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Boundedness of the parametric Marcinkiewicz integral operator and its commutators on generalized Morrey spaces

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Abstract. In this paper we study the boundedness of the parametric Marcinkiewicz operator μ_Ω^ρ on generalized Morrey spaces $M_{p,\varphi}$. We find the sufficient conditions on the pair (φ_1, φ_2) which ensure the boundedness of the operators μ_Ω^ρ from one generalized Morrey space M_{p,φ_1} to another M_{p,φ_2}, $1 < p < \infty$, and from the space M_{1,φ_1} to the weak space WM_{1,φ_2}. As an application of the above result, the boundedness of the commutator of Marcinkiewicz operators $[a, \mu_\Omega^\rho]$ on generalized Morrey spaces is also obtained. In the case $a \in \mathrm{BMO}(\mathbb{R}^n)$, we find the sufficient conditions on the pair (φ_1, φ_2) which ensure the boundedness of the operators $[a, \mu_\Omega^\rho]$ from one generalized Morrey space M_{p,φ_1} to another M_{p,φ_2}, $1 < p < \infty$, and from the space M_{1,φ_1} to the weak space WM_{1,φ_2}. In all the cases the conditions for boundedness are given in terms of Zygmund-type integral inequalities on (φ_1, φ_2) which do not require any assumption on the monotonicity of φ_1, φ_2.

Keywords. Parametric Marcinkiewicz operator, generalized Morrey space, commutator, BMO.

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1 Introduction

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at x of radius r, ${}^{\complement}B(x,r)$ denote its complement and $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$. Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere of $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$.

In [20], Stein defined the Marcinkiewicz integral for higher dimensions. Let Ω satisfy the following conditions.

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(i) Ω is a homogeneous function of degree zero on $\mathbb{R}^n$. That is,

$$\Omega(tx) = \Omega(x) \tag{1.1}$$

for all $t > 0$ and $x \in \mathbb{R}^n$.

(ii) Ω has mean zero on S^{n-1}. That is,

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1.2}$$

where $x' = x/|x|$ for any $x \neq 0$.

(iii) $\Omega \in L_1(S^{n-1})$.

The Marcinkiewicz integral operator μ_Ω of higher dimension is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3}\right)^{\frac{1}{2}},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

Remark 1.1. We easily see that the Marcinkiewicz integral operator of higher dimension μ_Ω can be regarded as a generalized version of the classical Marcinkiewicz integral in the one-dimensional case. Also, it is easy to see that μ_Ω^ρ is a special case of the Littlewood–Paley g-function if we take

$$g(x) = \Omega(x')|x|^{-n+1}\chi_{|x|\leq 1}(|x|).$$

We say that $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, $0 < \alpha \leq 1$, if there exists a constant $C > 0$ such that $|\Omega(x') - \Omega(y')| \leq C|x' - y'|^\alpha$ for all $x', y' \in S^{n-1}$.

In [20], Stein proved the following results.

Theorem 1.2 (E. M. Stein). (a) *If Ω satisfies* (1.1), $\Omega \in L_1(S^{n-1})$ *and Ω is odd, then μ_Ω is bounded on $L_p(\mathbb{R}^n)$ for* $1 < p < \infty$.

(b) *If Ω satisfies* (1.1), (1.2) *and* $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, $0 < \alpha \leq 1$, *then μ_Ω is of weak type* $(1,1)$. *That is, there exists a constant C such that for any $t > 0$ and $f \in L_1(\mathbb{R}^n)$,*

$$\left|\left\{x \in \mathbb{R}^n : \mu_\Omega(f)(x) > t\right\}\right| \leq \frac{C}{t}\int_{\mathbb{R}^n} |f(x)|dx.$$

(c) *If* Ω *satisfies* (1.1), (1.2) *and* $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, $0 < \alpha \le 1$, *then* μ_Ω *is of type* (p,p) *for* $1 < p \le 2$. *That is, there exists a constant* A_p *such that for any* $f \in L_p(\mathbb{R}^n)$,

$$\|\mu_\Omega(f)\|_{L_p} \le A_p \|f\|_{L_p}.$$

The L_p boundedness of μ_Ω has been studied extensively. See [2, 15, 20, 21] among others. A survey of the past studies can be found in [12]. Recently, the following result has been obtained in [1]:

Theorem 1.3. *If* $\Omega \in L(\log L)^{1/2}(S^{n-1})$ *and satisfies* (1.2), *then* μ_Ω *is bounded on* $L_p(\mathbb{R}^n)$ *for* $1 < p < \infty$. *The exponent* $1/2$ *is the best possible one.*

Let us turn to the parameter case. In 1960, Hörmander [15] considered the L_p boundedness for a class of parametric Marcinkiewicz integral $\mu_\Omega f(x)$, which is defined by

$$\mu_\Omega^\rho(f)(x) = \left(\int_0^\infty \left| \frac{1}{t^\rho} \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y)dy \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}},$$

where $0 < \rho < n$. It is easy to see that when $\rho = 1$, μ_Ω^ρ is just μ_Ω introduced by Stein in [20].

