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Riesz potential on the Heisenberg group and modified Morrey spaces

Vagif S. Guliyev and Yagub Y. Mammadov

Abstract

In this paper we study the fractional maximal operator M_α, $0 \le \alpha < Q$ and the Riesz potential operator $\mathcal{I}_\alpha$, $0 < \alpha < Q$ on the Heisenberg group in the modified Morrey spaces $\widetilde{L}_{p,\lambda}(\mathbb{H}_n)$, where $Q = 2n + 2$ is the homogeneous dimension on $\mathbb{H}_n$. We prove that the operators M_α and $\mathcal{I}_\alpha$ are bounded from the modified Morrey space $\widetilde{L}_{1,\lambda}(\mathbb{H}_n)$ to the weak modified Morrey space $W\widetilde{L}_{q,\lambda}(\mathbb{H}_n)$ if and only if, $\alpha/Q \le 1-1/q \le \alpha/(Q-\lambda)$ and from $\widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ to $\widetilde{L}_{q,\lambda}(\mathbb{H}_n)$ if and only if, $\alpha/Q \le 1/p - 1/q \le \alpha/(Q-\lambda)$.

In the limiting case $\frac{Q-\lambda}{\alpha} \le p \le \frac{Q}{\alpha}$ we prove that the operator M_α is bounded from $\widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ to $L_\infty(\mathbb{H}_n)$ and the modified fractional integral operator $\widetilde{I}_\alpha$ is bounded from $\widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ to $BMO(\mathbb{H}_n)$.

As applications of the properties of the fundamental solution of sub-Laplacian $\mathcal{L}$ on $\mathbb{H}_n$, we prove two Sobolev-Stein embedding theorems on modified Morrey and Besov-modified Morrey spaces in the Heisenberg group setting. As an another application, we prove the boundedness of $\mathcal{I}_\alpha$ from the Besov-modified Morrey spaces $B\widetilde{L}^s_{p\theta,\lambda}(\mathbb{H}_n)$ to $B\widetilde{L}^s_{q\theta,\lambda}(\mathbb{H}_n)$.

1 Introduction

Heisenberg group appear in quantum physics and many parts of mathematics, including Fourier analysis, several complex variables, geometry and topology.

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Analysis on the groups is also motivated by their role as the simplest and the most important model in the general theory of vector fields satisfying Hormander's condition. In the present paper we will prove the boundedness of the Riesz potential on the Heisenberg group in modified Morrey spaces.

We state some basic results about Heisenberg group. More detailed information can be found in [9, 11, 30] and the references therein. Let $\mathbb{H}_n$ be the $2n+1$-dimensional Heisenberg group. That is, $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$, with multiplication

$$(z,t) \cdot (w,s) = (z+w, t+s+2Im(z \cdot \bar{w})),$$

where $z \cdot \bar{w} = \sum_{j=1}^{n} z_j \bar{w_j}$. The inverse element of $u=(z,t)$ is $u^{-1} = (-z,-t)$ and we write the identity of $\mathbb{H}_n$ as $0 = (0,0)$. The Heisenberg group is a connected, simply connected nilpotent Lie group. We define one-parameter dilations on $\mathbb{H}_n$, for $r > 0$, by $\delta_r(z,t) = (rz, r^2 t)$. These dilations are group automorphisms and the Jacobian determinant is r^Q, where $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}_n$. A homogeneous norm on $\mathbb{H}_n$ is given by $|(z,t)| = (|z|^2 + |t|)^{1/2}$. With this norm, we define the Heisenberg ball centered at $u=(z,t)$ with radius r by $B(u,r) = \{v \in \mathbb{H}_n : |u^{-1}v| < r\}$, ${}^{\complement}B(u,r) = \mathbb{H}_n \setminus B(u,r)$, and we denote by $B_r = B(0,r) = \{v \in \mathbb{H}_n \, : \, |v| < r\}$ the open ball centered at 0, the identity element of $\mathbb{H}_n$, with radius r. The volume of the ball $B(u,r)$ is $C_Q r^Q$, where C_Q is the volume of the unit ball B_1.

Using coordinates $u = (z,t) = (x+iy, t)$ for points in $\mathbb{H}_n$, the left-invariant vector fields X_j, Y_j and T on $\mathbb{H}_n$ equal to $\frac{\partial}{\partial x_j}$, $\frac{\partial}{\partial y_j}$ and $\frac{\partial}{\partial t}$ at the origin are given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t},$$

respectively. These $2n+1$ vector fields form a basis for the Lie algebra of $\mathbb{H}_n$ with commutation relations

$$[Y_j, X_j] = 4T$$

for $j = 1, ..., n$, and all other commutators equal to 0.

Given a function f which is integrable on any ball $B(u,r) \subset \mathbb{H}_n$, the fractional maximal function $M_\alpha f$, $0 \le \alpha < Q$ of f is defined by

$$M_\alpha f(u) = \sup_{r>0} |B(u,r)|^{-1+\frac{\alpha}{Q}} \int_{B(u,r)} |f(v)| dV(v).$$

The fractional maximal function $M_\alpha f$ coincides for $\alpha = 0$ with the Hardy-Littlewood maximal function $Mf \equiv M_0 f$ (see [9, 30]) and is intimately related to the fractional integral

$$I_\alpha f(u) = \int_{\mathbb{H}_n} |v^{-1}u|^{\alpha-Q} f(v)dV(v), \qquad 0 < \alpha < Q.$$

The operators M_α and I_α play important role in real and harmonic analysis (see, for example [8, 9, 30]).

The classical Riesz potential is an important technical tool in harmonic analysis, theory of functions and partial differential equations. The classical Riesz potential $\mathcal{I}_\alpha$ is defined on $\mathbb{R}^n$ by

$$\mathcal{I}_\alpha f = (-\Delta)^{-\alpha/2} f, \quad 0 < \alpha < n,$$

where Δ is the Laplacian operator. It is known, that $\mathcal{I}_\alpha f(x) = \gamma(\alpha)^{-1} \int_{\mathbb{R}^n} |x - y|^{\alpha-n} f(y)dy \equiv I_\alpha f(x)$.

The potential and related topics on the Heisenberg group we consider the sub-Laplacian defined by

$$\mathcal{L} = -\sum_{j=1}^{n} \big(X_j^2 + Y_j^2\big).$$

The Riesz potential on the Heisenberg group is defined in terms of the sub-Laplacian $\mathcal{L}$.

Definition 1.1. *For $0 < \alpha < Q$, the Riesz potential $\mathcal{I}_\alpha$ is defined, initially on the Schwartz space $\mathbb{S}(\mathbb{H}_n)$, by*

$$\mathcal{I}_\alpha f(z,t) = \mathcal{L}^{-\frac{\alpha}{2}} f(z,t).$$

In [33] the relation between Riesz potential and heat kernel on the Heisenberg group is studied. The following theorem give expression of $\mathcal{I}_\alpha$, which provides a bridge to discuss the boundedness of the Riesz potential (see [33], Theorem 1).

Theorem A. *Let $q_s(z,t)$ be the heat kernel on $\mathbb{H}_n$. For $0 \le \alpha < Q$, we have for $f \in \mathbb{S}(\mathbb{H}_n)$*

$$\mathcal{I}_\alpha f(z,t) = \int_0^\infty \Gamma\big(\frac{\alpha}{2}\big)^{-1} s^{\frac{\alpha}{2}-1} q_s(\cdot) ds * f(z,t).$$

The Riesz potential $\mathcal{I}_\alpha$ satisfies the following estimate (see [33], Theorem 2)

$$|\mathcal{I}_\alpha f(z,t)| \lesssim I_\alpha f(z,t). \tag{1}$$

Inequality (1) gives a suitable estimate for the Riesz potential on the Heisenberg group.
In the theory of partial differential equations, together with weighted $L_{p,w}(\mathbb{R}^n)$ spaces, Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ play an important role. Morrey spaces were introduced by C. B. Morrey in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations (see [22]). Later, Morrey spaces found important applications to Navier-Stokes ([20], [32]) and Schrödinger ([23], [24], [25], [27], [28]) equations, elliptic problems with discontinuous coefficients ([3], [6]), and potential theory ([1], [2]). An exposition of the Morrey spaces can be found in the book [19].

Definition 1.2. *Let $1 \le p < \infty$, $0 \le \lambda \le Q$, $[t]_1 = \min\{1, t\}$. We denote by $L_{p,\lambda}(\mathbb{H}_n)$ the Morrey space, and by $\widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ the modified Morrey space, the set of locally integrable functions $f(u)$, $u \in \mathbb{H}_n$, with the finite norms*

$$\|f\|_{L_{p,\lambda}} = \sup_{u\in\mathbb{H}_n,\, t>0} \left(t^{-\lambda} \int_{B(u,t)} |f(y)|^p dV(v) \right)^{1/p},$$

$$\|f\|_{\widetilde{L}_{p,\lambda}} = \sup_{u\in\mathbb{H}_n,\, t>0} \left([t]_1^{-\lambda} \int_{B(u,t)} |f(y)|^p dV(v) \right)^{1/p}$$

respectively.

