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Boundedness of the Anisotropic Maximal and Anisotropic Singular Integral Operators in Generalized Morrey Spaces

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Abstract In this paper we give the conditions on the pair (ω_1, ω_2) which ensures the boundedness of the anisotropic maximal operator and anisotropic singular integral operators from one generalized Morrey space $\mathcal{M}_{p,\omega_1}$ to another $\mathcal{M}_{p,\omega_2}$, $1 < p < \infty$, and from the space $\mathcal{M}_{1,\omega_1}$ to the weak space $W\mathcal{M}_{1,\omega_2}$.

Keywords Generalized Morrey spaces, anisotropic maximal operator, Hardy operator, anisotropic singular integral operator

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1 Introduction and main results

The theory of boundedness of classical operators of real analysis, such as maximal operator and singular integral operators etc, from one weighted Lebesgue space to another one is well studied by now. These results have good applications in the theory of partial differential equations. However, in the theory of partial differential equations, along with weighted Lebesgue spaces, general Morrey-type spaces also play an important role.

Let $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \geq 1$, $i = 1, \dots, n$, $|\alpha| = \sum_{i=1}^n \alpha_i$ and $t^\alpha x \equiv (t^{\alpha_1} x_1, \dots, t^{\alpha_n} x_n)$. Following [1, 2], the function $F(x, \rho) = \sum_{i=1}^n x_i^2 \rho^{-2\alpha_i}$, considered for any fixed $x \in \mathbb{R}^n$, is a decreasing one with respect to $\rho > 0$ and the equation $F(x, \rho) = 1$ is uniquely solvable in $\rho(x)$. It is a simple matter to check that $\rho(x-y)$ defines a distance between any two points x, $y \in \mathbb{R}^n$. Thus $\mathbb{R}^n$, endowed with the metric ρ, results in a homogeneous metric space (see [1–3]). The balls with respect to $\rho(x)$, centered at the origin and of radius r are simply the ellipsoids

$$\mathcal{E}_r(0) = \left\{ x \in \mathbb{R}^n : \frac{x_1^2}{r^{2\alpha_1}} + \cdots + \frac{x_n^2}{r^{2\alpha_n}} < 1 \right\},$$

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with Lebesgue measure $|\mathcal{E}_r(0)| = C(n)r^{|\alpha|}$. It is easy to see that $\mathcal{E}_1(0) \equiv \mathbb{S}^{n-1}$ with respect to the Euclidean one.

Let $f \in L_1^{\rm loc}(\mathbb{R}^n)$. The anisotropic maximal function Mf is defined by

$$Mf(x) = \sup_{t>0} \frac{1}{|\mathcal{E}(x,t)|} \int_{\mathcal{E}(x,t)} |f(y)|dy,$$

where $|\mathcal{E}(x,t)|$ is the Lebesgue measure of the ellipsoid $\mathcal{E}(x,t)$ centered at x.

The boundedness of the maximal operator M in Morrey spaces $\mathcal{M}_{p,\lambda}$ was proven in [4] (isotropic case) and in generalized Morrey spaces $\mathcal{M}_{p,\omega}$, $p \in (1,\infty)$ with a function $\omega(x,r)$ satisfying suitable doubling and integral conditions $\widetilde{\mathcal{Z}}_{p,|\alpha|}$ (see Section 2) in [5]. In more general substations, namely in local and global Morrey type spaces, the boundedness of the maximal operator M has been investigated in [6–14].

Definition 1.1 *The function $k(x;\xi) : \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\}) \to \mathbb{R}$ is called a variable Calderón–Zygmund type kernel with mixed homogeneity if*

i) *For every fixed x the function $k(x;\cdot)$ is a constant kernel satisfying*:

$\mathrm{i_a}$) $k(x;\cdot) \in C^\infty(\mathbb{R}^n\backslash\{0\})$;

$\mathrm{i_b}$) $k(x;t^\alpha\xi) = \mu^{-|\alpha|}k(x;\xi),\ t>0$;

$\mathrm{i_c}$) $\int_{\mathbb{S}^{n-1}} k(x;\xi)d\sigma_\xi = 0$ *and* $\int_{\mathbb{S}^{n-1}} |k(x;\xi)|d\sigma_\xi < \infty$;

ii) *For every multiindex β, the inequality* $\sup_{\xi\in\mathbb{S}^{n-1}} |D_\xi^\beta k(x;\xi)| \le C(\beta)$ *is satisfied independently of x.*

Note that in the isotropic case $\alpha_i = 1$, $i = 1,\ldots,n$ and thus $|\alpha| = n$, Definition 1.1 gives rise to the classical Calderón–Zygmund kernels (see, for example, [15] and [16]). One more example is when $\alpha_1 = \cdots = \alpha_{n-1} = 1$, $\alpha_n = \overline{\alpha} \ge 1$. In this case we obtain the parabolic kernels studied by Jones in [17] and discussed in [2].

