infty \left(\int_{\mathcal{E}_d(x,t)} |f(y)|^{p_1} \mathrm{d}y\right)^{\frac{1}{p_1}} \frac{\mathrm{d}t}{t^{\frac{|d|}{p_1}-(\alpha+\delta)+\frac{1}{p_1}+\delta+\frac{1}{p_1'}}} \\ &\lesssim r^{\frac{|d|}{p_2}} \left(\int_r^\infty \left(\int_{\mathcal{E}_d(x,t)} |f(y)|^{p_1} \mathrm{d}y\right) \frac{\mathrm{d}t}{t^{|d|-(\alpha+\delta)p_1+1}}\right)^{\frac{1}{p_1}} \left(\int_r^\infty \frac{\mathrm{d}t}{t^{p_1'\delta+1}}\right)^{\frac{1}{p_1'}} \\ &\lesssim r^{\frac{|d|}{p_2}-\delta} \left(\int_r^\infty \left(\int_{\mathcal{E}_d(x,t)} |f(y)|^{p_1} \mathrm{d}y\right) \frac{\mathrm{d}t}{t^{|d|-(\alpha+\delta)p_1+1}}\right)^{\frac{1}{p_1}}.\end{aligned}$$

Lemma 3.8 *Let $0 < p < \infty$, $0 < \alpha < |d|$. Then the inequality*

$$\begin{aligned}\|I_\alpha^d(|f|)\|_{WL_p(\mathcal{E}_d(x,r))} &\gtrsim r^{\frac{|d|}{p}} \int_r^\infty \|f\|_{L_1(\mathcal{E}_d(x,t))} \frac{\mathrm{d}t}{t^{|d|-\alpha+1}} \\ &\gtrsim r^{\alpha-|d|\left(1-\frac{1}{p}\right)} \|f\|_{L_1(\mathcal{E}_d(x,2r))}\end{aligned}$$

*holds for any ball $\mathcal{E}_d(x,r) \subset \mathbb{R}^n$ and for all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.*

*Proof* For all $y \in \mathcal{E}_d(x,r)$ $\rho(y-z) \le 2r$ if $z \in \mathcal{E}_d(x,r)$ and $\rho(y-z) \le 2\rho(x-z)$ if $z \in {}^{\complement}\mathcal{E}_d(x,r)$, therefore

$$\begin{aligned}I_\alpha^d(|f|)(y) &= \int_{\mathcal{E}_d(x,r)} \frac{|f(z)|}{\rho(y-z)^{|d|-\alpha}} \mathrm{d}z + \int_{^{\complement}\mathcal{E}_d(x,r)} \frac{|f(z)|}{\rho(y-z)^{|d|-\alpha}} \mathrm{d}z \\ &\ge (2r)^{\alpha-|d|} \int_{\mathcal{E}_d(x,r)} |f(z)| \mathrm{d}z + 2^{\alpha-|d|} \int_{^{\complement}\mathcal{E}_d(x,r)} \frac{|f(z)|}{\rho(x-z)^{|d|-\alpha}} \mathrm{d}z \\ &= (|d|-\alpha) 2^{\alpha-|d|} \int_r^\infty \left(\int_{\mathcal{E}_d(x,r)} |f(z)| \mathrm{d}z\right) \frac{\mathrm{d}t}{t^{|d|-\alpha+1}} \\ &\quad + (|d|-\alpha) 2^{\alpha-|d|} \int_{^{\complement}\mathcal{E}_d(x,r)} \left(\int_{\rho(x-z)}^\infty \frac{\mathrm{d}t}{t^{|d|-\alpha+1}}\right) |f(z)| \mathrm{d}z \\ &= (|d|-\alpha) 2^{\alpha-|d|} \left(\int_r^\infty \left(\int_{\mathcal{E}_d(x,r)} |f(z)| \mathrm{d}z\right) \frac{\mathrm{d}t}{t^{|d|-\alpha+1}} \right. \\ &\quad \left. + \int_r^\infty \left(\int_{\mathcal{E}_d(x,t) \setminus \mathcal{E}_d(x,r)} |f(z)| \mathrm{d}z\right) \frac{\mathrm{d}t}{t^{|d|-\alpha+1}}\right) \\ &= (|d|-\alpha) 2^{\alpha-|d|} \int_r^\infty \|f\|_{L_1(\mathcal{E}_d(x,t))} \frac{\mathrm{d}t}{t^{|d|-\alpha+1}}.\end{aligned}$$

Hence the first of the desired inequalities follows.[2]

The second one follows since

$$\begin{aligned}r^{\frac{|d|}{p}} \int_r^\infty \|f\|_{L_1(\mathcal{E}_d(x,t))} \frac{\mathrm{d}t}{t^{|d|-\alpha+1}} &\ge r^{\frac{|d|}{p}} \int_{2r}^\infty \|f\|_{L_1(\mathcal{E}_d(x,t))} \frac{\mathrm{d}t}{t^{|d|-\alpha+1}} \\ &\gtrsim r^{\alpha-|d|\left(1-\frac{1}{p}\right)} \|f\|_{L_1(\mathcal{E}_d(x,2r))}.\end{aligned}$$

■

Theorem 3.9 (1) *Let $0 < p < \infty$ and $|d|\left(1-\frac{1}{p}\right)_+ < \alpha < |d|$. Then the equivalences*

$$\begin{aligned}\|I_\alpha^d(|f|)\|_{WL_p(\mathcal{E}_d(x,r))} &\approx \|I_\alpha^d(|f|)\|_{L_p(\mathcal{E}_d(x,r))} \\ &\approx r^{\frac{|d|}{p}} \bar{I}_{\alpha,r}^d(|f|)(x) + r^{\alpha-|d|\left(1-\frac{1}{p}\right)} \|f\|_{L_1(\mathcal{E}_d(x,2r))} \qquad (3.13) \\ &\approx r^{\frac{|d|}{p}} \int_r^\infty \|f\|_{L_1(\mathcal{E}_d(x,t))} \frac{\mathrm{d}t}{t^{|d|-\alpha+1}}\end{aligned}$$

*hold for any ball $\mathcal{E}_d(x,r) \subset \mathbb{R}^n$ and for all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.*