Note that

$$\widetilde{L}_{p,0}(\mathbb{H}_n) = L_{p,0}(\mathbb{H}_n) = L_p(\mathbb{H}_n),$$

$$\widetilde{L}_{p,\lambda}(\mathbb{H}_n) \subset\!\succ L_{p,\lambda}(\mathbb{H}_n) \cap L_p(\mathbb{H}_n) \quad \text{and} \quad \max\{\|f\|_{L_{p,\lambda}}, \|f\|_{L_p}\} \le \|f\|_{\widetilde{L}_{p,\lambda}} \qquad (2)$$

and if $\lambda < 0$ or $\lambda > Q$, then $L_{p,\lambda}(\mathbb{H}_n) = \widetilde{L}_{p,\lambda}(\mathbb{H}_n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{H}_n$.

Definition 1.3. *[15] Let $1 \le p < \infty$, $0 \le \lambda \le Q$. We denote by $WL_{p,\lambda}(\mathbb{H}_n)$ the weak Morrey space and by $W\widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ the modified weak Morrey space the set of locally integrable functions $f(u)$, $u \in \mathbb{H}_n$ with finite norms*

$$\|f\|_{WL_{p,\lambda}} = \sup_{r>0} r \sup_{u\in\mathbb{H}_n,\, t>0} \left(t^{-\lambda} \ |\{v \in B(u,t) :\ |f(v)| > r\}| \right)^{1/p},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda}} = \sup_{r>0} r \sup_{u\in\mathbb{H}_n,\, t>0} \left([t]_1^{-\lambda} \ |\{v \in B(u,t) :\ |f(v)| > r\}| \right)^{1/p}$$

respectively.

Note that

$$W L_p(\mathbb{H}_n) = W L_{p,0}(\mathbb{H}_n) = W \widetilde{L}_{p,0}(\mathbb{H}_n),$$
$$L_{p,\lambda}(\mathbb{H}_n) \subset W L_{p,\lambda}(\mathbb{H}_n) \text{ and } \|f\|_{W L_{p,\lambda}} \le \|f\|_{L_{p,\lambda}},$$
$$\widetilde{L}_{p,\lambda}(\mathbb{H}_n) \subset W \widetilde{L}_{p,\lambda}(\mathbb{H}_n) \text{ and } \|f\|_{W \widetilde{L}_{p,\lambda}} \le \|f\|_{\widetilde{L}_{p,\lambda}}.$$

The classical result by Hardy-Littlewood-Sobolev states that if $1 < p < q < \infty$, then I_α is bounded from $L_p(\mathbb{H}_n)$ to $L_q(\mathbb{H}_n)$ if and only if $\alpha = Q\left(\frac{1}{p} - \frac{1}{q}\right)$ and for $p = 1 < q < \infty$, I_α is bounded from $L_1(\mathbb{H}_n)$ to $WL_q(\mathbb{H}_n)$ if and only if $\alpha = Q\left(1 - \frac{1}{q}\right)$.

Spanne (see [29]) and Adams [1] studied boundedness of I_α on $\mathbb{R}^n$ in Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$. Later on Chiarenza and Frasca [5] was reproved boundedness of I_α in these spaces $L_{p,\lambda}(\mathbb{R}^n)$. By more general results of Guliyev [12] (see, also [13, 14, 15]) one can obtain the following generalization of the results in [1, 5, 29] to the Heisenberg group case (see, also [21]).

Theorem B. *Let $0 < \alpha < Q$ and $0 \le \lambda < Q - \alpha$, $1 \le p < \frac{Q-\lambda}{\alpha}$.*

1) If $1 < p < \frac{Q-\lambda}{\alpha}$, then condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator I_α from $L_{p,\lambda}(\mathbb{H}_n)$ to $L_{q,\lambda}(\mathbb{H}_n)$.

2) If $p = 1$, then condition $1 - \frac{1}{q} = \frac{\alpha}{Q-\lambda}$ is necessary and sufficient for the boundedness of the operator I_α from $L_{1,\lambda}(\mathbb{H}_n)$ to $WL_{q,\lambda}(\mathbb{H}_n)$.

If $\alpha = \frac{Q}{p} - \frac{Q}{q}$, then $\lambda = 0$ and the statement of Theorem B reduces to the aforementioned result by Hardy-Littlewood-Sobolev.

Recall that, for $0 < \alpha < Q$,

$$M_\alpha f(u) \le v_n^{\frac{\alpha}{Q}-1} I_\alpha(|f|)(u), \tag{3}$$

hence Theorem B also implies the boundedness of the fractional maximal operator M_α, where $v_n = |B(e,1|$ is the volume of the unit ball in $\mathbb{H}_n$.

We define $BMO(\mathbb{H}_n)$, the set of locally integrable functions f with finite norms

$$\|f\|_* = \sup_{r>0,\ u\in\mathbb{H}_n} |B_r|^{-1} \int_{B_r} |f(v^{-1}u) - f_{B_r}(u)| dV(v) < \infty,$$

where $f_{B_r}(u) = |B_r|^{-1} \int_{B_r} f(v^{-1}u) dV(v)$.

In this paper we study the fractional maximal operator M_α and the Riesz potential operator $\mathcal{I}_\alpha$ on the Heisenberg group in the modified Morrey space. In the case $p = 1$ we prove that the operators M_α and $\mathcal{I}_\alpha$ are bounded from $\widetilde{L}_{1,\lambda}(\mathbb{H}_n)$ to $W\widetilde{L}_{q,\lambda}(\mathbb{H}_n)$ if and only if, $\alpha/Q \le 1 - 1/q \le \alpha/(Q-\lambda)$. In the case $1 < p < (Q-\lambda)/\alpha$ we prove that the operators M_α and $\mathcal{I}_\alpha$ are bounded from $\widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ to $\widetilde{L}_{q,\lambda}(\mathbb{H}_n)$ if and only if, $\alpha/Q \le 1/p - 1/q \le \alpha/(Q-\lambda)$.

In the limiting case $\frac{Q-\lambda}{\alpha} \le p \le \frac{Q}{\alpha}$ we prove that the operator M_α is bounded from $\widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ to $L_\infty(\mathbb{H}_n)$ and the modified fractional integral operator $\widetilde{I}_\alpha$ is bounded from $\widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ to $BMO(\mathbb{H}_n)$.

As an application of the properties of the fundamental solution of sub-Laplacian $\mathcal{L}$ on $\mathbb{H}_n$, in Theorem 3.3 we prove the following modified Morrey version of Sobolev inequality on $\mathbb{H}_n$:

$$\|u\|_{\widetilde{L}_{q,\lambda}} \le C\|\nabla_{\mathcal{L}} u\|_{\widetilde{L}_{p,\lambda}}, \quad \text{for every} \quad u \in C_0^\infty(\mathbb{H}_n),$$

where $0 \le \lambda < Q$, $1 < p < Q-\lambda$ and $\frac{1}{Q} \le \frac{1}{p} - \frac{1}{q} \le \frac{1}{Q-\lambda}$.

In Theorem 7 we obtain boundedness of the operator $\mathcal{I}_\alpha$ from the Besov-modified Morrey spaces $B\widetilde{L}^s_{p\theta,\lambda}(\mathbb{H}_n)$ to $B\widetilde{L}^s_{q\theta,\lambda}(\mathbb{H}_n)$, $1 < p < q < \infty$, $\alpha/Q \le 1/p - 1/q \le \alpha/(Q-\lambda)$, $1 \le \theta \le \infty$ and $0 < s < 1$.

As an another application, in Theorem 3.5 we obtain the following Sobolev-Stein embedding inequality in Besov-modified Morrey space on $\mathbb{H}_n$.

$$\|u\|_{B\widetilde{L}^s_{q\theta,\lambda}} \le C\|\nabla_{\mathcal{L}} u\|_{B\widetilde{L}^s_{p\theta,\lambda}}, \quad \text{for every} \quad u \in C_0^\infty(\mathbb{H}^n)$$

where $0 \le \lambda < Q-1$, $1 < p < Q-\lambda$, $1 \le \theta \le \infty$, $0 < s < 1$ and $1/Q \le 1/p - 1/q \le 1/(Q-\lambda)$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Main Results

The following Hardy-Littlewood-Sobolev inequality in modified Morrey spaces on the Heisenberg group is valid.