We consider the following anisotropic singular integral

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} k(x;x-y)f(y)dy \tag{1.1}$$

with a variable Calderón–Zygmund type kernel $k(x,\xi)$, $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n \setminus \{0\}$, satisfying a mixed homogeneity condition $\mathrm{i_b}$). The boundedeness of the operator T in $L_p(\mathbb{R}^n)$, $p \in (1,\infty)$ was proven in [1, 2] and in Morrey spaces $\mathcal{M}_{p,\lambda}$ in [18] (isotropic case). The boundedness of the operator T in generalized Morrey spaces $\mathcal{M}_{p,\omega}$, $p \in (1,\infty)$ with a function $\omega(x,r)$ satisfying suitable doubling and integral conditions $\widetilde{\mathcal{Z}}_{p,|\alpha|}$ in [19] (isotropic case in [5]), and the boundedness of the operator T from $\mathcal{M}_{p,\omega_1}$ to $\mathcal{M}_{p,\omega_2}$, $1<p<\infty$ satisfying integral conditions $(\omega_1,\omega_2) \in \widetilde{\mathcal{Z}}_{p,|\alpha|}$ were proven in [12, 13]. Our goal is to extend results in [6, 12–14] with a pair (ω_1,ω_2) satisfying more large integral conditions $\mathcal{Z}_{p,|\alpha|}$. In [7–13] the boundedness of the singular integral operators in local and global Morrey-type spaces has been investigated. Note that the global Morrey-type space is a more general space than the generalized Morrey space.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of the appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Generalized Morrey Spaces and Preliminary Results

Morrey spaces $\mathcal{M}_{p,\lambda}$ were introduced by Morrey in 1938 [20] and defined as follows:

For $0 \le \lambda \le n$, $1 \le p \le \infty$, $f \in \mathcal{M}_{p,\lambda}$ if $f \in L_p^{\rm loc}(\mathbb{R}^n)$ and

$$\|f\|_{\mathcal{M}_{p,\lambda}} \equiv \|f\|_{\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty,$$

where $B(x,r)$ is the open ball centered at x of radius r. Note that $\mathcal{M}_{p,0} = L_p(\mathbb{R}^n)$ and $\mathcal{M}_{p,n} = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $\mathcal{M}_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

These spaces appeared to be quite useful in the study of the local behaviour of solutions to partial differential equations, apriori estimates and other topics in the theory of partial differential equations.

We also denote by $W\mathcal{M}_{p,\lambda}$ the weak Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{W\mathcal{M}_{p,\lambda}} \equiv \|f\|_{W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where WL_p denotes the weak L_p-space.

If in place of the power function r^λ in the definition of $\mathcal{M}_{p,\lambda}$ we consider any positive measurable function $\omega(x,r)$, then it becomes the generalized Morrey space $\mathcal{M}_{p,\omega}$.

Definition 2.1 *Let $\omega(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $\mathcal{M}_{p,\omega}$ the generalized Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{\mathcal{M}_{p,\omega}} \equiv \|f\|_{\mathcal{M}_{p,\omega}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \omega(x,r)^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))}.$$

Denote by $W\mathcal{M}_{p,\omega}$ the weak generalized Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{W\mathcal{M}_{p,\omega}} \equiv \|f\|_{W\mathcal{M}_{p,\omega}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} \omega(x,r)^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r))} < \infty.$$

Definition 2.2 *Let $\omega_1(x,r)$, $\omega_2(x,r)$ be two positive measurable functions on $\mathbb{R}^n \times (0,\infty)$. We say that (ω_1,ω_2) belongs to the class $\mathcal{Z}_{p,m}$, $p \in [0,\infty)$, $m > 0$, if there is a constant C such that, for any $x \in \mathbb{R}^n$ and for any $t > 0$,*

$$\left(\int_t^\infty \left(\frac{\operatorname{ess\,inf}_{r<s<\infty} \omega_1(x,s)}{r^m}\right)^{\frac{1}{p}} \frac{dr}{r}\right)^p \le C\,\frac{\omega_2(x,t)}{t^m}, \quad \text{if } \ p \in (0,\infty) \tag{2.1}$$

and

$$\operatorname*{ess\,sup}_{t<r<\infty} \frac{\operatorname{ess\,inf}_{r<s<\infty} \omega_1(x,s)}{r^m} \le C\frac{\omega_2(x,t)}{t^m}, \quad \text{if } \ p = 0. \tag{2.2}$$