(2) *Let* $1<p<\infty$ *and* $\alpha = |d|\left(1-\frac{1}{p}\right)$. *Then the equivalences*

$$\left\| I^d_{|d|\left(1-\frac{1}{p}\right)}(|f|)\right\|_{WL_p(\mathcal{E}_d(x,r))} \approx r^{\frac{|d|}{p}} \overline{I}^{\,d}_{|d|\left(1-\frac{1}{p}\right),r}(|f|)(x) + \|f\|_{L_1(\mathcal{E}_d(x,2r))}$$

$$\approx r^{\frac{|d|}{p}} \int_r^\infty \|f\|_{L_1(\mathcal{E}_d(x,t))} \frac{\mathrm{d}t}{t^{\frac{|d|}{p}+1}} \tag{3.14}$$

*hold for any ball* $\mathcal{E}_d(x,r)\subset\mathbb{R}^n$ *and for all* $f\in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

*Proof* The second equivalence in (3.13) for both $\|I^d_\alpha(|f|)\|_{WL_p(\mathcal{E}_d(x,r))}$ and $\|I^d_\alpha(|f|)\|_{L_p(\mathcal{E}_d(x,r))}$ and the first equivalence in (3.14) follow by Lemma 3.5 (estimate above) and Lemmas 3.1 and 3.8 (estimate below). The third equivalence in (3.13) for both $\|I^d_\alpha(|f|)\|_{WL_p(\mathcal{E}_d(x,r))}$ and $\|I^d_\alpha(|f|)\|_{L_p(\mathcal{E}_d(x,r))}$ and the second equivalence in (3.14) follow by Lemmas 3.6 and 3.8. ∎

## 4. Anisotropic Riesz potential and Hardy operator

Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset consisting of all non-negative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\downarrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$, which are non-increasing on $(0,\infty)$ and we set

$$\mathbb{A} = \left\{\varphi\in\mathfrak{M}^+(0,\infty;\downarrow) : \lim_{t\to\infty}\varphi(t)=0\right\}.$$

Let $H$ be the Hardy operator

$$(Hg)(t) := \int_0^t g(r)\mathrm{d}r, \quad 0<t<\infty.$$

Lemma 4.1 *Let condition* (3.8) *or condition* (3.9) *be satisfied. Moreover, let* $0<\theta_2\le\infty$ *and* $w_2\in\Omega_{\theta_2}$.

*Then*

$$\|I^d_\alpha f\|_{LM_{p_2\theta_2,w_2,d}} \lesssim \|Hg_{p_1}\|_{L_{\theta_2,\upsilon_2}(0,\infty)} \tag{4.1}$$

*for all* $f\in L^{\mathrm{loc}}_{p_1}(\mathbb{R}^n)$, *where*

$$g_{p_1}(t) = \left(\int_{\mathcal{E}_d(0,,t^{-\frac{1}{\sigma}})} |f(y)|^{p_1}\mathrm{d}y\right)^{\frac{1}{p_1}}, \quad \sigma = \frac{|d|}{p_1} - \alpha > 0,$$

*and*

$$\upsilon_2(r) = w_2(r^{-\frac{1}{\sigma}})r^{-\frac{|d|}{\sigma p_2}-\frac{1}{\theta_2\sigma}-\frac{1}{\theta_2}}. \tag{4.2}$$

*Moreover, if* $p_1=1$, $0<p_2<\infty$ *and* $|d|\left(1-\frac{1}{p_2}\right)_+ < \alpha < |d|$, *then*

$$\|I^d_\alpha f\|_{WLM_{p_2\theta_2,w_2,d}} \approx \|I^d_\alpha f\|_{LM_{p_2\theta_2,w_2,d}} \approx \|Hg_1\|_{L_{\theta_2,\upsilon_2}(0,\infty)}$$

*for all non-negative functions* $f\in L^{\mathrm{loc}}_1(\mathbb{R}^n)$.

*Also if* $1<p_2<\infty$ *and* $\alpha=|d|\big(1-\frac{1}{p_2}\big)$, *then*

$$\|I^d_\alpha f\|_{WLM_{p_2\theta_2,w_2,d}}\approx\|Hg_1\|_{L_{\theta_2,\upsilon_2}(0,\infty)}$$

*for all non-negative functions* $f\in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

*Proof* By Lemma 3.6 we have

$$\begin{aligned}\|I^d_\alpha f\|_{LM_{p_2\theta_2,w_2,d}} &\lesssim \left\|w_2(r)r^{\frac{|d|}{p_2}}\int_r^\infty \|f\|_{L_{p_1}(\mathcal{E}_d(0,t))}\frac{\mathrm{d}t}{t^{\sigma+1}}\right\|_{L_{\theta_2(0,\infty)}}\\
&\approx \left\|w_2(r)r^{\frac{|d|}{p_2}}\int_0^{r^{-\sigma}} \|f\|_{L_{p_1}(\mathcal{E}_d(0,\tau^{-\frac{1}{\sigma}}))}\mathrm{d}\tau\right\|_{L_{\theta_2(0,\infty)}}\\
&= \left\|w_2(r)r^{\frac{|d|}{p_2}}\int_0^{r^{-\sigma}} g_{p_1}(\tau)\mathrm{d}\tau\right\|_{L_{\theta_2(0,\infty)}}\\
&= \left\|w_2(\rho^{-\frac{1}{\sigma}})\rho^{-\frac{|d|}{\sigma p_2}-\frac{1}{\theta_2\sigma}-\frac{1}{\theta_2}}(Hg_{p_1}(\rho))\right\|_{L_{\theta_2(0,\infty)}}\\
&= \|Hg_{p_1}\|_{L_{\theta_2,\upsilon_2}(0,\infty)}.\end{aligned}$$

The second and third statements of the lemma follow by applying Theorem 3.9 also. ∎

THEOREM 4.2 *Let condition* (3.8) *or condition* (3.9) *be satisfied. Moreover, let* $0<\theta_1$, $\theta_2\le\infty$, $w_1\in\Omega_{\theta_1}$ *and* $w_2\in\Omega_{\theta_2}$.

*Then* $I^d_\alpha$ *is bounded from* $LM_{p\theta,w,d}$ *to* $LM_{p_1\theta_1,w_1,d}$ *if, and in the case* $p_1=1$, $0<p_2<\infty$ *and* $|d|\big(1-\frac{1}{p_2}\big)_+<\alpha<|d|$ *only if the operator* $H$ *is bounded from* $L_{\theta_1,\upsilon_1}(0,\infty)$ *to* $L_{\theta_2,\upsilon_2}(0,\infty)$ *on the cone* $\mathbb{A}$, *that is*

$$\|Hg\|_{L_{\theta_2,\upsilon_2}(0,\infty)}\lesssim\|g\|_{L_{\theta_1,\upsilon_1}(0,\infty)} \tag{4.3}$$

*for all functions* $g\in\mathbb{A}$, *where*

$$\upsilon_1(r)=w_1(r^{-\frac{1}{\sigma}})r^{-\frac{1}{\theta_1\sigma}-\frac{1}{\theta_1}} \tag{4.4}$$

*and* $\upsilon_2$ *is defined by equality* (4.2).