Theorem 2.1. *Let $0 < \alpha < Q$, $0 \le \lambda < Q-\alpha$ and $1 \le p < \frac{Q-\lambda}{\alpha}$.*

1) If $1 < p < \frac{Q-\lambda}{\alpha}$, then condition $\frac{\alpha}{Q} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{Q-\lambda}$ is necessary and sufficient for the boundedness of the operators $\mathcal{I}_\alpha$ and I_α from $\widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ to $\widetilde{L}_{q,\lambda}(\mathbb{H}_n)$.

2) If $p = 1 < \frac{Q-\lambda}{\alpha}$, then condition $\frac{\alpha}{Q} \le 1 - \frac{1}{q} \le \frac{\alpha}{Q-\lambda}$ is necessary and sufficient for the boundedness of the operators $\mathcal{I}_\alpha$ and I_α from $\widetilde{L}_{1,\lambda}(\mathbb{H}_n)$ to $W\widetilde{L}_{q,\lambda}(\mathbb{H}_n)$.

Corollary 2.1. *Let $0 < \alpha < Q$, $0 \le \lambda < Q-\alpha$ and $1 \le p \le \frac{Q-\lambda}{\alpha}$.*

1) If $1 < p < \frac{Q-\lambda}{\alpha}$, then condition $\frac{\alpha}{Q} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{Q-\lambda}$ is necessary and sufficient for the boundedness of the operator M_α from $\widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ to $\widetilde{L}_{q,\lambda}(\mathbb{H}_n)$.

2) If $p = 1 < \frac{Q-\lambda}{\alpha}$*, then condition* $\frac{\alpha}{Q} \leq 1 - \frac{1}{q} \leq \frac{\alpha}{Q-\lambda}$ *is necessary and sufficient for the boundedness of the operator* M_α *from* $\widetilde{L}_{1,\lambda}(\mathbb{H}_n)$ *to* $W\widetilde{L}_{q,\lambda}(\mathbb{H}_n)$*.*

3) If $\frac{Q-\lambda}{\alpha} \leq p \leq \frac{Q}{\alpha}$*, then the operator* M_α *is bounded from* $\widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ *to* $L_\infty(\mathbb{H}_n)$*.*

The examples show that the operator I_α are not defined for all functions $f \in \widetilde{L}_{p,\lambda}(\mathbb{H}_n)$, $0 \leq \lambda < Q - \alpha$, if $p \geq \frac{Q-\lambda}{\alpha}$.

We consider the modified fractional integral

$$\widetilde{I}_\alpha f(u) = \int_{\mathbb{H}_n} \left(|uv^{-1}|^{\alpha-Q} - |v|^{\alpha-Q} \chi_{\complement_{B(e,1)}}(v) \right) f(v) dV(v),$$

where $\chi_{\complement_{B(e,1)}}$ is the characteristic function of the set $\complement B(e,1) = \mathbb{H}_n \setminus B(e,1)$.

Note that in the limiting case $\frac{Q-\lambda}{\alpha} \leq p \leq \frac{Q}{\alpha}$ statement 1) in Theorem B does not hold. Moreover, there exists $f \in \widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ such that $I_\alpha f(u) = \infty$ for all $u \in \mathbb{H}_n$. However, as will be proved, statement 1) holds for the modified fractional integral $\widetilde{I}_\alpha$ if the space $L_\infty(\mathbb{H}_n)$ is replaced by a wider space $BMO(\mathbb{H}_n)$.

The following theorem we obtain conditions ensuring that the operator $\widetilde{I}_\alpha$ is bounded from the space $\widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ to $BMO(\mathbb{H}_n)$.

Theorem 2.2. *Let* $0 < \alpha < Q$, $0 \leq \lambda < Q - \alpha$*, and* $\frac{Q-\lambda}{\alpha} \leq p \leq \frac{Q}{\alpha}$*, then the operator* $\widetilde{I}_\alpha$ *is bounded from* $\widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ *to* $BMO(\mathbb{H}_n)$*. Moreover, if the integral* $I_\alpha f$ *exists almost everywhere for* $f \in \widetilde{L}_{p,\lambda}(\mathbb{H}_n)$, $\frac{Q-\lambda}{\alpha} \leq p \leq \frac{Q}{\alpha}$*, then* $I_\alpha f \in BMO(\mathbb{H}_n)$ *and the following inequality is valid*

$$\|I_\alpha f\|_{BMO} \leq C \|f\|_{\widetilde{L}_{p,\lambda}},$$

where $C > 0$ *is independent of* f*.*

3 Some applications

It is known that (see [4], p. 247) if $|\cdot|$ is a homogeneous norm on $\mathbb{H}_n$, then there exists a positive constant C_0 such that $\Gamma(x) = C_0 |x|^{2-Q}$ is the fundamental solution of $\mathcal{L}$.

From Theorem 2.1, one easily obtains an inequality extending the classical Sobolev embedding theorem to the Heisenberg groups.

Theorem 3.3. (Sobolev-Stein embedding on modified Morrey spaces) *Let* $0 \le \lambda < Q$, $1 < p < Q-\lambda$ *and* $\frac{1}{Q} \le \frac{1}{p} - \frac{1}{q} \le \frac{1}{Q-\lambda}$. *Then there exists a positive constant* C *such that*

$$\|u\|_{\widetilde{L}_{q,\lambda}} \le C\|\nabla_{\mathcal{L}} u\|_{\widetilde{L}_{p,\lambda}}, \quad \textit{for every} \quad u \in C_0^\infty(\mathbb{H}_n).$$

Proof. Let $u \in C_0^\infty(\mathbb{H}_n)$. By using the integral representation formula for the fundamental solution (see [4], p. 237), we have

$$u(x) = \int_{\mathbb{H}_n} \Gamma(x^{-1}y)\mathcal{L}u(y)dy \tag{4}$$

Keeping in mind that $\mathcal{L} = \sum_{i=1}^n (X_i^2 + Y_i^2)$ and $X_i^* = -X_i$, $Y_i^* = -Y_i$, by integrating by parts at the right-hand side (4), we obtain

$$u(x) = \int_{\mathbb{H}_n} (\nabla_{\mathcal{L}}\Gamma)(x^{-1}y)\nabla_{\mathcal{L}} u(y)dy. \tag{5}$$

On the other hand, out of the origin, we have

$$\nabla_{\mathcal{L}}\Gamma(x) = C_0\,\nabla_{\mathcal{L}}\left(|x|^{2-Q}\right) = (2-Q)C_0\,|x|^{1-Q}\,\nabla_{\mathcal{L}}|x|,$$

so that, since $\nabla_{\mathcal{L}}|\cdot|$ is smooth in $\mathbb{H}_n \setminus \{0\}$ and δ_λ-homogeneous of degree zero,

$$\nabla_{\mathcal{L}}\Gamma(x) \le C|x|^{1-Q},$$

for a suitable constant $C > 0$ depending only on $\mathcal{L}$. Using this inequality in (5), we get

$$|u(x)| \le C\int_{\mathbb{H}_n} |\nabla_{\mathcal{L}} u(y)||x|^{1-Q}dy = CI_1(|\nabla_{\mathcal{L}} u|)(x). \tag{6}$$

Then, by Theorem 2.1,

$$\|u\|_{\widetilde{L}_{q,\lambda}} \le C\|I_1(|\nabla_{\mathcal{L}} u|)\|_{\widetilde{L}_{q,\lambda}} \le C\|\nabla_{\mathcal{L}} u\|_{\widetilde{L}_{p,\lambda}}.$$

□

In the following theorem we prove the boundedness of $\mathcal{I}_\alpha$ in the Besov-modified Morrey spaces on $\mathbb{H}_n$

$$B\widetilde{L}^s_{p\theta,\lambda}(\mathbb{H}_n) = \Big\{f : \|f\|_{B\widetilde{L}^s_{p\theta,\lambda}} = \|f\|_{\widetilde{L}_{p,\lambda}} + \Big(\int_{\mathbb{H}_n} \frac{\|f(x\cdot) - f(\cdot)\|^\theta_{\widetilde{L}_{p,\lambda}}}{|x|^{Q+s\theta}}dx\Big)^{\frac{1}{\theta}} < \infty\Big\} \tag{7}$$

where $1 \le p, \theta \le \infty$ and $0 < s < 1$.

Besov spaces $B^s_{p\theta}(\mathbb{H}_n)$ in the setting Lie groups were studied by many authors (see, for example [8, 10, 16, 26, 31]).