Theorem 1.4 (Hörmander). *If* $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, $0 < \alpha \le 1$, *satisfies conditions* (1.1), (1.2), *then* μ_Ω^ρ *is bounded on* $L_p(\mathbb{R}^n)$ *for* $1 < p < \infty$. *Moreover, there exists a constant* A_p *such that for any* $f \in L_p(\mathbb{R}^n)$ *and* $0 < \rho < n$,

$$\|\mu_\Omega^\rho(f)\|_{L_p} \le A_p \|f\|_{L_p}.$$

In the present work, we shall prove the boundedness of the Marcinkiewicz operator μ_Ω^ρ from one generalized Morrey space M_{p,φ_1} to another M_{p,φ_2}, $1 < p < \infty$, and from the space M_{1,φ_1} to the weak space WM_{1,φ_2}. In the case $a \in \mathrm{BMO}(\mathbb{R}^n)$, we find the sufficient conditions on the pair (φ_1, φ_2) which ensure the boundedness of the operators $[a, \mu_\Omega^\rho]$ from one generalized Morrey space M_{p,φ_1} to another M_{p,φ_2}, $1 < p < \infty$ and from the space M_{1,φ_1} to the weak space WM_{1,φ_2}.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Generalized Morrey spaces

The classical Morrey spaces $M_{p,\lambda}$ were originally introduced by Morrey in [17] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [14, 16].

We denote by $M_{p,\lambda} \equiv M_{p,\lambda}(\mathbb{R}^n)$ the Morrey space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

where $1 \le p < \infty$ and $0 \le \lambda \le n$.

Note that $M_{p,0} = L_p(\mathbb{R}^n)$ and $M_{p,n} = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $M_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

We also denote by $WM_{p,\lambda} \equiv WM_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which (see, for example, [18, 19])

$$\|f\|_{WM_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(B(x,r))$ denotes the weak L_p-space of measurable functions f for which

$$\begin{aligned} \|f\|_{WL_p(B(x,r))} &\equiv \|f\chi_{B(x,r)}\|_{WL_p(\mathbb{R}^n)} \\ &= \sup_{t>0} t \left|\left\{y \in B(x,r) : |f(y)| > t\right\}\right|^{\frac{1}{p}} \\ &= \sup_{t>0} t^{\frac{1}{p}} (f\chi_{B(x,r)})^*(t) < \infty. \end{aligned}$$

Here g^* denotes the non-increasing rearrangement of the function g.

Definition 2.1. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))}.$$

Also, we denote by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ the weak generalized Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r))} < \infty.$$

According to this definition, we recover the spaces $M_{p,\lambda}$ and $WM_{p,\lambda}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$M_{p,\lambda} = M_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}, \qquad WM_{p,\lambda} = WM_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

3 Marcinkiewicz operator in the spaces $M_{p,\varphi}$

In this section we are going to use the following statement on the boundedness of the Hardy operator

$$(Hg)(t) := \frac{1}{t}\int_0^t g(r)dr, \quad 0 < t < \infty.$$

Theorem 3.1 ([5]). *The inequality*

$$\operatorname*{ess\,sup}_{t>0} w(t)Hg(t) \le c \operatorname*{ess\,sup}_{t>0} v(t)g(t)$$

holds for all non-negative and non-increasing g on $(0,\infty)$ if and only if

$$A := \sup_{t>0} \frac{w(t)}{t}\int_0^t \frac{dr}{\operatorname{ess\,sup}_{0<s<r} v(s)} < \infty,$$

and $c \approx A$.

Lemma 3.2. *Let $1 \le p < \infty$ and let $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, $0 < \alpha \le 1$, satisfy conditions (1.1), (1.2). Then, for $1 < p < \infty$ the inequality*

$$\|\mu_\Omega^\rho(f)\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^\infty t^{-\frac{n}{p}-1}\|f\|_{L_p(B(x_0,t))}dt$$

holds for any ball $B(x_0,r)$, $0<\rho<n$, and for all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.