Definition 2.3 *Let $\omega_1(x,r)$, $\omega_2(x,r)$ be two positive measurable functions on $\mathbb{R}^n \times (0,\infty)$. We say that (ω_1,ω_2) belongs to the class $\widetilde{\mathcal{Z}}_{p,m}$, $p \in [0,\infty)$, $m > 0$ if there is a constant C such that, for any $x \in \mathbb{R}^n$ and for any $t > 0$,*

$$\left(\int_t^\infty \left(\frac{\omega_1(x,r)}{r^m}\right)^{\frac{1}{p}} \frac{dr}{r}\right)^p \le C\frac{\omega_2(x,t)}{t^m}, \quad \text{if } \ p \in (0,\infty) \tag{2.3}$$

and

$$\operatorname*{ess\,sup}_{t<r<\infty} \frac{\omega_1(x,r)}{r^m} \le C\frac{\omega_2(x,t)}{t^m}, \quad \text{if } \ p = 0. \tag{2.4}$$

Note that $\widetilde{\mathcal{Z}}_{p,m} \subset \mathcal{Z}_{p,m}$ for $p \in [0,\infty)$, $m > 0$.

The following property for the class $\mathcal{Z}_{p,m}$, $p \in [0,\infty)$, $m > 0$ is valid.

Lemma 2.4 ([6]) *Let $m > 0$. Then*

$$\bigcup_{0<p<\infty} \mathcal{Z}_{p,m} \subset \mathcal{Z}_{0,m}.$$

Proof Assume that $(\omega_1, \omega_2) \in \mathcal{Z}_{p,m}$ for some $p \in (0, \infty)$. Then for any $s \in (t, \infty)$,

$$\begin{aligned}
\frac{\omega_2(x,t)}{t^m} &\gtrsim \left(\int_t^\infty \left(\frac{\operatorname{ess\,inf}_{r<\tau<\infty} \omega_1(x,\tau)}{r^m} \right)^{\frac{1}{p}} \frac{dr}{r} \right)^p \\
&\gtrsim \left(\int_s^\infty \left(\frac{\operatorname{ess\,inf}_{r<\tau<\infty} \omega_1(x,\tau)}{r^m} \right)^{\frac{1}{p}} \frac{dr}{r} \right)^p \\
&\gtrsim \operatorname*{ess\,inf}_{s<\tau<\infty} \omega_1(x,\tau) \left(\int_s^\infty \frac{dr}{r^{\frac{m}{p}+1}} \right)^p \\
&\approx \frac{\operatorname{ess\,inf}_{s<\tau<\infty} \omega_1(x,\tau)}{s^m}.
\end{aligned}$$

Thus,

$$\frac{\omega_2(x,t)}{t^m} \gtrsim \operatorname*{ess\,sup}_{t<s<\infty} \frac{\operatorname{ess\,inf}_{s<\tau<\infty} \omega_1(x,\tau)}{s^m}.$$

This proves that

$$\bigcup_{0<p<\infty} \mathcal{Z}_{p,m} \subset \mathcal{Z}_{0,m}. \qquad \square$$

Remark Let $w(t) = t^n$. Then $(\omega, \omega) \in \mathcal{Z}_{0,n}$, but $(\omega, \omega) \notin \mathcal{Z}_{p,n}$ for any $p \in (0, \infty)$.

In [19] Softova proved the following statement, containing in the isotropic case Nakai's result in [5].

Theorem 2.5 *Let $1 \le p < \infty$. Moreover, let $\omega(t)$, $t > 0$, be a positive measurable function satisfying the following conditions: there exists $c > 0$ such that*

$$0 < r \le t \le 2r \Rightarrow c^{-1}\omega(r) \le \omega(t) \le c\omega(r) \tag{2.5}$$

and $(\omega, \omega) \in \widetilde{\mathcal{Z}}_{1,|\alpha|}$.