*Moreover, if* $p_1=1$, $0<p_2<\infty$ *and* $|d|\big(1-\frac{1}{p_2}\big)_+<\alpha<|d|$ *or* $1<p_2<\infty$ *and* $\alpha=|d|\big(1-\frac{1}{p_2}\big)$, *then* $I^d_\alpha$ *is bounded from* $LM_{1\theta_1,w_1,d}$ *to* $WLM_{p_2\theta_2,w_2,d}$ *if and only if the operator* $H$ *is bounded from* $L_{\theta_1,\upsilon_1}(0,\infty)$ *to* $L_{\theta_2,\upsilon_2}(0,\infty)$ *on the cone* $\mathbb{A}$.

*Proof* Assume that the operator $H$ is bounded from $L_{\theta_1,\upsilon_1}(0,\infty)$ to $L_{\theta_2,\upsilon_2}(0,\infty)$ on the cone $\mathbb{A}$. Since $g_{p_1}\in\mathbb{A}$, by Lemma 4.1 we have

$$\|I^d_\alpha f\|_{LM_{p_2\theta_2,w_2,d}}\lesssim\|Hg_{p_1}\|_{L_{\theta_2,\upsilon_2}(0,\infty)}\lesssim\|g_{p_1}\|_{L_{\theta_1,\upsilon_1}(0,\infty)}.$$

Note that

$$\begin{aligned}\|g_{p_1}\|_{L_{\theta_1,\upsilon_1}(0,\infty)} &= \left\|\upsilon_1(t)\|f\|_{L_{p_1}(\mathcal{E}_d(0,,t^{-\frac{1}{\sigma}}))}\right\|_{L_{\theta_1,\upsilon_1}(0,\infty)}\\
&\approx \left\|\upsilon_1(\rho^{-\sigma})\rho^{-\frac{\sigma+1}{\theta_1}}\|f\|_{L_{p_1}(\mathcal{E}_d(0,\rho))}\right\|_{L_{\theta_1,\upsilon_1}(0,\infty)}\\
&= \left\|w_1(\rho)\|f\|_{L_{p_1}(\mathcal{E}_d(0,\rho))}\right\|_{L_{\theta_1,\upsilon_1}(0,\infty)}\\
&= \|f\|_{LM_{p_1\theta_1,w_1,d}}.\end{aligned}$$

Hence it follows that $I^d_\alpha$ is bounded from $LM_{p\theta,w,d}$ to $LM_{p_1\theta_1,w_1,d}$.

Assume that $I^d_\alpha$ is bounded from $LM_{1\theta_1,w_1,d}$ to $LM_{p_1\theta_1,w_1,d}$. Then for all non-negative $f \in L_1^{\text{loc}}(\mathbb{R}^n)$

$$\|Hg_{p_1}\|_{L_{\theta_2,v_2}(0,\infty)} \approx \|I^d_\alpha f\|_{LM_{p_2\theta_2,w_2,d}} \lesssim \|f\|_{LM_{1\theta_1,w_1,d}} \approx \|g_{p_1}\|_{L_{\theta_1,v_1}(0,\infty)}. \tag{4.5}$$

Let $g \in \mathbb{A}$ be locally absolutely continuous on $(0,\infty)$. Consider the non-negative measurable function $h$ on $(0,\infty)$ defined uniquely up to equivalence by the equality

$$g(t) = \|h(|\cdot|)\|_{L_{p_1}(\mathcal{E}_d(0,t^{-\frac{1}{\sigma}}))} = (|d|v_n)^{\frac{1}{p_1}}\left(\int_0^{t^{-\frac{1}{\sigma}}} h(\rho)^{p_1}\rho^{|d|-1}\mathrm{d}\rho\right)^{\frac{1}{p_1}}.$$

If we take in (4.5) $f(x) = h(\rho(x))$ then $g_{p_1} = g$ and (4.5) implies that

$$\|Hg\|_{L_{\theta_2,v_2}(0,\infty)} \lesssim \|g\|_{L_{\theta_1,v_1}(0,\infty)}. \tag{4.6}$$

Finally if $g$ is an arbitrary function in $\mathbb{A}$, then there exist functions $g_n \in \mathbb{A}$ which are locally absolutely continuous on $(0,\infty)$ and $g_n \nearrow g$ on $(0,\infty)$ as $|d| \to \infty$. Therefore by passing to the limit it follows that inequality (4.6) holds for all $g \in \mathbb{A}$. ■

## 5. Necessary and sufficient conditions

In order to obtain sufficient conditions on the weight functions ensuring boundedness of $I^d_\alpha$, we shall apply Theorem 4.2 and the known necessary and sufficient conditions ensuring boundedness of the Hardy operator $H$ from one weighted Lebesgue space to another one on the cone $\mathbb{A}$ (see, e.g. [28,29]).

Theorem 5.1 *Let condition* (3.8) *or condition* (3.9) *be satisfied. Moreover, let* $0 < \theta_1$, $\theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$, $w_2 \in \Omega_{\theta_2}$.

*Then the operator* $I^d_\alpha$ *is bounded from* $LM_{p_1\theta_1,w_1,d}$ *to* $LM_{p_2\theta_2,w_2,d}$ *if and in the case* $p_1 = 1$ *only if*,

(a) $1 < \theta_1 \le \theta_2 < \infty$, *then*

$$B_1^1 := \sup_{t>0}\left(\int_t^\infty w_2^{\theta_2}(r) r^{\theta_2\left(\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)\right)}\mathrm{d}r\right)^{\frac{1}{\theta_2}}\left(\int_t^\infty w_1^{\theta_1}(r)\mathrm{d}r\right)^{-\frac{1}{\theta_1}} < \infty, \tag{5.1}$$

*and*

$$B_2^1 := \sup_{t>0}\left(\int_0^t w_2^{\theta_2}(r) r^{\theta_2\frac{|d|}{p_2}}\mathrm{d}r\right)^{\frac{1}{\theta_2}}\left(\int_t^\infty \frac{w_1^{\theta_1}(r) r^{\theta_1'\left(\alpha-\frac{|d|}{p_1}\right)}}{\left(\int_r^\infty w_1^{\theta_1}(\rho)\mathrm{d}\rho\right)^{\theta_1'}}\mathrm{d}r\right)^{\frac{1}{\theta_1'}} < \infty. \tag{5.2}$$