Moreover, for $p=1$ the inequality

$$\|\mu_\Omega^\rho(f)\|_{WL_1(B(x_0,r))} \lesssim r^n \int_{2r}^\infty t^{-n-1}\|f\|_{L_1(B(x_0,t))}dt \tag{3.1}$$

holds for any ball $B(x_0,r)$, $0<\rho<n$, and for all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Let $p \in (1,\infty)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at x_0 and of radius r. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{\complement(2B)}(y), \quad r>0, \tag{3.2}$$

and have

$$\|\mu_\Omega^\rho(f)\|_{L_p(B)} \le \|\mu_\Omega^\rho(f_1)\|_{L_p(B)} + \|\mu_\Omega^\rho(f_2)\|_{L_p(B)}.$$

Since $f_1 \in L_p(\mathbb{R}^n)$, $\mu_\Omega^\rho(f_1) \in L_p(\mathbb{R}^n)$ and from the boundedness of T in $L_p(\mathbb{R}^n)$ we have

$$\|\mu_\Omega^\rho(f_1)\|_{L_p(B)} \le \|\mu_\Omega^\rho(f_1)\|_{L_p(\mathbb{R}^n)} \lesssim \|f_1\|_{L_p(\mathbb{R}^n)} = \|f\|_{L_p(2B)}.$$

It is clear that $x \in B$, $y \in {}^{\complement}(2B)$ implies $\frac{1}{2}|x_0 - y| \le |x - y| \le \frac{3}{2}|x_0 - y|$. Then by the Minkowski inequality and conditions on Ω, we get

$$\begin{aligned} \mu_\Omega^\rho(f_2)(x) &\le \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\rho}} |f_2(y)| \left(\int_{|x-y|}^\infty \frac{dt}{t^{1+2\rho}} \right)^{\frac{1}{2}} dy \\ &\lesssim \int_{{}^{\complement}(2B)} \frac{|f(y)|}{|x-y|^n} dy \\ &\lesssim \int_{{}^{\complement}(2B)} \frac{|f(y)|}{|x_0-y|^n} dy. \end{aligned} \tag{3.3}$$

By Fubini's theorem we have

$$\begin{aligned} \int_{{}^{\complement}(2B)} \frac{|f(y)|}{|x_0-y|^n} dy &\approx \int_{{}^{\complement}(2B)} |f(y)| \int_{|x_0-y|}^\infty \frac{dt}{t^{n+1}} dy \\ &= \int_{2r}^\infty \int_{2r \le |x_0-y| < t} |f(y)| dy \frac{dt}{t^{n+1}} \\ &\lesssim \int_{2r}^\infty \int_{B(x_0,t)} |f(y)| dy \frac{dt}{t^{n+1}}. \end{aligned}$$

Applying Hölder's inequality, we get

$$\int_{{}^{\complement}(2B)} \frac{|f(y)|}{|x_0-y|^n} dy \lesssim \int_{2r}^\infty \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}. \tag{3.4}$$

Moreover, for all $p \in [1, \infty)$ the inequality

$$\|\mu_\Omega^\rho(f_2)\|_{L_p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^\infty \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}. \tag{3.5}$$

is valid. Thus

$$\|\mu_\Omega^\rho(f)\|_{L_p(B)} \lesssim \|f\|_{L_p(2B)} + r^{\frac{n}{p}} \int_{2r}^\infty \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.$$

On the other hand,

$$\begin{aligned} \|f\|_{L_p(2B)} &\approx r^{\frac{n}{p}} \|f\|_{L_p(2B)} \int_{2r}^\infty \frac{dt}{t^{\frac{n}{p}+1}} \\ &\lesssim r^{\frac{n}{p}} \int_{2r}^\infty \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}. \end{aligned} \tag{3.6}$$

Thus

$$\|\mu_\Omega^\rho(f)\|_{L_p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^\infty \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.$$

Let $p = 1$. From the weak $(1,1)$ boundedness of μ_Ω^ρ it follows that

$$\begin{aligned}\|\mu_\Omega^\rho(f_1)\|_{WL_1(B)} &\le \|\mu_\Omega^\rho(f_1)\|_{WL_1(\mathbb{R}^n)} \lesssim \|f_1\|_{L_1(\mathbb{R}^n)} \\ &= \|f\|_{L_1(2B)} \lesssim r^n \int_{2r}^\infty \int_{B(x_0,t)} |f(y)|dy \frac{dt}{t^{n+1}}. \qquad (3.7)\end{aligned}$$