Then for $1 < p < \infty$ the operators M and T are bounded from $\mathcal{M}_{p,\omega}$ to $\mathcal{M}_{p,\omega}$ and for $p = 1$ the operators M and T are bounded from $\mathcal{M}_{1,\omega}$ to $W\mathcal{M}_{1,\omega}$.

The following statement, containing Softova results in [19] was proved by Guliyev in [12] for singular integrals defined on homogeneous Folland–Stein groups [21] (see also [13, 14]).

Theorem 2.6 *Let $1 \le p < \infty$ and $(\omega_1, \omega_2) \in \widetilde{\mathcal{Z}}_{p,|\alpha|}(\mathbb{R}^n)$. Then for $1 < p < \infty$ the operators M and T are bounded from $\mathcal{M}_{p,\omega_1}$ to $\mathcal{M}_{p,\omega_2}$ and for $p = 1$ the operators M and T are bounded from $\mathcal{M}_{1,\omega_1}$ to $W\mathcal{M}_{1,\omega_2}$.*

Sufficient conditions on ω for the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$ have been obtained in [5–14, 19, 22–28].

Let $\mathfrak{M}(0, \infty)$ be the set of all Lebesgue-measurable functions on $(0, \infty)$ and $\mathfrak{M}^+(0, \infty)$ its subset consisting of all nonnegative functions on $(0, \infty)$. We denote by $\mathfrak{M}^+(0, \infty; \uparrow)$ the cone of all functions in $\mathfrak{M}^+(0, \infty)$ which are non-decreasing on $(0, \infty)$ and

$$\mathbb{A} = \left\{ \varphi \in \mathfrak{M}^+(0, \infty; \uparrow) : \lim_{t \to 0+} \varphi(t) = 0 \right\}.$$

Let u be a continuous and non-negative function on $(0,\infty)$. We define the supremal operator $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(t,\infty)}, \quad t \in (0,\infty).$$

The following theorem was proved in [10].

Theorem 2.7 *Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L_\infty(t,\infty)} < \infty$ for any $t > 0$ and u be a continuous non-negative function on $(0,\infty)$.*

Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(0,\infty)$ to $L_{\infty,v_2}(0,\infty)$ on the cone $\mathbb{A}$ if and only if

$$\|v_2 \overline{S}_u(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)})\|_{L_\infty(0,\infty)} < \infty. \tag{2.6}$$

We are going to use the following statement on the boundedness of the Hardy operator

$$(Hg)(t) := \frac{1}{t}\int_0^t g(r)dr,\ 0 < t < \infty.$$

Theorem 2.8 ([29]) *The inequality*

$$\operatorname*{ess\,sup}_{t>0} w(t)Hg(t) \le c \operatorname*{ess\,sup}_{t>0} v(t)g(t) \tag{2.7}$$

holds for all non-negative and non-increasing g on $(0,\infty)$ if and only if

$$A := \sup_{t>0} \frac{w(t)}{t}\int_0^t \frac{ds}{\operatorname{ess\,sup}_{0<y<s} v(y)} < \infty, \tag{2.8}$$

and $c \approx A$.

3 The Anisotropic Maximal Operator in Generalized Morrey Spaces

We need the following two lemmas (see [14]).

Lemma 3.1 *Let $1 < p < \infty$. Then for any ellipsoid $\mathcal{E} = \mathcal{E}(x,r)$ in $\mathbb{R}^n$ the inequality*

$$\|Mf\|_{L_p(\mathcal{E}(x,r))} \lesssim \|f\|_{L_p(\mathcal{E}(x,2r))} + r^{\frac{|\alpha|}{p}} \sup_{t>2r} t^{-|\alpha|}\|f\|_{L_1(\mathcal{E}(x,t))} \tag{3.1}$$

holds for all $f \in L_p^{\rm loc}(\mathbb{R}^n)$.

Moreover, the inequality

$$\|Mf\|_{WL_1(\mathcal{E}(x,r))} \lesssim \|f\|_{L_1(\mathcal{E}(x,2r))} + r^{|\alpha|} \sup_{t>2r} t^{-|\alpha|}\|f\|_{L_1(\mathcal{E}(x,t))} \tag{3.2}$$

holds for all $f \in L_1^{\rm loc}(\mathbb{R}^n)$.