Theorem 3.4. *Let $0 < \alpha < Q$, $0 \le \lambda < Q - \alpha$ and $1 \le p < \frac{Q-\lambda}{\alpha}$. If $\frac{\alpha}{Q} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{Q-\lambda}$, $1 \le \theta \le \infty$ and $0 < s < 1$, then the operator $\mathfrak{I}_\alpha$ is bounded from the spaces $B\widetilde{L}^s_{p\theta,\lambda}(\mathbb{H}_n)$ to $B\widetilde{L}^s_{q\theta,\lambda}(\mathbb{H}_n)$. More precisely, there is a constant $C > 0$ such that*

$$\|\mathfrak{I}_\alpha f\|_{B\widetilde{L}^s_{q\theta,\lambda}} \le C\|f\|_{B\widetilde{L}^s_{p\theta,\lambda}}$$

holds for all $f \in B\widetilde{L}^s_{p\theta,\lambda}(\mathbb{H}_n)$.

Proof. By the definition of the Besov-modified Morrey spaces on $\mathbb{H}_n$ it suffices to show that

$$\|\tau_y \mathfrak{I}_\alpha f - \mathfrak{I}_\alpha f\|_{\widetilde{L}_{q,\lambda}} \le C\|\tau_y f - f\|_{\widetilde{L}_{p,\lambda}},$$

where $\tau_y f(x) = f(yx)$.

It is easy to see that $\tau_y f$ commutes with $\mathfrak{I}_\alpha$, i.e., $\tau_y \mathfrak{I}_\alpha f = \mathfrak{I}_\alpha(\tau_y f)$. Hence we obtain

$$|\tau_y \mathfrak{I}_\alpha f - \mathfrak{I}_\alpha f| = |\mathfrak{I}_\alpha(\tau_y f) - \mathfrak{I}_\alpha f| \le \mathfrak{I}_\alpha(|\tau_y f - f|).$$

Taking $\widetilde{L}_{p,\lambda}$-norm on both sides of the last inequality, we obtain the desired result by using the boundedness of $\mathfrak{I}_\alpha$ from $\widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ to $\widetilde{L}_{q,\lambda}(\mathbb{H}_n)$.

Thus the proof of the Theorem 3.4 is completed. □

From Theorem 3.4 we obtain the following Sobolev-Stein embedding inequality on Besov-modified Morrey space.

Theorem 3.5. (Sobolev-Stein embedding on Besov-modified Morrey space) *Let $0 \le \lambda < Q - 1$, $1 < p < Q - \lambda$, $1 \le \theta \le \infty$ and $0 < s < 1$. Then there exists a positive constant C such that*

$$\|u\|_{B\widetilde{L}^s_{q\theta,\lambda}} \le C\|\nabla_{\mathcal{L}} u\|_{B\widetilde{L}^s_{p\theta,\lambda}}, \quad \text{for every} \quad u \in C_0^\infty(\mathbb{H}^n)$$

where $1/Q \le 1/p - 1/q \le 1/(Q - \lambda)$.

The Dirichlet problem for the Kohn-Laplacian on $\mathbb{H}_n$ belongs to Jerison [17, 18]. In particular, our results lead to the following apriori estimate for the sub-Laplacian equation $\mathcal{L}f = g$.

Theorem 3.6. *Let $0 < s < 1$, $1 \le \theta \le \infty$, $g \in B\widetilde{L}^s_{p\theta,\lambda}(\mathbb{H}_n)$ and $\mathcal{L}f = g$*

1) If $0 \le \lambda < Q-2$, $1 < p < \frac{Q-\lambda}{2}$ and $\frac{2}{Q} \le \frac{1}{p} - \frac{1}{q} \le \frac{2}{Q-\lambda}$, then there exists a constant $C > 0$ such that

$$\|f\|_{B\widetilde{L}^s_{q\theta,\lambda}} \le C\|g\|_{B\widetilde{L}^s_{p\theta,\lambda}}.$$

, 2) If $0 \le \lambda < Q-1$, $1 < p < Q-\lambda$ and $\frac{1}{Q} \le \frac{1}{p} - \frac{1}{q} \le \frac{1}{Q-\lambda}$, then there exists a constant $C > 0$ such that

$$\|X_i f\|_{B\widetilde{L}^s_{q\theta,\lambda}} \le C\|g\|_{B\widetilde{L}^s_{p\theta,\lambda}}, \quad i = 1, 2, \cdots, n,$$

$$\|Y_i f\|_{B\widetilde{L}^s_{q\theta,\lambda}} \le C\|g\|_{B\widetilde{L}^s_{p\theta,\lambda}}, \quad i = 1, 2, \cdots, n.$$

The proof of Theorems 3.5 and 3.6 are similar to Theorem 3.3.

4 Preliminaries

Define $f_t(u) =: f(\delta_t u)$ and $[t]_{1,+} = \max\{1, t\}$. Then

$$\|f_t\|_{L_{p,\lambda}} = t^{-\frac{Q}{p}} \sup_{u\in\mathbb{H}_n,\, r>0} \left(r^{-\lambda} \int_{B(\delta_t u, tr)} |f(v)|^p dV(v) \right)^{1/p} = t^{\frac{\lambda-Q}{p}} \|f\|_{L_{p,\lambda}},$$

$$\begin{aligned}
\|f_t\|_{\widetilde{L}_{p,\lambda}} &= \sup_{u\in\mathbb{H}_n,\, r>0} \left([r]_1^{-\lambda} \int_{B(u,r)} |f_t(v)|^p \, dV(v) \right)^{1/p} \\
&= t^{-\frac{Q}{p}} \sup_{u\in\mathbb{H}_n,\, r>0} \left([r]_1^{-\lambda} \int_{B(\delta_t u, tr)} |f(v)|^p dV(v) \right)^{1/p} \\
&= t^{-\frac{Q}{p}} \sup_{r>0} \left(\frac{[tr]_1}{[r]_1} \right)^{\lambda/p} \sup_{u\in\mathbb{H}_n,\, r>0} \left([tr]_1^{-\lambda} \int_{B(\delta_t u, tr)} |f(v)|^p \, dV(v) \right)^{1/p} \\
&= t^{-\frac{Q}{p}} [t]_{1,+}^{\frac{\lambda}{p}} \|f\|_{\widetilde{L}_{p,\lambda}}.
\end{aligned}$$

In this section we study the $\widetilde{L}_{p,\lambda}$-boundedness of the maximal operator M.

Lemma 4.1. *Let $1 \le p < \infty$, $0 \le \lambda \le Q$. Then*

$$\widetilde{L}_{p,\lambda}(\mathbb{H}_n) = L_{p,\lambda}(\mathbb{H}_n) \cap L_p(\mathbb{H}_n) \quad \text{and} \quad \|f\|_{\widetilde{L}_{p,\lambda}} = \max\left\{ \|f\|_{L_{p,\lambda}}, \|f\|_{L_p} \right\}.$$

Proof. Let $f \in \widetilde{L}_{p,\lambda}(\mathbb{H}_n)$. Then from (2) we have that $f \in L_{p,\lambda}(\mathbb{H}_n) \cap L_p(\mathbb{H}_n)$ and $\max\left\{ \|f\|_{L_{p,\lambda}}, \|f\|_{L_p} \right\} \le \|f\|_{\widetilde{L}_{p,\lambda}}$.

Let now $f \in L_{p,\lambda}(\mathbb{H}_n) \cap L_p(\mathbb{H}_n)$. Then

$$\|f\|_{\widetilde{L}_{p,\lambda}} = \sup_{u\in\mathbb{H}_n, t>0} \left([t]_1^{-\lambda} \int_{B(u,t)} |f(v)|^p dV(v)\right)^{1/p}$$
$$= \max\left\{ \sup_{u\in\mathbb{H}_n, 0<t\le 1} \left(t^{-\lambda} \int_{B(u,t)} |f(v)|^p dV(v)\right)^{1/p}, \right.$$
$$\left. \sup_{u\in\mathbb{H}_n, t>1} \left(\int_{B(u,t)} |f(v)|^p dV(v)\right)^{1/p} \right\}$$
$$\le \max\left\{\|f\|_{L_{p,\lambda}}, \|f\|_{L_p}\right\}.$$

Therefore, $f \in \widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ and the embedding $L_{p,\lambda}(\mathbb{H}_n) \cap L_p(\mathbb{H}_n) \subset\!\!\succ \widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ is valid.

Thus $\widetilde{L}_{p,\lambda}(\mathbb{H}_n) = L_{p,\lambda}(\mathbb{H}_n) \cap L_p(\mathbb{H}_n)$ and $\|f\|_{\widetilde{L}_{p,\lambda}} = \max\left\{\|f\|_{L_{p,\lambda}}, \|f\|_{L_p}\right\}$. □

Analogously proved the following statement.