Then by (3.5) and (3.7) we get the inequality (3.1). □

Theorem 3.3. *Let* $0 < \rho < n$*,* $1 \le p < \infty$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\varphi_2(x,r), \qquad (3.8)$$

where C *does not depend on* x *and* r*. Let* $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$*,* $0 < \alpha \le 1$*, satisfy conditions* (1.1), (1.2)*. Then the operator* μ_Ω^ρ *is bounded from* M_{p,φ_1} *to* M_{p,φ_2} *for* $p > 1$ *and from* M_{1,φ_1} *to* WM_{1,φ_2}*. Also for* $p > 1$

$$\|\mu_\Omega^\rho(f)\|_{M_{p,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}},$$

and for $p = 1$

$$\|\mu_\Omega^\rho(f)\|_{WM_{1,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}}.$$

Proof. By Lemma 3.2 and Theorem 3.1 we have for $p > 1$

$$\begin{aligned}\|\mu_\Omega^\rho(f)\|_{M_{p,\varphi_2}} &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \|f\|_{L_p(B(x,t))} \frac{dt}{t^{\frac{n}{p}+1}} \\ &\approx \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_0^{r^{-\frac{n}{p}}} \|f\|_{L_p(B(x,t^{-\frac{p}{n}}))} dt \\ &= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r^{-\frac{p}{n}})^{-1} \int_0^r \|f\|_{L_p(B(x,t^{-\frac{p}{n}}))} dt \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r^{-\frac{p}{n}})^{-1} r \|f\|_{L_p(B(x,r^{-\frac{p}{n}}))} = \|f\|_{M_{p,\varphi_1}}\end{aligned}$$

and for $p = 1$

$$\begin{aligned}\|\mu_\Omega^\rho(f)\|_{WM_{1,\varphi_2}} &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \|f\|_{L_1(B(x,t))} \frac{dt}{t^{n+1}} \\ &\approx \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_0^{r^{-n}} \|f\|_{L_1(B(x,t^{-n}))} dt\end{aligned}$$

$$= \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x, r^{-\frac{1}{n}})^{-1} \int_0^r \|f\|_{L_1(B(x,t^{-\frac{1}{n}}))} dt$$
$$\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x, r^{-\frac{1}{n}})^{-1} r \|f\|_{L_1(B(x,r^{-\frac{1}{n}}))}$$
$$= \|f\|_{M_{1,\varphi_1}}. \qquad \square$$

4 Commutators of the parametric Marcinkiewicz operator in the spaces $M_{p,\varphi}$

It is well known that the commutator is an important integral operator and plays a key role in harmonic analysis. In 1965, Calderón [3, 4] studied a kind of commutators appearing in Cauchy integral problems of Lip-line. Let K be a Calderón–Zygmund singular integral operator and $a \in \mathrm{BMO}(\mathbb{R}^n)$. A well-known result of Coifman, Rochberg and Weiss [10] states that the commutator operator $[a, K]f = K(af) - aKf$ is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. The commutator of Calderón–Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [6–9, 11, 13]).

First we introduce the definition of the space of $\mathrm{BMO}(\mathbb{R}^n)$.

Definition 4.1. Suppose that $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, let

$$\|f\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy < \infty,$$

where

$$f_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) dy.$$

Define

$$\mathrm{BMO}(\mathbb{R}^n) = \left\{ f \in L_1^{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_* < \infty \right\}.$$

Modulo constants, the space $\mathrm{BMO}(\mathbb{R}^n)$ is a Banach space with respect to the norm $\|\cdot\|_*$.

Remark 4.2. (1) John–Nirenberg inequality: There are constants C_1, $C_2 > 0$, such that for all $f \in \mathrm{BMO}(\mathbb{R}^n)$ and $\beta > 0$

$$\left| \left\{ x \in B : |f(x) - f_B| > \beta \right\} \right| \le C_1 |B| e^{-C_2 \beta / \|f\|_*}, \quad \forall B \subset \mathbb{R}^n.$$

(2) The John–Nirenberg inequality implies that

$$\|f\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}|^p dy \right)^{\frac{1}{p}} \tag{4.1}$$

for $1 < p < \infty$.

(3) Let $f \in \mathrm{BMO}(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$|f_{B(x,r)} - f_{B(x,t)}| \le C\|f\|_* \ln \frac{t}{r} \quad \text{for } 0 < 2r < t, \tag{4.2}$$

where C is independent of f, x, r and t.