Proof Let $1 < p < \infty$. It is obvious that for any ellipsoid $\mathcal{E} = \mathcal{E}(x,r)$ the following inequality holds

$$\|Mf\|_{L_p(\mathcal{E})} \le \|M(f\chi_{(2\mathcal{E})})\|_{L_p(\mathcal{E})} + \|M(f\chi_{\mathbb{R}^n\setminus(2\mathcal{E})})\|_{L_p(\mathcal{E})}.$$

By continuity of the operator $M : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)$, $1 < p < \infty$ we have

$$\|M(f\chi_{(2\mathcal{E})})\|_{L_p(\mathcal{E})} \lesssim \|f\|_{L_p(2\mathcal{E})}.$$

Let y be an arbitrary point from $\mathcal{E}$. If $\mathcal{E}(y,t) \cap \{\mathbb{R}^n\setminus(2\mathcal{E})\} \ne \emptyset$, then $t > r$. Indeed, if $z \in \mathcal{E}(y,t) \cap \{\mathbb{R}^n\setminus(2\mathcal{E})\}$, then $t > \rho(z,y) \ge \rho(z,x) - \rho(x,y) > 2r - r = r$.

On the other hand $\mathcal{E}(y,t)\cap\{\mathbb{R}^n\backslash(2\mathcal{E})\}\subset\mathcal{E}(x,2t)$. Indeed, $z\in\mathcal{E}(y,t)\cap\{\mathbb{R}^n\backslash(2\mathcal{E})\}$, then we get $\rho(z,x)\le\rho(z,y)+\rho(y,x)<t+r<2t$. Hence

$$\begin{aligned} M(f\chi_{\mathbb{R}^n\backslash(2\mathcal{E})})(y) &= \sup_{t>0}\frac{1}{|\mathcal{E}(y,t)|}\int_{\mathcal{E}(y,t)\cap\{\mathbb{R}^n\backslash(2\mathcal{E})\}}|f(z)|dz \\ &\le 2^{|\alpha|}\sup_{t>r}\frac{1}{|\mathcal{E}(x,2t)|}\int_{\mathcal{E}(x,2t)}|f(z)|dz \\ &= 2^{|\alpha|}\sup_{t>2r}\frac{1}{|\mathcal{E}(x,t)|}\int_{\mathcal{E}(x,t)}|f(z)|dz. \end{aligned}$$

Therefore, for all $y\in\mathcal{E}$ we have

$$M(f\chi_{\mathbb{R}^n\backslash(2\mathcal{E})})(y)\le 2^{|\alpha|}\sup_{t>2r}\frac{1}{|\mathcal{E}(x,t)|}\int_{\mathcal{E}(x,t)}|f(z)|dz. \tag{3.3}$$

Thus

$$\|Mf\|_{L_p(\mathcal{E})}\lesssim\|f\|_{L_p(2\mathcal{E})}+|\mathcal{E}|^{\frac{1}{p}}\bigg(\sup_{t>2r}\frac{1}{|\mathcal{E}(x,t)|}\int_{\mathcal{E}(x,t)}f(y)dy\bigg).$$

Let $p=1$. It is obvious that for any ellipsoid $\mathcal{E}=\mathcal{E}(x,r)$ the following inequality holds:

$$\|Mf\|_{WL_1(\mathcal{E})}\le\|M(f\chi_{(2\mathcal{E})})\|_{WL_1(\mathcal{E})}+\|M(f\chi_{\mathbb{R}^n\backslash(2\mathcal{E})})\|_{WL_1(\mathcal{E})}.$$

By continuity of the operator $M:L_1(\mathbb{R}^n)\to WL_1(\mathbb{R}^n)$ we have

$$\|M(f\chi_{(2\mathcal{E})})\|_{WL_1(\mathcal{E})}\lesssim\|f\|_{L_1(2\mathcal{E})}.$$

Then by (3.3), we get the inequality (3.2). □

Lemma 3.2 *Let $1<p<\infty$. Then for any ellipsoid $\mathcal{E}=\mathcal{E}(x,r)$ in $\mathbb{R}^n$, the inequality*

$$\|Mf\|_{L_p(\mathcal{E}(x,r))}\lesssim r^{\frac{|\alpha|}{p}}\sup_{t>2r}t^{-\frac{|\alpha|}{p}}\|f\|_{L_p(\mathcal{E}(x,t))} \tag{3.4}$$

holds for all $f\in L_p^{\rm loc}(\mathbb{R}^n)$.

Moreover, the inequality

$$\|Mf\|_{WL_1(\mathcal{E}(x,r))}\lesssim r^{|\alpha|}\sup_{t>2r}t^{-|\alpha|}\|f\|_{L_1(\mathcal{E}(x,t))} \tag{3.5}$$

holds for all $f\in L_1^{\rm loc}(\mathbb{R}^n)$.