Lemma 4.2. *Let $1 \le p < \infty$, $0 \le \lambda \le Q$. Then*

$$W\widetilde{L}_{p,\lambda}(\mathbb{H}_n) = WL_{p,\lambda}(\mathbb{H}_n) \cap WL_p(\mathbb{H}_n)$$

and

$$\|f\|_{W\widetilde{L}_{p,\lambda}} = \max\left\{\|f\|_{WL_{p,\lambda}}, \|f\|_{WL_p}\right\}.$$

Theorem 4.7. *[21] 1. If $f \in L_{1,\lambda}(\mathbb{H}_n)$, $0 \le \lambda < Q$, then $Mf \in WL_{1,\lambda}(\mathbb{H}_n)$ and*

$$\|Mf\|_{WL_{1,\lambda}} \le C_\lambda \|f\|_{L_{1,\lambda}},$$

where C_λ depends only on n and λ.

2. If $f \in L_{p,\lambda}(\mathbb{H}_n)$, $1 < p < \infty$, $0 \le \lambda < Q$, then $Mf \in L_{p,\lambda}(\mathbb{H}_n)$ and

$$\|Mf\|_{L_{p,\lambda}} \le C_{p,\lambda} \|f\|_{L_{p,\lambda}},$$

where $C_{p,\lambda}$ depends only on n, p and λ.

The following statement is valid:

Theorem 4.8. *1. If* $f \in \widetilde{L}_{1,\lambda}(\mathbb{H}_n)$, $0 \le \lambda < Q$, *then* $Mf \in W\widetilde{L}_{1,\lambda}(\mathbb{H}_n)$ *and*

$$\|Mf\|_{W\widetilde{L}_{1,\lambda}} \le \overline{C}_{1,\lambda} \|f\|_{\widetilde{L}_{1,\lambda}},$$

where $C_{1,\lambda}$ *depends only on* λ.

2. If $f \in \widetilde{L}_{p,\lambda}(\mathbb{H}_n)$, $1 < p < \infty$,$0 \le \lambda < Q$, *then* $Mf \in \widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ *and*

$$\|Mf\|_{\widetilde{L}_{p,\lambda}} \le \overline{C}_{p,\lambda} \|f\|_{\widetilde{L}_{p,\lambda}},$$

where $C_{p,\lambda}$ *depends only on* p *and* λ.

Proof. It is obvious that (see Lemmas 4.1 and 4.2)

$$\|Mf\|_{\widetilde{L}_{p,\lambda}} = \max\left\{\|Mf\|_{L_{p,\lambda}}, \|Mf\|_{L_p}\right\}$$

for $1 < p < \infty$ and

$$\|Mf\|_{W\widetilde{L}_{1,\lambda}} = \max\left\{\|Mf\|_{WL_{1,\lambda}}, \|Mf\|_{WL_1}\right\}$$

for $p = 1$.

Let $1 < p < \infty$. By the boundedness of M on $L_p(\mathbb{H}_n)$ and from Theorem 4.7 we have

$$\|Mf\|_{\widetilde{L}_{p,\lambda}} \le \max\{C_p, C_{p,\lambda}\} \|f\|_{\widetilde{L}_{p,\lambda}}.$$

Let $p = 1$. By the boundedness of M from $L_1(\mathbb{H}_n)$ to $WL_1(\mathbb{H}_n)$ and from Theorem 4.7 we have

$$\|Mf\|_{W\widetilde{L}_{1,\lambda}} \le \max\{C_1, C_{1,\lambda}\} \|f\|_{\widetilde{L}_{1,\lambda}}.$$

□

Lemma 4.3. *Let* $0 < \alpha < Q$. *Then for* $2|u| \le |v|$, $u, v \in \mathbb{H}_n$, *the following inequality is valid*

$$\left||v^{-1}u|^{\alpha-Q} - |v|^{\alpha-Q}\right| \le 2^{Q-\alpha+1} |v|^{\alpha-Q-1} |u|. \tag{8}$$

Proof. From the mean value theorem we get

$$\left||v^{-1}u|^{\alpha-Q} - |v|^{\alpha-Q}\right| \le \left||v^{-1}u| - |v|\right| \cdot \xi^{\alpha-Q-1},$$

where $\min\left\{|v^{-1}u|, |v|\right\} \le \xi \le \max\left\{|v^{-1}u|, |v|\right\}$.

We note that $\frac{1}{2}|v| \le |v^{-1}u| \le \frac{3}{2}|v|$, and $\left||v^{-1}u| - |v|\right| \le |u|$.

Thus the proof of the lemma is completed. □

5 Proof of the Theorems

Proof of Theorem 2.1.

1) *Sufficiency.* Let $0 < \alpha < Q$, $0 < \lambda < Q - \alpha$, $f \in \widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ and $1 < p < \frac{Q-\lambda}{\alpha}$. Then

$$I_\alpha f(u) = \left(\int_{B(u,t)} + \int_{\complement_{B(u,t)}} \right) f(v)|uv^{-1}|^{\alpha-Q} dV(v) \equiv A(u,t) + C(u,t).$$

For $A(u,t)$ we have

$$\begin{aligned} |A(u,t)| &\leq \int_{B(u,t)} |f(v)||uv^{-1}|^{\alpha-Q} dV(v) \\ &\leq \sum_{j=1}^{\infty} \left(2^{-j}t\right)^{\alpha-Q} \int_{B(u,2^{-j+1}t)\setminus B(u,2^{-j}t)} |f(v)| dV(v). \end{aligned}$$

Hence

$$|A(u,t)| \lesssim t^\alpha M f(u). \tag{9}$$

In the second integral by the Hölder's inequality we have

$$\begin{aligned} |C(u,t)| &\leq \left(\int_{\complement_{B(u,t)}} |uv^{-1}|^{-\beta} |f(v)|^p dV(v) \right)^{1/p} \\ &\times \left(\int_{\complement_{B(u,t)}} |uv^{-1}|^{\left(\frac{\beta}{p}+\alpha-Q\right)p'} dV(v) \right)^{1/p'} = J_1 \cdot J_2. \end{aligned}$$

Let $\lambda < \beta < Q - \alpha p$. For J_1 we get

$$
\begin{aligned}
J_1 &= \Big(\sum_{j=0}^{\infty} \int_{B(u,2^{j+1}t)\setminus B(u,2^j t)} |f(v)|^p |uv^{-1}|^{-\beta} dV(v) \Big)^{1/p} \\
&\le t^{-\frac{\beta}{p}} \, \|f\|_{\widetilde{L}_{p,\lambda}} \, \Big(\sum_{j=0}^{\infty} 2^{-\beta j} [2^{j+1}t]_1^{\lambda} \Big)^{1/p} \\
&= t^{-\frac{\beta}{p}} \, \|f\|_{\widetilde{L}_{p,\lambda}} \begin{cases} \Big(2^{\lambda} \, t^{\lambda} \sum\limits_{j=0}^{[\log_2 \frac{1}{2t}]} 2^{(\lambda-\beta)j} + \sum\limits_{j=[\log_2 \frac{1}{2t}]+1}^{\infty} 2^{-\beta j} \Big)^{1/p}, & 0 < t < \frac{1}{2}, \\ \Big(\sum\limits_{j=0}^{\infty} 2^{-\beta j} \Big)^{1/p}, & t \ge \frac{1}{2} \end{cases} \\
&\approx t^{-\frac{\beta}{p}} \, \|f\|_{\widetilde{L}_{p,\lambda}} \begin{cases} \Big(t^{\lambda} + t^{\beta} \Big)^{1/p}, & 0 < t < \frac{1}{2}, \\ 1, & t \ge \frac{1}{2} \end{cases} \\
&\approx \|f\|_{\widetilde{L}_{p,\lambda}} \begin{cases} t^{\frac{\lambda-\beta}{p}}, & 0 < t < \frac{1}{2}, \\ t^{-\frac{\beta}{p}}, & t \ge \frac{1}{2} \end{cases} \\
&= [2t]_1^{\frac{\lambda}{p}} \; t^{-\frac{\beta}{p}} \;\; \|f\|_{\widetilde{L}_{p,\lambda}} . \qquad (10)
\end{aligned}
$$

For J_2 we obtain

$$
J_2 = \left(\int_{\mathbb{S}} d\xi \int_t^{\infty} r^{Q-1+\left(\frac{\beta}{p}+\alpha-Q\right)p'} dr \right)^{\frac{1}{p'}} \approx t^{\frac{\beta}{p}+\alpha-\frac{Q}{p}} . \qquad (11)
$$

From (10) and (11) we have

$$
|C(u,t)| \lesssim [t]_1^{\frac{\lambda}{p}} \; t^{\alpha-\frac{Q}{p}} \;\; \|f\|_{\widetilde{L}_{p,\lambda}} . \qquad (12)
$$