(b) $0 < \theta_1 \le 1$, $0 < \theta_1 \le \theta_2 < \infty$, *then* $B_1^1 < \infty$ *and*

$$B_2^2 := \sup_{t>0} t^{\alpha-\frac{|d|}{p_1}}\left(\int_0^t w_2^{\theta_2}(r) r^{\theta_2\frac{|d|}{p_2}}\mathrm{d}r\right)^{\frac{1}{\theta_2}}\left(\int_t^\infty w_1^{\theta_1}(r)\mathrm{d}r\right)^{-\frac{1}{\theta_1}} < \infty. \tag{5.3}$$

(c) $1<\theta_1<\infty,\ 0<\theta_2<\theta_1<\infty,\ \theta_2\neq 1$, *then*

$$B_1^3 := \left(\int_0^\infty \left(\frac{\int_t^\infty w_2^{\theta_2}(r) r^{\theta_2\left(\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)\right)}\mathrm{d}r}{\int_t^\infty w_1^{\theta_1}(r)\mathrm{d}r}\right)^{\frac{\theta_2}{\theta_1-\theta_2}} w_2^{\theta_2}(t) t^{\theta_2\left(\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)\right)}\mathrm{d}t\right)^{\frac{\theta_1-\theta_2}{\theta_1\theta_2}} < \infty,$$

*and*

$$B_2^3 := \left(\int_0^\infty \left[\left(\int_0^t w_2^{\theta_2}(r) r^{\theta_2\frac{|d|}{p_2}}\,\mathrm{d}r\right)^{\frac{1}{\theta_2}} \left(\int_t^\infty \frac{w_1^{\theta_1}(r) r^{\theta_1'\left(\alpha-\frac{|d|}{p_1}\right)}}{\left(\int_r^\infty w_1^{\theta_1}(\rho)\mathrm{d}\rho\right)^{\theta_1'}}\mathrm{d}r\right)^{\frac{\theta_2-1}{\theta_2}}\right]^{\frac{\theta_1\theta_2}{\theta_1-\theta_2}}\right.$$

$$\left.\times \frac{w_1^{\theta_1}(t) t^{\theta_1'\left(\alpha-\frac{|d|}{p_1}\right)}}{\left(\int_t^\infty w_1^{\theta_1}(\rho)\mathrm{d}\rho\right)^{\theta_1'}}\mathrm{d}t\right)^{\frac{\theta_1-\theta_2}{\theta_1\theta_2}} < \infty.$$

(d) $1=\theta_2<\theta_1<\infty$, *then*

$$B_1^4 := \left(\int_0^\infty \left(\frac{\int_t^\infty w_2(r) r^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\mathrm{d}r}{\int_t^\infty w_1^{\theta_1}(r)\mathrm{d}r}\right)^{\frac{1}{\theta_1-1}} w_2(t) t^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\mathrm{d}t\right)^{\frac{\theta_1-1}{\theta_1}} < \infty,$$

*and*

$$B_2^4 := \left(\int_0^\infty \left(\frac{\int_t^\infty w_2(r) r^{\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\mathrm{d}r + t^{\alpha-\frac{|d|}{p_1}}\int_0^t w_2(r) r^{\frac{|d|}{p_2}}\mathrm{d}r}{\int_t^\infty w_1^{\theta_1}(r)\mathrm{d}r}\right)^{\theta_1'-1}\times\right.$$

$$\left.\times t^{\alpha-\frac{|d|}{p_1}}\left(\int_0^t w_2(r) r^{\frac{|d|}{p_2}}\mathrm{d}r\right)\frac{\mathrm{d}t}{t}\right)^{\theta_1'} < \infty.$$

(e) $0<\theta_2<\theta_1=1$, *then*

$$B_1^5 := \left(\int_0^\infty \left(\frac{\int_t^\infty w_2^{\theta_2}(r) r^{\theta_2\left(\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)\right)}\mathrm{d}r}{\int_t^\infty w_1(r)\mathrm{d}r}\right)^{\frac{\theta_2}{1-\theta_2}} w_2^{\theta_2}(t) t^{\theta_2\left(\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)\right)}\mathrm{d}t\right)^{\frac{1-\theta_2}{\theta_2}} < \infty,$$

*and*

$$B_2^5 := \left(\int_0^\infty \left(\int_0^t w_2^{\theta_2}(r) r^{\theta_2\frac{|d|}{p_2}}\mathrm{d}r\right)^{\frac{\theta_2}{1-\theta_2}} \left(\inf_{t<s<\infty} s^{\frac{|d|}{p_1}-\alpha}\int_s^\infty w_1(\rho)\mathrm{d}\rho\right)^{\frac{\theta_2}{\theta_2-1}} w_2^{\theta_2}(t) t^{\theta_2\frac{|d|}{p_2}}\mathrm{d}t\right)^{\frac{1-\theta_2}{\theta_2}} < \infty.$$

(f) $0<\theta_2<\theta_1<1$, *then* $B_1^3<\infty$ *and*

$$B_2^6 := \left(\int_0^\infty \sup_{t\le s<\infty} \frac{s^{\left(\alpha-\frac{|d|}{p_1}\right)\frac{\theta_1\theta_2}{\theta_1-\theta_2}}}{\left(\int_s^\infty w_1^{\theta_1}(\rho)\mathrm{d}\rho\right)^{\frac{\theta_2}{\theta_1-\theta_2}}} \left(\int_0^t w_2^{\theta_2}(r) r^{\theta_2\frac{|d|}{p_2}}\mathrm{d}r\right)^{\frac{\theta_2}{\theta_1-\theta_2}} w_2^{\theta_2}(t) t^{\theta_2\frac{|d|}{p_2}}\mathrm{d}t\right)^{\frac{\theta_1-\theta_2}{\theta_1\theta_2}} < \infty.$$

(g) $0<\theta_1\le 1$, $\theta_2=\infty$, *then*

$$B^7 := \operatorname*{ess\,sup}_{0<t\le s<\infty} \frac{w_2(t)t^{\frac{|d|}{p_2}}}{s^{\frac{|d|}{p_1}-\alpha}\left(\int_s^\infty w_1^{\theta_1}(r)\mathrm{d}r\right)^{\frac{1}{\theta_1}}} < \infty.$$