Proof Let $1<p<\infty$. Denote

$$\mathcal{M}_1:=|\mathcal{E}|^{\frac{1}{p}}\bigg(\sup_{t>2r}\frac{1}{|\mathcal{E}(x,t)|}\int_{\mathcal{E}(x,t)}|f(y)|dy\bigg),\quad \mathcal{M}_2:=\|f\|_{L_p(2\mathcal{E})}.$$

Applying Hölder's inequality, we get

$$\mathcal{M}_1\lesssim|\mathcal{E}|^{\frac{1}{p}}\bigg(\sup_{t>2r}\frac{1}{|\mathcal{E}(x,t)|^{\frac{1}{p}}}\bigg(\int_{\mathcal{E}(x,t)}|f(y)|^p dy\bigg)^{\frac{1}{p}}\bigg).$$

On the other hand,

$$\begin{aligned} &|\mathcal{E}|^{\frac{1}{p}}\bigg(\sup_{t>2r}\frac{1}{|\mathcal{E}(x,t)|^{\frac{1}{p}}}\bigg(\int_{\mathcal{E}(x,t)}|f(y)|^p dy\bigg)^{\frac{1}{p}}\bigg) \\ &\quad\gtrsim|\mathcal{E}|^{\frac{1}{p}}\bigg(\sup_{t>2r}\frac{1}{|\mathcal{E}(x,t)|^{\frac{1}{p}}}\bigg)\|f\|_{L_p(2\mathcal{E})}\approx\mathcal{M}_2. \end{aligned}$$

Since by Lemma 3.1,

$$\|Mf\|_{L_p(\mathcal{E})}\le\mathcal{M}_1+\mathcal{M}_2,$$

we arrive at (3.4).

Let $p = 1$. The inequality (3.5) directly follows from (3.2). □

Theorem 3.3 *Let $p \in [1, \infty)$ and $(\omega_1, \omega_2) \in \mathcal{Z}_{0,|\alpha|}(\mathbb{R}^n)$. Then for $p > 1$ the operator M is bounded from $\mathcal{M}_{p,\omega_1}$ to $\mathcal{M}_{p,\omega_2}$, and for $p = 1$ the operator M is bounded from $\mathcal{M}_{1,\omega_1}$ to $W\mathcal{M}_{1,\omega_2}$.*

Proof By Lemma 3.2 and Theorem 2.7, we get

$$\|Mf\|_{\mathcal{M}_{p,\omega_2}(\mathbb{R}^n)} \lesssim \sup_{x\in\mathbb{R}^n, r>0} \omega_2(x,r)^{-\frac{1}{p}} r^{\frac{|\alpha|}{p}} \Big(\sup_{t>r} t^{-\frac{|\alpha|}{p}} \|f\|_{L_p(\mathcal{E}(x,t))} \Big)$$
$$\lesssim \sup_{x\in\mathbb{R}^n, r>0} \omega_1(x,r)^{-\frac{1}{p}} \|f\|_{L_p(\mathcal{E}(x,t))} = \|f\|_{\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)},$$

if $p \in (1, \infty)$; and

$$\|Mf\|_{W\mathcal{M}_{1,\omega_2}(\mathbb{R}^n)} \lesssim \sup_{x\in\mathbb{R}^n, r>0} \omega_2(x,r)^{-1} r^{|\alpha|} \Big(\sup_{t>r} t^{-|\alpha|} \|f\|_{L_1(\mathcal{E}(x,t))} \Big)$$
$$\lesssim \sup_{x\in\mathbb{R}^n, r>0} \omega_1(x,r)^{-1} \|f\|_{L_1(\mathcal{E}(x,t))} = \|f\|_{\mathcal{M}_{1,\omega_1}(\mathbb{R}^n)},$$

if $p = 1$. □

4 The Anisotropic Singular Integral Operator in Generalized Morrey Spaces

The following Lemma has been proved in [12] (see also [9, 13, 14]). For the sake of completeness, we give the proof.