For $b \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, the commutator $[b, \mu_\Omega^\rho]$ formed by b and the parametric Marcinkiewicz integral μ_Ω^ρ, $0 < \rho < n$, is defined by

$$[b, \mu_\Omega^\rho] f(x) = \left(\int_0^\infty \left| \frac{1}{t^\rho} \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} (b(x) - b(y)) f(y) dy \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

Lemma 4.3. *Let* $1 \le p < \infty$, $a \in \mathrm{BMO}(\mathbb{R}^n)$, *and let* $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, $0 < \alpha \le 1$, *satisfy conditions* (1.1), (1.2). *Then, for* $1 < p < \infty$ *the inequality*

$$\big\| [a, \mu_\Omega^\rho] f \big\|_{L_p(B(x_0,r))} \lesssim \|a\|_* r^{\frac{n}{p}} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right) t^{-\frac{n}{p}-1} \|f\|_{L_p(B(x_0,t))} dt$$

holds for any ball $B(x_0, r)$, $0 < \rho < n$, *and for all* $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.

Moreover, for $p = 1$ *the inequality*

$$\big\| [a, \mu_\Omega^\rho] f \big\|_{WL_1(B(x_0,r))} \lesssim \|a\|_* r^n \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right) t^{-n-1} \|f\|_{L_1(B(x_0,t))} dt \tag{4.3}$$

holds for any ball $B(x_0, r)$, $0 < \rho < n$, *and for all* $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement(2B)}$. Hence

$$\big\| [a, \mu_\Omega^\rho] f \big\|_{L_p(B)} \le \big\| [a, \mu_\Omega^\rho] f_1 \big\|_{L_p(B)} + \big\| [a, \mu_\Omega^\rho] f_2 \big\|_{L_p(B)}.$$

From the boundedness of $[a,\mu_\Omega^\rho]$ in $L_p(\mathbb{R}^n)$ it follows that

$$\begin{aligned}\big\|[a,\mu_\Omega^\rho]f_1\big\|_{L_p(B)} &\le \big\|[a,\mu_\Omega^\rho]f_1\big\|_{L_p(\mathbb{R}^n)}\\ &\lesssim \|a\|_*\|f_1\|_{L_p(\mathbb{R}^n)} = \|a\|_*\|f\|_{L_p(2B)}.\end{aligned}$$

For $x\in B$ we have

$$\begin{aligned}\big|[a,\mu_\Omega^\rho]f_2(x)\big| &\lesssim \int_{\mathbb{R}^n}\frac{|a(y)-a(x)|}{|x-y|^n}|f(y)|dy\\ &\approx \int_{\complement(2B)}\frac{|a(y)-a(x)|}{|x_0-y|^n}|f(y)|dy.\end{aligned}$$

Then

$$\begin{aligned}\big\|[a,\mu_\Omega^\rho]f_2\big\|_{L_p(B)} &\lesssim \left(\int_B\left(\int_{\complement(2B)}\frac{|a(y)-a(x)|}{|x_0-y|^n}|f(y)|dy\right)^p dx\right)^{\frac1p}\\ &\lesssim \left(\int_B\left(\int_{\complement(2B)}\frac{|a(y)-a_B|}{|x_0-y|^n}|f(y)|dy\right)^p dx\right)^{\frac1p}\\ &\quad+\left(\int_B\left(\int_{\complement(2B)}\frac{|a(x)-a_B|}{|x_0-y|^n}|f(y)|dy\right)^p dx\right)^{\frac1p}\\ &= I_1+I_2.\end{aligned}$$

Let us estimate I_1.

$$\begin{aligned}I_1 &\approx r^{\frac np}\int_{\complement(2B)}\frac{|a(y)-a_B|}{|x_0-y|^n}|f(y)|dy\\ &\approx r^{\frac np}\int_{\complement(2B)}|a(y)-a_B|\,|f(y)|\int_{|x_0-y|}^\infty\frac{dt}{t^{n+1}}dy\\ &\approx r^{\frac np}\int_{2r}^\infty\int_{2r\le|x_0-y|\le t}|a(y)-a_B|\,|f(y)|dy\frac{dt}{t^{n+1}}\\ &\lesssim r^{\frac np}\int_{2r}^\infty\int_{B(x_0,t)}|a(y)-a_B|\,|f(y)|dy\frac{dt}{t^{n+1}}.\end{aligned}$$