(h) $1<\theta_1<\infty$, $\theta_2=\infty$, *then*

$$B^8 := \operatorname*{ess\,sup}_{t>0} w_2(t)t^{\frac{|d|}{p_2}}\left(\int_t^\infty \frac{r^{\theta_1'\left(\alpha-\frac{|d|}{p_1}\right)}}{\left(\int_r^\infty w_1^{\theta_1}(s)\mathrm{d}s\right)^{\theta_1'-1}}\frac{\mathrm{d}r}{r}\right)^{\frac{1}{\theta_1'}} < \infty.$$

(i) $\theta_1=\infty$, $0<\theta_2<\infty$, *then*

$$B^{10} := \left(\int_0^\infty\left(t^{\frac{|d|}{p_1}-\alpha}\int_t^\infty \frac{s^{\alpha-\frac{|d|}{p_1}-1}\mathrm{d}s}{\operatorname{ess\,sup}_{s<y<\infty} w_1(y)}\right)^{\theta_2} w_2^{\theta_2}(t)t^{\theta_2\left(\alpha-|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)\right)}\mathrm{d}t\right)^{\frac{1}{\theta_2}} < \infty.$$

(j) $\theta_1=\theta_2=\infty$, *then*

$$B^9 := \operatorname*{ess\,sup}_{t>0} w_2(t)t^{\frac{|d|}{p_2}}\int_t^\infty \frac{s^{\alpha-\frac{|d|}{p_1}-1}}{\operatorname{ess\,sup}_{s<y<\infty} w_1(y)}\mathrm{d}s < \infty.$$

*Moreover, if* $p_1=1$, $0<p_2<\infty$ *and* $|d|\left(1-\frac{1}{p_2}\right)_+<\alpha<|d|$ *or* $1<p_2<\infty$ *and* $\alpha=|d|\left(1-\frac{1}{p_2}\right)$, *then* $I_\alpha^d$ *is bounded from* $LM_{1\theta_1,w_1,d}$ *to* $WLM_{p_2\theta_2,w_2,d}$ *if and only if conditions (a)–(j) are satisfied.*

*Proof* From results in [28,29] it follows that conditions (a)–(j) are necessary and sufficient for inequality (4.3) to hold, where $\upsilon_1$ and $\upsilon_2$ are defined by (4.2) and (4.4) respectively.

For example, let $1<\theta_1\le\theta_2<\infty$, then by [28,29] inequality (4.3) holds if and only if

$$A_1^1 := \sup_{t>0}\left(\int_0^t v_2^{\theta_2}(s)\mathrm{d}s\right)^{\frac{1}{\theta_2}}\left(\int_0^t v_1^{\theta_1}(s)\mathrm{d}s\right)^{-\frac{1}{\theta_1}} < \infty,$$

and

$$A_2^1 := \sup_{t>0}\left(\int_t^\infty v_2^{\theta_2}(s)\right)^{\frac{1}{\theta_2}}\left(\int_0^t \frac{v_1^{\theta_1}(s)s^{\theta_1'}}{\left(\int_0^s v_2^{\theta_2}(\tau)\mathrm{d}s\right)^{\theta_1'}}\mathrm{d}s\right)^{\frac{1}{\theta_1'}} < \infty.$$

If $\upsilon_1$ and $\upsilon_2$ are defined by (4.2) and (4.4), respectively, then by using the substitute $r=s^{-\frac{1}{\sigma}}$ we get

$$A_1^1 := \sup_{t>0}\left(\int_0^t w_2^{\theta_2}(s^{-\frac{1}{\sigma}})s^{-\frac{|d|\theta_2}{\sigma p_2}-\frac{1}{\sigma}-1}\mathrm{d}s\right)^{\frac{1}{\theta_2}}\left(\int_0^t w_1^{\theta_1}(s^{-\frac{1}{\sigma}})r^{-\frac{1}{\sigma}-1}\mathrm{d}s\right)^{-\frac{1}{\theta_1}} \approx B_1^1$$

and similarly $A_2^1\approx B_2^1$.

Hence the statement follows by Theorem 4.2. ■

*Remark 5.2* Note that two conditions (5.1) and (5.3) are equivalent to anisotropic variant of the Burenkov–Guliyevs condition

$$\left\| w_2(r)\left(\frac{r}{t+r}\right)^{|d|/p_2} \right\|_{L_{\theta_2}(0,\infty)} \le c\|w_1\|_{L_{\theta_1}(t,\infty)} \tag{5.4}$$

for all $t>0$, where $c>0$ is independent of $t$.

COROLLARY 5.3 *Let condition* (3.8) *or condition* (3.9) *be satisfied. Moreover, let functions* $w_1 \in \Omega_{p_1,\infty,d}$ *and* $w_2 \in \Omega_{p_2,\infty,d}$ *satisfy the following condition*:

$$\sup_{t>0} w_2(t) t^{\frac{|d|}{p_2}} \int_t^\infty \frac{s^{\alpha-\frac{|d|}{p_1}-1}}{\operatorname{ess\,sup}_{s<\tau<\infty} w_1(\tau)} \mathrm{d}s < \infty. \tag{5.5}$$

*Then* $I_\alpha^d$ *is bounded from* $\mathcal{M}_{p_1,w_1,d}$ *to* $\mathcal{M}_{p_2,w_2,d}$.

*Proof* Clearly boundedness of $I_\alpha^d$ from $LM_{p_1\infty,w_1,d}$ to $LM_{p_2\infty,w_2,d}$ implies boundedness of $I_\alpha^d$ from $GM_{p_1\infty,w_1,d} = \mathcal{M}_{p_1,w_1,d}$ to $GM_{p_2\infty,w_2,d} = \mathcal{M}_{p_2,w_2,d}$. ■

*Remark 5.4* Let $1<p_1<p_2<\infty$, $\alpha = |d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)$. It is obvious that if condition (1.3) holds, then condition (5.5) holds too. Moreover for non-increasing continuous functions $w_1$ conditions (1.3) and (5.5) coincide. However, in general, condition (5.5) does not imply condition (1.3). For example, the functions

$$w_1(r) = \chi_{(1,\infty)}(r) r^{-\beta}, \quad w_2(t) = \frac{1}{t^\beta+1}, \quad 0<\beta<\frac{|d|}{p_1}-\alpha$$

satisfy condition (5.5) but do not satisfy condition (1.3).

THEOREM 5.5 (1) *Let* $1<p_1<p_2<\infty$, $\alpha = |d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)$, $0<\theta_1<\infty$ *and* $\theta_1 \le \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$, *then condition* (5.4) *is necessary and sufficient for boundedness of* $I_\alpha^d$ *from* $LM_{p_1\theta_1,w_1,d}$ *to* $LM_{p_2\theta_2,w_2,d}$.