Thus for all $t > 0$ we get

$$
|I_\alpha f(u)| \lesssim \left(t^{\alpha} Mf(u) + [t]_1^{\frac{\lambda}{p}} \; t^{\alpha-\frac{Q}{p}} \;\; \|f\|_{\widetilde{L}_{p,\lambda}} \right)
$$

$$
\lesssim \min \left\{ t^{\alpha} Mf(u) + t^{\alpha-\frac{Q}{p}} \;\; \|f\|_{\widetilde{L}_{p,\lambda}} \, , \; t^{\alpha} Mf(u) + t^{\alpha-\frac{Q-\lambda}{p}} \;\; \|f\|_{\widetilde{L}_{p,\lambda}} \right\} .
$$

Minimizing with respect to t, at

$$
t = \left[(Mf(u))^{-1} \, \|f\|_{\widetilde{L}_{p,\lambda}} \right]^{p/(Q-\lambda)}
$$

and

$$
t = \left[(Mf(u))^{-1} \, \|f\|_{\widetilde{L}_{p,\lambda}} \right]^{p/Q}
$$

we have

$$|I_\alpha f(u)| \le C_{11} \min\left\{ \left(\frac{Mf(u)}{\|f\|_{\widetilde{L}_{p,\lambda}}}\right)^{1-\frac{p\alpha}{Q-\lambda}}, \left(\frac{Mf(u)}{\|f\|_{\widetilde{L}_{p,\lambda}}}\right)^{1-\frac{p\alpha}{Q}} \right\} \|f\|_{\widetilde{L}_{p,\lambda}} .$$

Then

$$|I_\alpha f(u)| \lesssim (Mf(u))^{p/q} \, \|f\|_{\widetilde{L}_{p,\lambda}}^{1-p/q} .$$

Hence, by Theorem 4.8, we have

$$\begin{aligned}\int_{B(u,t)} |I_\alpha f(v)|^q \, dV(v) &\lesssim \|f\|_{\widetilde{L}_{p,\lambda}}^{q-p} \int_{B(u,t)} (Mf(v))^p \, dV(v)\\ &\lesssim [t]_1^\lambda \, \|f\|_{\widetilde{L}_{p,\lambda}}^q ,\end{aligned}$$

which implies that I_α is bounded from $\widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ to $\widetilde{L}_{q,\lambda}(\mathbb{H}_n)$.

Necessity. Let $1 < p < \frac{Q-\lambda}{\alpha}$, $f \in \widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ and the operators $\mathcal{I}_\alpha$ and I_α are bounded from $\widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ to $\widetilde{L}_{q,\lambda}(\mathbb{H}_n)$.

Define $f_t(u) =: f(\delta_t u)$, $[t]_{1,+} = \max\{1, t\}$. Then

$$\begin{aligned}\|f_t\|_{\widetilde{L}_{p,\lambda}} &= \sup_{r>0,\, u\in\mathbb{H}_n} \left([r]_1^{-\lambda} \int_{B(u,r)} |f_t(v)|^p \, dV(v)\right)^{1/p}\\ &= t^{-\frac{Q}{p}} \sup_{u\in\mathbb{H}_n,\, r>0} \left([r]_1^{-\lambda} \int_{B(u,tr)} |f(v)|^p dV(v)\right)^{1/p}\\ &= t^{-\frac{Q}{p}} \sup_{r>0} \left(\frac{[tr]_1}{[r]_1}\right)^{\lambda/p} \sup_{r>0,\, u\in\mathbb{H}_n} \left([tr]_1^{-\lambda} \int_{B(u,tr)} |f(v)|^p \, dV(v)\right)^{1/p}\\ &= t^{-\frac{Q}{p}} [t]_{1,+}^{\frac{\lambda}{p}} \, \|f\|_{\widetilde{L}_{p,\lambda}} ,\end{aligned}$$

and

$$I_\alpha f_t(u) = t^{-\alpha} I_\alpha f(\delta_t u), \qquad \mathcal{I}_\alpha f_t(u) = t^{-\alpha} I_\alpha f(\delta_t u),$$

$$\begin{aligned}\|I_\alpha f_t\|_{\widetilde{L}_{q,\lambda}} &= t^{-\alpha} \sup_{u\in\mathbb{H}_n,\, r>0} \left([r]_1^{-\lambda} \int_{B(u,r)} |I_\alpha f(\delta_t v)|^q \, dV(v)\right)^{1/q}\\ &= t^{-\alpha-\frac{Q}{q}} \sup_{r>0} \left(\frac{[tr]_1}{[r]_1}\right)^{\lambda/q} \sup_{r>0,\, u\in\mathbb{H}_n} \left([tr]_1^{-\lambda} \int_{B(\delta_t u,tr)} |I_\alpha f(v)|^q \, dV(v)\right)^{1/q}\\ &= t^{-\alpha-\frac{Q}{q}} [t]_{1,+}^{\frac{\lambda}{q}} \, \|I_\alpha f\|_{\widetilde{L}_{q,\lambda}} .\end{aligned}$$

Also

$$\|\mathfrak{I}_\alpha f_t\|_{\widetilde{L}_{q,\lambda}} = t^{-\alpha-\frac{Q}{q}}[t]_{1,+}^{\frac{\lambda}{q}} \|I_\alpha f\|_{\widetilde{L}_{q,\lambda}}.$$

By the boundedness of $\mathfrak{I}_\alpha$ from $\widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ to $\widetilde{L}_{q,\lambda}(\mathbb{H}_n)$

$$\begin{aligned}\|\mathfrak{I}_\alpha f\|_{\widetilde{L}_{q,\lambda}} &= t^{\alpha+\frac{Q}{q}}\ [t]_{1,+}^{-\frac{\lambda}{q}}\ \|I_\alpha f_t\|_{\widetilde{L}_{q,\lambda}} \\ &\le t^{\alpha+\frac{Q}{q}}\ [t]_{1,+}^{-\frac{\lambda}{q}}\ \|f_t\|_{\widetilde{L}_{p,\lambda}} \\ &\le C_{p,q,\lambda} t^{\alpha+\frac{Q}{q}-\frac{Q}{p}}[t]_{1,+}^{\frac{\lambda}{p}-\frac{\lambda}{q}}\|f\|_{\widetilde{L}_{p,\lambda}},\end{aligned}$$

where $C_{p,q,\lambda}$ depends only on p, q, λ and n.

If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{Q}$, then in the case $t \to 0$ we have $\|\mathfrak{I}_\alpha f\|_{\widetilde{L}_{q,\lambda}} = 0$ for all $f \in \widetilde{L}_{p,\lambda}(\mathbb{H}_n)$.

As well as if $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{Q-\lambda}$, then at $t \to \infty$ we obtain $\|\mathfrak{I}_\alpha f\|_{\widetilde{L}_{q,\lambda}} = 0$ for all $f \in \widetilde{L}_{p,\lambda}(\mathbb{H}_n)$.

Therefore $\frac{\alpha}{Q} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{Q-\lambda}$. Analogously we get the last equality for the I_α.

2) *Sufficiency.* Let $f \in \widetilde{L}_{1,\lambda}(\mathbb{H}_n)$. We have

$$\begin{aligned}&|\{v \in B(u,t)\ :\ |I_\alpha f(v)| > 2\beta\}| \\ &\le |\{v \in B(u,t)\ :\ |A(v,t)| > \beta\}| + |\{v \in B(u,t)\ :\ |C(v,t)| > \beta\}|.\end{aligned}$$

Then

$$\begin{aligned}C(v,t) &= \sum_{j=0}^{\infty} \int_{B(v,2^{j+1}t)\setminus B(v,2^j t)} |f(z)||vw^{-1}|^{\alpha-Q} dV(w) \\ &\le t^{\alpha-Q}\ \|f\|_{\widetilde{L}_{1,\lambda}} \sum_{j=0}^{\infty} 2^{-(Q-\alpha)j}[2^{j+1}t]_1^\lambda\end{aligned}$$

$$= t^{\alpha-Q}\ \|f\|_{\widetilde{L}_{1,\lambda}} \begin{cases} 2^\lambda\, t^\lambda \sum\limits_{j=0}^{[\log_2 \frac{1}{2t}]} 2^{(\lambda-Q+\alpha)j} + \sum\limits_{j=[\log_2 \frac{1}{2t}]+1}^{\infty} 2^{-(Q-\alpha)j}, & 0 < t < \frac{1}{2}, \\ \sum\limits_{j=0}^{\infty} 2^{-(Q-\alpha)j}, & t \ge \frac{1}{2} \end{cases}$$

$$\approx t^{\alpha-Q}\ \|f\|_{\widetilde{L}_{1,\lambda}} \begin{cases} t^\lambda + C_{15} t^{Q-\alpha}, & 0 < t < \frac{1}{2}, \\ 1, & t \ge \frac{1}{2} \end{cases}$$

$$\approx \|f\|_{\widetilde{L}_{1,\lambda}} \begin{cases} t^{\lambda+\alpha-Q}, & 0 < t < \frac{1}{2}, \\ t^{\alpha-Q}, & t \ge \frac{1}{2} \end{cases} = [2t]_1^\lambda\ t^{\alpha-Q}\ \ \|f\|_{\widetilde{L}_{1,\lambda}}.$$

Taking into account inequality (9) and Theorem 2.1, we have

$$|\{v\in B(u,t)\ :\ |A(v,t)|>\beta\}| \le \left|\left\{v\in B(u,t)\ :\ Mf(v)>\frac{\beta}{C_1t^\alpha}\right\}\right| \le \frac{C_2t^\alpha}{\beta}\cdot [t]_1^\lambda\, \|f\|_{\widetilde{L}_{1,\lambda}},$$

where $C_2=C_1\cdot C_{1,\lambda}$ and thus if $C_2[2t]_1^\lambda\, t^{\alpha-Q}\ \|f\|_{\widetilde{L}_{1,\lambda}}=\beta$, then $|C(v,t)|\le\beta$ and consequently, $|\{v\in B(u,t)\ :\ |C(v,t)|>\beta\}|=0$.