Lemma 4.1 *Let $p \in [1, \infty)$, $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ and for any $x_0 \in \mathbb{R}^n$,*

$$\int_1^\infty t^{-\frac{|\alpha|}{p}-1} \|f\|_{L_p(\mathcal{E}(x_0,t))} dt < \infty.$$

Then Tf exists for a.e. $x \in \mathbb{R}^n$ and for any $x_0 \in \mathbb{R}^n$, $r > 0$ and $p \in (1, \infty)$

$$\|Tf\|_{L_p(\mathcal{E}(x_0,r))} \le C\, r^{\frac{|\alpha|}{p}} \int_{2r}^\infty t^{-\frac{|\alpha|}{p}-1} \|f\|_{L_p(\mathcal{E}(x_0,t))} dt, \tag{4.1}$$

where constant $C > 0$ does not depend on x_0, r and f.

Moreover, for any $x_0 \in \mathbb{R}^n$ and $r > 0$,

$$\|Tf\|_{WL_1(\mathcal{E}(x_0,r))} \le C\, r^{|\alpha|} \int_{2r}^\infty t^{-|\alpha|-1} \|f\|_{L_1(\mathcal{E}(x_0,t))} dt, \tag{4.2}$$

where constant $C > 0$ does not depend on x_0, r and f.

Proof Let $p \in (1, \infty)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $\mathcal{E} = \mathcal{E}(x_0, r)$ for the ellipsoid centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2\mathcal{E}}$ and $f_2 = f\chi_{\mathbb{R}^n\setminus(2\mathcal{E})}$. Since $f_1 \in L_p(\mathbb{R}^n)$, $Tf_1(x)$ exists for a.e. $x \in \mathbb{R}^n$ and from the boundedness of T in $L_p(\mathbb{R}^n)$ (see [18]) it follows that

$$\|Tf_1\|_{L_p(\mathcal{E})} \le \|Tf_1\|_{L_p(\mathbb{R}^n)} \le C\|f_1\|_{L_p(\mathbb{R}^n)} = C\|f\|_{L_p(2\mathcal{E})},$$

where the constant $C > 0$ is independent of f.

Now we prove that the non-singular integral $Tf_2(x)$ exists for all $x \in \mathcal{E}$.

It is clear that $x \in \mathcal{E}$, $y \in \mathbb{R}^n\setminus(2\mathcal{E})$ implies $\rho(x - y) \sim \rho(x_0 - y)$ and we get

$$|Tf_2(x)| \lesssim \int_{\mathbb{R}^n\setminus(2\mathcal{E})} \frac{|f(y)|}{\rho(x_0 - y)^{|\alpha|}} dy.$$

By Fubini's theorem, we have

$$\int_{\mathbb{R}^n\setminus(2\mathcal{E})} \frac{|f(y)|}{\rho(x_0-y)^{|\alpha|}}dy \approx \int_{\mathbb{R}^n\setminus(2\mathcal{E})} |f(y)| \int_{\rho(x_0-y)}^{\infty} \frac{dt}{t^{|\alpha|+1}}dy$$

$$\approx \int_{2r}^{\infty} \int_{2r\le\rho(x_0-y)<t} |f(y)|dy \frac{dt}{t^{|\alpha|+1}}$$

$$\lesssim \int_{2r}^{\infty} \int_{\mathcal{E}(x_0,t)} |f(y)|dy \frac{dt}{t^{|\alpha|+1}}.$$

Applying Hölder's inequality, we get

$$\int_{\mathbb{R}^n\setminus(2\mathcal{E})} \frac{|f(y)|}{\rho(x_0-y)^{|\alpha|}}dy \lesssim \int_{2r}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{|\alpha|}{p}+1}}.$$

Therefore $Tf_2(x)$ exists for all $x\in\mathcal{E}$. Since $\mathbb{R}^n=\bigcup_{r>0}\mathcal{E}(x_0,r)$, we get the existence of $Tf(x)$ for a.e. $x_0\in\mathbb{R}^n$.

Moreover, for all $p\in[1,\infty)$, the inequality

$$\|Tf_2\|_{L_p(\mathcal{E})} \lesssim r^{\frac{|\alpha|}{p}} \int_{2r}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{|\alpha|}{p}+1}} \tag{4.3}$$

is valid. Thus

$$\|Tf\|_{L_p(\mathcal{E})} \lesssim \|f\|_{L_p(2\mathcal{E})} + r^{\frac{|\alpha|}{p}} \int_{2r}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{|\alpha|}{p}+1}}.$$

On the other hand,

$$\|f\|_{L_p(2\mathcal{E})} \approx r^{\frac{|\alpha|}{p}} \|f\|_{L_p(2\mathcal{E})} \int_{2r}^{\infty} \frac{dt}{t^{\frac{|\alpha|}{p}+1}} \lesssim r^{\frac{|\alpha|}{p}} \int_{2r}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{|\alpha|}{p}+1}}.$$