(2) *Let* $1 \le p_1<p_2<\infty$, $\alpha = |d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)$, $0<\theta_1<\infty$ *and* $\theta_1 \le \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$, *then condition* (5.4) *is necessary and sufficient for boundedness of* $I_\alpha^d$ *from* $LM_{p_1\theta_1,w_1,d}$ *to* $WLM_{p_2\theta_2,w_2,d}$.

*Proof* If $0<\theta_1\le 1$, $0<\theta_1\le\theta_2\le\infty$, then the statement of the theorem is proved in [24,25]. Let $1<\theta_1\le\theta_2<\infty$. Since

$$\begin{aligned}
B_2^1 &= \sup_{t>0}\left(\int_0^t w_2^{\theta_2}(r) r^{\theta_2\frac{|d|}{p_2}} \mathrm{d}r\right)^{\frac{1}{\theta_2}} \left(\int_t^\infty \frac{w_1^{\theta_1}(r) r^{\theta_1'\left(\alpha-\frac{|d|}{p_1}\right)}}{\left(\int_r^\infty w_1^{\theta_1}(\rho)\mathrm{d}\rho\right)^{\theta_1'}} \mathrm{d}r\right)^{\frac{1}{\theta_1'}} \\
&\le \sup_{t>0} t^{\alpha-\frac{|d|}{p_1}} \left(\int_0^t w_2^{\theta_2}(r) r^{\theta_2\frac{|d|}{p_2}} \mathrm{d}r\right)^{\frac{1}{\theta_2}} \left(\int_t^\infty \frac{w_1^{\theta_1}(r)}{\left(\int_r^\infty w_1^{\theta_1}(\rho)\mathrm{d}\rho\right)^{\theta_1'}} \mathrm{d}r\right)^{\frac{1}{\theta_1'}} \\
&\approx \sup_{t>0} t^{\alpha-\frac{|d|}{p_1}} \left(\int_0^t w_2^{\theta_2}(r) r^{\theta_2\frac{|d|}{p_2}} \mathrm{d}r\right)^{\frac{1}{\theta_2}} \left(-\int_t^\infty \mathrm{d}\left(\int_r^\infty w_1^{\theta_1}(\rho)\mathrm{d}\rho\right)^{1-\theta_1'}\right)^{\frac{1}{\theta_1'}} \\
&= \sup_{t>0} t^{\alpha-\frac{|d|}{p_1}} \left(\int_0^t w_2^{\theta_2}(r) r^{\theta_2\frac{|d|}{p_2}} \mathrm{d}r\right)^{\frac{1}{\theta_2}} \left(\int_t^\infty w_1^{\theta_1}(r)\mathrm{d}r\right)^{-\frac{1}{\theta_1}} = B_2^2,
\end{aligned}$$

sufficiency of (5.1) and (5.3) follows by Theorem 5.1, part (a). Hence condition (5.4) is sufficient (by Remark 5.2) and necessary (by Theorem 1.3, part 1) for boundedness of $I_\alpha^d$ from $LM_{p_1\theta_1,w_1,d}$ to $LM_{p_2\theta_2,w_2,d}$. The case $1<\theta_1<\infty$, $\theta_2=\infty$ is similar, because in this case $B^8 \le B_2^2$ by the same argument as above.

The proof of sufficiency for the second statement is similar. As for necessity one should note that boundedness of $I_\alpha^d$ from $LM_{p_1\theta_1,w_1,d}$ to $WLM_{p_2\theta_2,w_2,d}$ implies boundedness of the fractional maximal operator $M_\alpha^d$ from $LM_{p_1\theta_1,w_1,d}$ to $WLM_{p_2\theta_2,w_2,d}$ and that condition (5.4) is necessary for boundedness of $M_\alpha^d$ from $LM_{p_1\theta_1,w_1,d}$ to $WLM_{p_2\theta_2,w_2,d}$ [16]. ■

Corollary 5.6 *Let* $1<p_1\le p_2<\infty$, $\alpha=|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)$, $0<\theta_1<\infty$ *and* $\theta_1\le\theta_2\le\infty$, $w_2\in\Omega_{\theta_2}$ *and*

$$\left\| w_2(r)\left(\frac{r}{t+r}\right)^{\frac{|d|}{p_2}}\right\|_{L_{\theta_2}(0,\infty)} < \infty$$

*for all* $t>0$. *Moreover, if* $\theta_2=\infty$ *and* $\theta_1<\infty$ *it is also assumed that*

$$\lim_{t\to\infty}\left\| w_2(r)\left(\frac{r}{t+r}\right)^{\frac{|d|}{p_2}}\right\|_{L_\infty(0,\infty)} = 0.$$

*Then*

(1) $I_\alpha^d$ *is bounded from* $LM_{p_1\theta_1,w_1^*,d}$ *to* $LM_{p_2\theta_2,w_2,d}$, *where* $w_1^*$ *is a non-increasing continuous function on* $(0,\infty)$ *defined by*

$$\|w_1^*\|_{L_{\theta_1}(t,\infty)} = \left\| w_2(r)\left(\frac{r}{t+r}\right)^{\frac{|d|}{p_2}}\right\|_{L_{\theta_2}(0,\infty)}, \quad t\in(0,\infty).$$

(2) *If* $w_1\in\Omega_{\theta_1}$ *and* $I_\alpha^d$ *is bounded from* $LM_{p_1\theta_1,w_1,d}$ *to* $LM_{p_2\theta_2,w_2,d}$, *then*

$$LM_{p_1\theta_1,w_1,d} \subset LM_{p_1\theta_1,w_1^*,d}.$$

(*Hence* $LM_{p_1\theta_1,w_1^*,d}$ *is the maximal among spaces* $LM_{p_1\theta_1,w_1,d}$ *for which* $I_\alpha^d$ *is bounded from* $LM_{p_1\theta_1,w_1,d}$ *to* $LM_{p_2\theta_2,w_2,d}$.)

*Proof* Since condition (5.4) is also necessary and sufficient for boundedness of the fractional maximal operator $M_\alpha^d$ [19], the proof of Corollary 5.6 is also the same as for the case of $M_\alpha^d$. ■

An analogue of Corollary 5.6 also holds for the case in which $LM_{p_2\theta_2,w_2,d}$ is replaced by $WLM_{p_2\theta_2,w_2,d}$.