Then

$$|\{v\in B(u,t)\ :\ |I_\alpha f(v)|>2\beta\}| \lesssim \frac{1}{\beta}\,[t]_1^\lambda\, t^\alpha\ \|f\|_{\widetilde{L}_{1,\lambda}} \lesssim [t]_1^\lambda \left(\frac{\|f\|_{\widetilde{L}_{1,\lambda}}}{\beta}\right)^{\frac{Q-\lambda}{Q-\lambda-\alpha}},\ \text{if } 2t<1$$

and

$$|\{v\in B(u,t)\ :\ |I_\alpha f(v)|>2\beta\}| \lesssim \frac{1}{\beta}\,[t]_1^\lambda\, t^\alpha\ \|f\|_{\widetilde{L}_{1,\lambda}} \lesssim [t]_1^\lambda \left(\frac{\|f\|_{\widetilde{L}_{1,\lambda}}}{\beta}\right)^{\frac{Q}{Q-\alpha}},\ \text{if } 2t\ge 1.$$

Finally we have

$$|\{v\in B(u,t)\ :\ |I_\alpha f(v)|>2\beta\}| \lesssim [t]_1^\lambda \min\left\{\left(\frac{\|f\|_{\widetilde{L}_{1,\lambda}}}{\beta}\right)^{\frac{Q-\lambda}{Q-\lambda-\alpha}}, \left(\frac{\|f\|_{\widetilde{L}_{1,\lambda}}}{\beta}\right)^{\frac{Q}{Q-\alpha}}\right\} \lesssim [t]_1^\lambda \left(\frac{1}{\beta}\,\|f\|_{\widetilde{L}_{1,\lambda}}\right)^q.$$

Necessity. Let the operators $\mathcal{I}_\alpha$ and I_α are bounded from $\widetilde{L}_{1,\lambda}(\mathbb{H}_n)$ to

$W\widetilde{L}_{q,\lambda}(\mathbb{H}_n)$. We have

$$\begin{aligned}\|\mathcal{I}_\alpha f_t\|_{W\widetilde{L}_{q,\lambda}} &= \sup_{r>0} r \sup_{u\in\mathbb{H}_n,\,\tau>0}\left([\tau]_1^{-\lambda}\int_{\{v\in B(u,\tau)\,:\,|\mathcal{I}_\alpha f_t(v)|>r\}} dV(v)\right)^{1/q}\\ &= \sup_{r>0} r \sup_{u\in\mathbb{H}_n,\,\tau>0}\left([\tau]_1^{-\lambda}\int_{\{v\in B(\delta_t u,\tau)\,:\,|\mathcal{I}_\alpha f(\delta_t v)|>rt^\alpha\}} dV(v)\right)^{1/q}\\ &= t^{-\alpha-\frac{Q}{q}}\sup_{\tau>0}\left(\frac{[t\tau]_1}{[\tau]_1}\right)^{\lambda/q}\sup_{r>0} rt^\alpha\\ &\times \sup_{u\in\mathbb{H}_n,\,\tau>0}\left([t\tau]_1^{-\lambda}\int_{\{v\in B(u,t\tau)\,:\,|\mathcal{I}_\alpha f(v)|>rt^\alpha\}} dV(v)\right)^{1/q}\\ &= t^{-\alpha-\frac{Q}{q}}[t]_{1,+}^{\frac{\lambda}{q}}\,\|\mathcal{I}_\alpha f\|_{W\widetilde{L}_{q,\lambda}}\,.\end{aligned}$$

By the boundedness of $\mathcal{I}_\alpha$ from $\widetilde{L}_{1,\lambda}(\mathbb{H}_n)$ to $W\widetilde{L}_{q,\lambda}(\mathbb{H}_n)$

$$\begin{aligned}\|\mathcal{I}_\alpha f\|_{W\widetilde{L}_{q,\lambda}} &= t^{\alpha+\frac{Q}{q}}[t]_{1,+}^{-\frac{\lambda}{q}}\,\|\mathcal{I}_\alpha f_t\|_{W\widetilde{L}_{q,\lambda}}\\ &\lesssim t^{\alpha+\frac{Q}{q}}[t]_{1,+}^{-\frac{\lambda}{q}}\,\|f_t\|_{\widetilde{L}_{1,\lambda}}\\ &= t^{\alpha+\frac{Q}{q}-Q}[t]_{1,+}^{\lambda-\frac{\lambda}{q}}\,\|f\|_{\widetilde{L}_{1,\lambda}}.\end{aligned}$$

If $1<\frac{1}{q}+\frac{\alpha}{Q}$, then in the case $t\to 0$ we have $\|\mathcal{I}_\alpha f\|_{W\widetilde{L}_{q,\lambda}}=0$ for all $f\in\widetilde{L}_{1,\lambda}(\mathbb{H}_n)$.

Similarly, if $1>\frac{1}{q}+\frac{\alpha}{Q-\lambda}$, then for $t\to\infty$ we obtain $\|\mathcal{I}_\alpha f\|_{W\widetilde{L}_{q,\lambda}}=0$ for all $f\in\widetilde{L}_{1,\lambda}(\mathbb{H}_n)$.

Therefore $\frac{\alpha}{Q}\le 1-\frac{1}{q}\le\frac{\alpha}{Q-\lambda}$. Analogously we get the last equality for the I_α.

Thus Theorem 2.1 is proved.

Proof of Corollary 2.1.

Sufficiency of Corollary 2.1 follows from Theorem 2.1 and inequality (3).

Necessity. (1) Let M_α be bounded from $\widetilde{L}_{p,\lambda}(\mathbb{H}_n)$ to $\widetilde{L}_{q,\lambda}(\mathbb{H}_n)$ for $1<p<\frac{Q-\lambda}{\alpha}$. Then we have

$$M_\alpha f_t(u)=t^{-\alpha}M_\alpha f(\delta_t u),$$

and

$$\|M_\alpha f_t\|_{\widetilde{L}_{q,\lambda}}=t^{-\alpha-\frac{Q}{q}}[t]_{1,+}^{\frac{\lambda}{q}}\,\|M_\alpha f\|_{\widetilde{L}_{q,\lambda}}\,.$$

By the same argument in Theorem 2.1 we obtain $\frac{\alpha}{Q} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{Q-\lambda}$.
(2) Let M_α be bounded from $\widetilde{L}_{1,\lambda}(\mathbb{H}_n)$ to $W\widetilde{L}_{q,\lambda}(\mathbb{H}_n)$. Then

$$\|M_\alpha f_t\|_{W\widetilde{L}_{q,\lambda}} = t^{-\alpha-\frac{Q}{q}} [t]_{1,+}^{\frac{\lambda}{q}} \|M_\alpha f\|_{W\widetilde{L}_{q,\lambda}}.$$

Hence we obtain $\frac{\alpha}{Q} \le 1 - \frac{1}{q} \le \frac{\alpha}{Q-\lambda}$.
(3) Let $\frac{Q-\lambda}{\alpha} \le p \le \frac{Q}{\alpha}$. Then by the Hölder's inequality we have

$$\begin{aligned}
\|M_\alpha f\|_{L_\infty} &= 2^{-Q} \sup_{u\in\mathbb{H}_n,\, t>0} t^{\alpha-Q} \int_{B(u,t)} |f(v)| dV(v) \\
&\le 2^{-\frac{Q}{p}} \sup_{u\in\mathbb{H}_n,\, t>0} t^{\alpha-\frac{Q}{p}} [t]_1^{\frac{\lambda}{p}} \left([t]_1^{-\lambda} \int_{B(u,t)} |f(v)|^p dV(v) \right)^{1/p} \\
&\le 2^{-\frac{Q}{p}} \|f\|_{\widetilde{L}_{p,\lambda}} \sup_{t>0} t^{\alpha-\frac{Q}{p}} [t]_1^{\frac{\lambda}{p}} \\
&= 2^{-\frac{Q}{p}} \|f\|_{\widetilde{L}_{p,\lambda}} \max\left\{ \sup_{0<t\le 1} t^{\alpha-\frac{Q-\lambda}{p}}, \sup_{t>1} t^{\alpha-\frac{Q}{p}} \right\} = 2^{-\frac{Q}{p}} \|f\|_{\widetilde{L}_{p,\lambda}}.
\end{aligned}$$

Thus Corollary 2.1 is proved.