Applying Hölder's inequality, by (4.1), (4.2) we get

$$\begin{aligned}I_1 &\lesssim r^{\frac np}\int_{2r}^\infty\int_{B(x_0,t)}|a(y)-a_{B(x_0,t)}|\,|f(y)|dy\frac{dt}{t^{n+1}}\\ &\quad+r^{\frac np}\int_{2r}^\infty|a_{B(x_0,r)}-a_{B(x_0,t)}|\int_{B(x_0,t)}|f(y)|dy\frac{dt}{t^{n+1}}\end{aligned}$$

$$\lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \left(\int_{B(x_0,t)} |a(y) - a_{B(x_0,t)}|^{p'} dy \right)^{\frac{1}{p'}} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{n+1}}$$
$$+ r^{\frac{n}{p}} \int_{2r}^{\infty} |a_{B(x_0,r)} - a_{B(x_0,t)}| \int_{B(x_0,t)} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}$$
$$\lesssim \|a\|_* r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.$$

In order to estimate I_2 note that

$$I_2 \approx \left(\int_B |a(x) - a_B|^p dx \right)^{\frac{1}{p}} \int_{\complement(2B)} \frac{|f(y)|}{|x_0 - y|^n} dy.$$

By (4.1), we get

$$I_2 \lesssim \|a\|_* r^{\frac{n}{p}} \int_{\complement(2B)} \frac{|f(y)|}{|x_0 - y|^n} dy.$$

Thus, by (3.4)

$$I_2 \lesssim \|a\|_* r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.$$

Summing I_1 and I_2, for all $p \in [1, \infty)$ we get

$$\left\| [a, \mu_\Omega^\rho] f_2 \right\|_{L_p(B)} \lesssim \|a\|_* r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}. \tag{4.4}$$

Finally,

$$\left\| [a, \mu_\Omega^\rho] f \right\|_{L_p(B)}$$
$$\lesssim \|a\|_* \|f\|_{L_p(2B)} + \|a\|_* r^{\frac{n}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}},$$

and the statement of Lemma 4.3 follows by (3.6).

Let $p = 1$. From the weak $(1,1)$ boundedness of $[a, \mu_\Omega^\rho]$ and (3.6) it follows that

$$\left\| [a, \mu_\Omega^\rho] f_1 \right\|_{WL_1(B)} \le \left\| [a, \mu_\Omega^\rho] f_1 \right\|_{WL_1(\mathbb{R}^n)}$$
$$\lesssim \|a\|_* \|f_1\|_{L_1(\mathbb{R}^n)} = \|a\|_* \|f\|_{L_1(2B)}$$
$$\lesssim \|a\|_* r^n \int_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \frac{dt}{t^{n+1}}. \tag{4.5}$$

Then from (4.4) and (4.5) we get the inequality (4.3). □

The following theorem is true.

Theorem 4.4. *Let* $1 \le p < \infty$, $0 < \rho < n$, $a \in \mathrm{BMO}(\mathbb{R}^n)$ *and let* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\varphi_2(x,r), \tag{4.6}$$

where C *does not depend on* x *and* r. *Let* $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, $0 < \alpha \le 1$, *satisfy conditions* (1.1), (1.2). *Then the operator* $[a, \mu_\Omega^\rho]$ *is bounded from* M_{p,φ_1} *to* M_{p,φ_2} *for* $p > 1$ *and bounded from* M_{1,φ_1} *to* WM_{1,φ_2}.

Moreover, for $p > 1$

$$\big\| [a, \mu_\Omega^\rho] f \big\|_{M_{p,\varphi_2}} \lesssim \|a\|_* \|f\|_{M_{p,\varphi_1}},$$

and for $p = 1$,

$$\big\| [a, \mu_\Omega^\rho] f \big\|_{WM_{1,\varphi_2}} \lesssim \|a\|_* \|f\|_{M_{1,\varphi_1}}.$$

Proof. The statement of Theorem 4.4 follows by Lemma 4.3 and Theorem 3.1 in the same manner as in the proof of Theorem 3.3. □

Corollary 4.5. *Let* $1 \le p < \infty$, (φ_1, φ_2) *satisfy condition* (4.6), $a \in \mathrm{BMO}(\mathbb{R}^n)$ *and let* $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, $0 < \alpha \le 1$, *satisfy conditions* (1.1), (1.2). *Then the operator* $[a, \mu_\Omega]$ *is bounded from* M_{p,φ_1} *to* M_{p,φ_2} *for* $p > 1$ *and from* M_{1,φ_1} *to* WM_{1,φ_2}.

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