Corollary 5.7 *Let* $1<p_1\le p_2<\infty$, $\alpha=|d|\left(\frac{1}{p_1}-\frac{1}{p_2}\right)$, $w_1\in\Omega_{p_1}$ *and* $w_2\in\Omega_{p_2}$, *then condition* (5.4) *is necessary and sufficient for boundedness of* $I_\alpha^d$ *from* $L_{p_1,W_1}$ *to* $L_{p_2,W_2}$, *where* $W_1(x)=\|w_1\|_{L_{p_1}(\rho(x),\infty)}$, $W_2(x)=\|w_2\|_{L_{p_2}(\rho(x),\infty)}$.

*Proof* It suffices to take into account that for $0<p\le\infty$

$$\|f\|_{LM_{pp,w,d}} = \|f\|_{L_{p,W}},$$

where for all $x\in\mathbb{R}^n$ $W(x)=\|w\|_{L_p(\rho(x),\infty)}$ [21]. ■

It is interesting to note that condition (5.4) has the form that differs from the known, necessary and sufficient conditions discussed in detail, for example, in [30].

*Example 5.8* Let the condition (3.8) or condition (3.9) be satisfied. Moreover, let $1<\theta_1\le\theta_2<\infty$, and $\beta$ be such that

$$\beta+\frac{1}{\theta_2}<0, \quad \beta+\frac{|d|}{p_2}+\frac{1}{\theta_2}>0, \quad \beta+\frac{|d|}{p_2}+\frac{1}{\theta_2}+\alpha-\frac{|d|}{p_1}<0,$$

then it is easy to calculate that the functions $w_1(t)=t^{\beta+\frac{|d|}{p_2}+\frac{1}{\theta_2}+\alpha-\frac{|d|}{p_1}-\frac{1}{\theta_1}}$, $w_2(t)=t^{\beta}$ satisfy the condition (a) of Theorem 5.1. Thus $I^d_\alpha$ is bounded from $LM_{p\theta,w,d}$ to $LM_{p_1\theta_1,w_1,d}$.

## 6. Concluding remarks

The assumption made at the beginning of this article $d_i\ge 1$, $i=1,\dots,n$, is not essential. One may assume that $d_i>0$, $i=1,\dots,n$. However, under this assumption the function $\rho(x-y)$, $x,y\in\mathbb{R}^n$ is in general a quasi-distance, which does note cause any problem.

Also note that if $\nu>0$ then

$$I^{\nu d}_{\nu\alpha}=I^d_\alpha \quad \forall\nu>0.$$

$$\|f\|_{L_p(\mathcal{E}_d(0,r))}=\|f\|_{L_p(\mathcal{E}_{\nu d}(0,r^{1/\nu}))}.$$

Lemma 6.1 *Let* $1<p_1<p_2<\infty$, $0<\theta_1,\theta_2\le\infty$, $w_1\in\Omega_{\theta_1}$ *and* $w_2\in\Omega_{\theta_2}$. *Then for* $\nu>0$

$$\|I^d_\alpha f\|_{LM_{p_1\theta_1,w_1,d}\to LM_{p_2\theta_2,w_2,d}}=\|I^{\nu d}_{\nu\alpha}f\|_{LM_{p_1\theta_1,w_1(\rho^\nu)\rho^{\frac{\nu-1}{\theta_1}},\nu d}\to LM_{p_2\theta_2,w_2(\rho^\nu)\rho^{\frac{\nu-1}{\theta_2}},\nu d}}.$$

*Proof*

$$\begin{aligned}\|I^d_\alpha f\|_{LM_{p_1\theta_1,w_1,d}\to LM_{p_2\theta_2,w_2,d}}&=\sup_{f\not\sim 0, f\in LM_{p_1\theta_1,w_1,d}}\frac{\|I^d_\alpha f\|_{LM_{p_2\theta_2,w_2,d}}}{\|f\|_{LM_{p_1\theta_1,w_1,d}}}\\&=\sup_{f\not\sim 0, f\in LM_{p_1\theta_1,w_1,d}}\frac{\|w_2(r)\|I^d_\alpha f\|_{L_p(\mathcal{E}_d(0,r))}\|_{L_{\theta_2}(0,\infty)}}{\|w_1(r)\|f\|_{L_p(\mathcal{E}_d(0,r))}\|_{L_{\theta_1}(0,\infty)}}\\&=\sup_{f\not\sim 0, f\in LM_{p_1\theta_1,w_1,d}}\frac{\|w_2(r)\|I^{\nu d}_{\nu\alpha} f\|_{L_p(\mathcal{E}_{\nu d}(0,r^{1/\nu}))}\|_{L_{\theta_2}(0,\infty)}}{\|w_1(r)\|f\|_{L_p(\mathcal{E}_{\nu d}(0,r^{1/\nu}))}\|_{L_{\theta_1}(0,\infty)}}\\&=\nu^{1/\theta_2-1/\theta_1}\sup_{f\not\sim 0, f\in LM_{p_1\theta_1,w_1,d}}\frac{\|w_2(\rho^\nu)\rho^{\frac{\nu-1}{\theta_2}}\|I^{\nu d}_{\nu\alpha} f\|_{L_p(\mathcal{E}_{\nu d}(0,\rho))}\|_{L_{\theta_2}(0,\infty)}}{\|w_1(\rho^\nu)\rho^{\frac{\nu-1}{\theta_1}}\|f\|_{L_p(\mathcal{E}_{\nu d}(0,\rho))}\|_{L_{\theta_1}(0,\infty)}}\\&=\|I^{\nu d}_{\nu\alpha}f\|_{LM_{p_1\theta_1,w_1(\rho^\nu)\rho^{\frac{\nu-1}{\theta_1}},\nu d}\to LM_{p_2\theta_2,w_2(\rho^\nu)\rho^{\frac{\nu-1}{\theta_2}},\nu d}}.\end{aligned}$$

■

## Acknowledgements

The research of Akbulut and Guliyev were partially supported by the grant of Ahi Evran University Science Research Project (Kirsehir, Turkey), FBA-10-05 and by the Scientific and

Technological Research Council of Turkey (TUBITAK Project No: 110T695). The research of Guliyev was partially supported by the grant of Science Development Foundation under the President of the Republic of Azerbaijan project EIF-2010-1(1)-40/06-1.

## Notes

1. We apply the following simple statement. If $\Omega$ is a measurable set in $\mathbb{R}^n$, $M>0$ and for almost all $y\in\Omega$ $g(y)\geq M$, then for any $0<p\leq\infty$ $\|g\|_{WL_p(\Omega)}\geq M\,|\Omega|^{\frac{1}{p}}$.
2. See endnote 1.