Proof of Theorem 2.2.
Let $0 < \alpha < Q$, $0 \le \lambda < Q - \alpha$, $\frac{Q-\lambda}{\alpha} \le p \le \frac{Q}{\alpha}$ and $f \in \widetilde{L}_{p,\lambda}(\mathbb{H}_n)$. For given $t > 0$ we denote

$$f_1(u) = f(u)\chi_{B(e,2t)}(u), \quad f_2(u) = f(u) - f_1(u). \tag{13}$$

Then

$$\widetilde{I}_\alpha f(u) = \widetilde{I}_\alpha f_1(u) + \widetilde{I}_\alpha f_2(u) = F_1(u) + F_2(u),$$

where

$$F_1(u) = \int_{B(e,2t)} \left(|v^{-1}u|^{\alpha-Q} - |v|^{\alpha-Q} \chi_{\complement B(e,1)}(v) \right) dV(v),$$

$$F_2(u) = \int_{\complement B(e,2t)} \left(|v^{-1}u|^{\alpha-Q} - |v|^{\alpha-Q} \chi_{\complement B(e,1)}(v) \right) dV(v)$$

Note that the function f_1 has compact (bounded) support and thus

$$a_1 = -\int_{B(e,2t)\setminus B(e,\min\{1,2t\})} |v|^{\alpha-Q} dV(v)$$

is finite.

Note also that

$$F_1(u) - a_1 = \int_{B(e,2t)} |v^{-1}u|^{\alpha-Q} f(v)dV(v) - \int_{B(e,2t)\setminus B(e,\min\{1,2t\})} |v|^{\alpha-Q} f(v)dV(v)$$

$$+ \int_{B(e,2t)\setminus B(e,\min\{1,2t\})} |v|^{\alpha-Q} f(v)dV(v) = \int_{\mathbb{H}_n} |v^{-1}u|^{\alpha-Q} f_1(v)dV(v) = I_\alpha f_1(u).$$

Therefore

$$|F_1(u) - a_1| \le \int_{\mathbb{H}_n} |v|^{\alpha-Q} \left|f_1(v^{-1}u)\right| dV(v) = \int_{B(u,2t)} |v|^{\alpha-Q} \left|f(v^{-1}u)\right| dV(v).$$

Further, for $u \in B(e,t)$, $v \in B(u,2t)$ we have

$$|v| \le |u| + |v^{-1}u| < 3t.$$

Consequently for $u \in B(e,t)$

$$|F_1(u) - a_1| \le \int_{B(e,3t)} |v|^{\alpha-Q} \left|f(v^{-1}u)\right| dV(v). \tag{14}$$

Then

$$\begin{aligned}
&|B(e,t)|^{-1} \int_{B(e,t)} \left|F_1(w^{-1}u) - a_1\right| dV(w) \\
&\lesssim t^{-Q} \int_{B(e,t)} \left(\int_{B(e,3t)} |v|^{\alpha-Q} |f(v^{-1}w^{-1}u)| dV(v) \right) dV(w) \\
&\lesssim t^{-Q} \int_{B(e,3t)} \left(\int_{B(e,t)} |f(v^{-1}w^{-1}u)| dV(w) \right) |v|^{\alpha-Q} dV(v) \\
&= t^{-Q} \int_{B(e,3t)} \left(\int_{B(u,t)} |f(wv)| dV(w) \right) |v|^{\alpha-Q} dV(v) \\
&\lesssim t^{-Q} t^{Q-\alpha} \|f\|_{\widetilde{L}_{1,Q-\alpha}} \int_{B(e,3t)} |v|^{\alpha-Q} dV(v) \\
&\approx \|f\|_{L_{1,Q-\alpha}} .
\end{aligned} \tag{15}$$

Denote

$$a_2 = \int_{B(e,\max\{1,2t\})\setminus B(e,2t)} |v|^{\alpha-Q} f(v)dV(v).$$

and estimate $|F_2(u) - a_2|$ for $u \in B(e,t)$:

$$|F_2(u) - a_2| \le \int_{\complement_{B(e,2t)}} |f(v)| \left| |v^{-1}u|^{\alpha-Q} - |v|^{\alpha-Q} \right| dV(v).$$

Applying Lemma 4.3 we have

$$
\begin{aligned}
|F_2(u) - a_2| &\leq 2^{Q-\alpha+1}|u| \int_{\complement_{B(e,2t)}} |f(v)||v|^{\alpha-Q-1} dV(v) \\
&= 2^{Q-\alpha+1}|u| \sum_{j=1}^{\infty} \int_{B(e,2^{j+1}t)\setminus B(e,2^j t)} |f(v)||v|^{\alpha-Q-1} dV(v) \\
&\lesssim |u| \sum_{j=1}^{\infty} \left(2^j t\right)^{\alpha-Q-1} \left(2^{j+1} t\right)^{Q-\alpha} \|f\|_{L_{1,Q-\alpha}} \\
&\approx |u|\, t^{-1} \|f\|_{L_{1,Q-\alpha}} . \qquad (16)
\end{aligned}
$$

Denote

$$
a_f = a_1 + a_2 = \int_{B(e,\max\{1,2t\})} |v|^{\alpha-Q} dV(v).
$$

Finally, from (15) and (16) we have

$$
\sup_{u,t} \frac{1}{|B(e,t)|} \int_{B(e,t)} \left| \widetilde{I}_\alpha f(v^{-1}u) - a_f \right| dV(v) \lesssim \|f\|_{L_{1,Q-\alpha}}. \qquad (17)
$$

By the Hölder's inequality we have

$$
\begin{aligned}
\|f\|_{L_{1,Q-\alpha}} &= \sup_{u\in\mathbb{H}_n,\, t>0} t^{\alpha-Q} \int_{B(u,t)} |f(v)| dV(v) \\
&\leq 2^{\frac{n}{p'}} \sup_{u\in\mathbb{H}_n,\, t>0} t^{\alpha-\frac{Q}{p}} [t]_1^{\frac{\lambda}{p}} \left([t]_1^{-\lambda} \int_{B(u,t)} |f(v)|^p dV(v) \right)^{1/p} \\
&\leq 2^{\frac{n}{p'}} \|f\|_{\widetilde{L}_{p,\lambda}} \sup_{t>0} t^{\alpha-\frac{Q}{p}} [t]_1^{\frac{\lambda}{p}} \qquad (18) \\
&= 2^{\frac{n}{p'}} \|f\|_{\widetilde{L}_{p,\lambda}} \max\left\{ \sup_{0<t\leq 1} t^{\alpha-\frac{Q-\lambda}{p}}, \sup_{t>1} t^{\alpha-\frac{Q}{p}} \right\} \\
&= 2^{\frac{n}{p'}} \|f\|_{\widetilde{L}_{p,\lambda}} .
\end{aligned}
$$

Finally for $\frac{Q-\lambda}{\alpha} \leq p \leq \frac{Q}{\alpha}$ and $f \in \widetilde{L}_{p,\lambda}(\mathbb{H}_n)$, from (17) and (18) we get

$$
\left\| \widetilde{I}_\alpha f \right\|_{BMO} \leq 2 \sup_{u,t} \frac{1}{|B(e,t)|} \int_{B(e,t)} \left| \widetilde{I}_\alpha f(v^{-1}u) - a_f \right| dV(v) \lesssim \|f\|_{\widetilde{L}_{p,\lambda}}.
$$

Thus Theorem 2.2 is proved.

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Vagif S. Guliyev,
Ahi Evran University, Kirsehir, Turkey
and
Institute of Mathematics and Mechanics,
Baku, Azerbaijan
Email: vagif@guliyev.com

Yagub Y. Mammadov,
Institute of Mathematics and Mechanics,
Baku, Azerbaijan
and
Nakhchivan Teacher-Training Institute,
Nakhchivan, Azerbaijan
Email: yagubmammadov@yahoo.com