## References


[1] O.V. Besov, V.P. Il'in, and P.I. Lizorkin, *The $L_p$-estimates of a certain class of non-isotropically singular integrals*, Dokl. Akad. Nauk SSSR 169 (1966), pp. 1250–1253 (in Russian).

[2] E.B. Fabes and N. Rivère, *Singular integrals with mixed homogeneity*, Stud. Math. 27 (1966), pp. 19–38.

[3] M. Bramanti and M.C. Cerutti, *Commutators of singular integrals on homogeneous spaces*, Boll. Un. Mat. Ital. B 10(7) (1996), pp. 843–883.

[4] E.M. Stein, *Singular Integrals and Differentiability of Functions*, Princeton University Press, Princeton, NJ, 1970.

[5] E.M. Stein, *Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, NJ, 1993.

[6] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Am. Math. Soc. 43 (1938), pp. 126–166.

[7] S. Spanne, *Sur l'interpolation entre les espaces $\mathcal{L}_k^{p,\Phi}$*, Ann. Schola Norm. Sup. Pisa 20 (1966), pp. 625–648.

[8] D.R. Adams, *A note on Riesz potentials*, Duke Math. 42 (1975), pp. 765–778.

[9] F. Chiarenza and M. Frasca, *Morrey spaces and Hardy–Littlewood maximal function*, Rend. Math. 7 (1987), pp. 273–279.

[10] V.S. Guliyev, *Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{R}^n$*, DSc dissertation, Mat. Inst. Steklova, Moscow, 1994, pp. 1–329 (in Russian).

[11] V.S. Guliyev, *Function spaces, integral operators and two weighted inequalities on homogeneous groups*, Some Applications, Elm, Baku, 1999, pp. 1–332 (in Russian).

[12] V.S. Guliyev, *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl. (2009), p. 20, Art. ID 503948.

[13] D. Fan, S. Lu, and D. Yang, *Boundedness operators in Morrey spaces on homogeneous spaces and its applications*, Acta Math. Sin. (Engl. Ser.) 14 (1998), pp. 625–634.

[14] V.S. Guliyev and R.Ch. Mustafayev, *Integral operators of potential type in spaces of homogeneous type*, Dokl. Ross. Akad. Nauk 354(6) (1997), pp. 730–732 (in Russian).

[15] V.S. Guliyev and R.Ch. Mustafayev, *Fractional integrals in spaces of functions defined on spaces of homogeneous type*, Anal. Math. 24(3) (1998), pp. 181–200 (in Russian).

[16] A. Akbulut, I. Ekincioglu, A. Serbetci, and T.V. Tararykova, *Boundedness of the anisotropic fractional maximal operator in anisotropic local Morrey-type spaces,* (submitted).

[17] E.V. Guliyev, Sh.A. Muradova, and A. Eroglu, *Necessary and sufficient conditions for boundedness of the anisotropic maximal operator in the local anisotropic Morrey-type spaces*, Proc. A. Razmadze Math. Inst. 153 (2010), pp. 1–24.

[18] V. Burenkov, A. Gogatishvili, V.S. Guliyev, and R. Mustafayev, *Boundedness of the fractional maximal operator in local Morrey-type spaces*, Complex Var. Elliptic Eqns. 55(8–10) (2010), pp. 739–758.

[19] V. Burenkov, A. Gogatishvili, V.S. Guliyev, and R. Mustafayev, *Boundedness of the fractional maximal operator in local Morrey-type spaces*, Potential Anal. 35 (2011), pp. 1–21.

[20] V.I. Burenkov and H.V. Guliyev, *Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces*, Dokl. Math. 68(1) (2003), pp. 107–110; Transl. Dokl. Ross. Akad. Nauk. Matematika, 391(5) (2003), pp. 591–594.

[21] V.I. Burenkov and H.V. Guliyev, *Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces*, Stud. Math. 163(2) (2004), pp. 157–176.

[22] V.I. Burenkov, H.V. Guliyev, and V.S. Guliyev, *Necessary and sufficient conditions for the boundedness of the fractional maximal operator in the local Morrey-type spaces*, Dokl. Math. 74(1) (2006), pp. 540–544; Trans. Dokl. Ross. Akad. Nauk. Mat. 409(4) (2006), pp. 443–447.

[23] V.I. Burenkov, H.V. Guliyev, and V.S. Guliyev, *Necessary and sufficient conditions for boundedness of the fractional maximal operators in the local Morrey-type spaces*, J. Comput. Appl. Math. 208(1) (2007), pp. 280–301.

[24] V.I. Burenkov, H.V. Guliyev, and V.S. Guliyev, *Necessary and sufficient conditions for the boundedness of the Riesz operator in the local Morrey-type spaces*, Dokl. Math. 75(1) (2007), pp. 103–107; Trans. Dokl. Ross. Akad. Nauk. Mat. 412(5) (2007), pp. 585–589.

[25] V.I. Burenkov and V.S. Guliyev, *Necessary and sufficient conditions for the boundedness of the Riesz potential in local Morrey-type spaces*, Potential Anal. 30(3) (2009), pp. 211–249.

[26] V.I. Burenkov, V.S. Guliyev, A. Serbetci, and T.V. Tararykova, *Necessary and sufficient conditions for the boundedness of genuine singular integral operators in local Morrey-type spaces*, Eurasian Math. J. 1(1) (2010), pp. 32–53.

[27] V.I. Burenkov, V.S. Guliyev, T.V. Tararykova, and A. Serbetci, *Necessary and sufficient conditions for the boundedness of genuine singular integral operators in the local Morrey-type spaces*, Dokl. Math. 78(2) (2008), pp. 651–654; Trans. Dokl. Ross. Akad. Nauk. Mat. 422(1) (2008), pp. 11–14.

[28] M. Carro, L. Pick, J. Soria, and V.D. Stepanov, *On embeddings between classical Lorentz spaces*, Math. Ineq. Appl. 4(3) (2001), pp. 397–428.

[29] M. Carro, A. Gogatishvili, J. Martin, and L. Pick, *Weighted inequalities involving two Hardy operators with applications to embeddings of function spaces*, J. Oper. Theory 59(2) (2008), pp. 309–332.

[30] I. Genebashvili, A. Gogatishvili, V. Kokilashvili, and M. Krbec, *Weight Theory for Integral Transforms on Spaces of Homogeneous Type*, *Pitman Monogr aphs and Surveys in Pure Applied Mathematics*, Vol. 92, Longman, Harlow, 1998.