Thus

$$\|Tf\|_{L_p(\mathcal{E})} \lesssim r^{\frac{|\alpha|}{p}} \int_{2r}^{\infty} \|f\|_{L_p(\mathcal{E}(x_0,t))} \frac{dt}{t^{\frac{|\alpha|}{p}+1}}.$$

Let $p=1$. From the weak $(1,1)$ boundedness of T (see [3]), it follows that

$$\|Tf_1\|_{WL_1(\mathcal{E})} \le \|Tf_1\|_{WL_1(\mathbb{R}^n)} \le C\|f_1\|_{L_1(\mathbb{R}^n)} = C\|f\|_{L_1(2\mathcal{E})},$$

where the constant $C>0$ is independent of f.

Then by (4.3) we get the inequality (4.2). □

Theorem 4.2 *Let $p\in[1,\infty)$ and $(\omega_1,\omega_2)\in\mathcal{Z}_{p,|\alpha|}$. Then the anisotropic singular integral Tf exists for a.e. $x\in\mathbb{R}^n$; and for $p>1$ the operator T is bounded from $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}_{p,\omega_2}(\mathbb{R}^n)$, and for $p=1$ the operator T is bounded from $\mathcal{M}_{1,\omega_1}(\mathbb{R}^n)$ to $W\mathcal{M}_{1,\omega_2}(\mathbb{R}^n)$. Moreover, for $p>1$,*

$$\|Tf\|_{\mathcal{M}_{p,\omega_2}} \lesssim \|f\|_{\mathcal{M}_{p,\omega_1}};$$

and for $p=1$,

$$\|Tf\|_{W\mathcal{M}_{1,\omega_2}} \lesssim \|f\|_{\mathcal{M}_{1,\omega_1}}.$$

Proof By Lemma 4.1 and Theorem 2.8, we have, for $p > 1$,

$$\begin{aligned}
\|Tf\|_{\mathcal{M}_{p,\omega_2}(\mathbb{R}^n)} &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \omega_2(x,r)^{-\frac{1}{p}} r^{\frac{|\alpha|}{p}} \int_r^\infty \|f\|_{L_p(\mathcal{E}(x,t))} \frac{dt}{t^{\frac{|\alpha|}{p}+1}} \\
&\approx \sup_{x\in\mathbb{R}^n,\, r>0} \omega_2(x,r)^{-\frac{1}{p}} r^{\frac{|\alpha|}{p}} \int_0^{r^{-\frac{|\alpha|}{p}}} \|f\|_{L_p(\mathcal{E}(x,t^{-\frac{p}{|\alpha|}}))} dt \\
&= \sup_{x\in\mathbb{R}^n,\, r>0} \omega_2(x,r^{-\frac{p}{|\alpha|}})^{-\frac{1}{p}} \frac{1}{r} \int_0^r \|f\|_{L_p(\mathcal{E}(x,t^{-\frac{p}{|\alpha|}}))} dt \\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0} \omega_1(x,r^{-\frac{p}{|\alpha|}})^{-\frac{1}{p}} \|f\|_{L_p(\mathcal{E}(x,r^{-\frac{p}{|\alpha|}}))} \\
&= \|f\|_{\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)};
\end{aligned}$$

and for $p = 1$,

$$\begin{aligned}
\|Tf\|_{W\mathcal{M}_{1,\omega_2}(\mathbb{R}^n)} &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \omega_2(x,r)^{-1} r^{|\alpha|} \int_r^\infty \|f\|_{L_1(\mathcal{E}(x,t))} \frac{dt}{t^{n+1}} \\
&\approx \sup_{x\in\mathbb{R}^n,\, r>0} \omega_2(x,r)^{-1} r^{|\alpha|} \int_0^{r^{-|\alpha|}} \|f\|_{L_1(\mathcal{E}(x,t^{-|\alpha|}))} dt \\
&= \sup_{x\in\mathbb{R}^n,\, r>0} \omega_2(x,r^{-\frac{1}{|\alpha|}})^{-1} \frac{1}{r} \int_0^r \|f\|_{L_1(\mathcal{E}(x,t^{-\frac{1}{|\alpha|}}))} dt \\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0} \omega_1(x,r^{-\frac{1}{|\alpha|}})^{-1} \|f\|_{L_1(\mathcal{E}(x,r^{-\frac{1}{|\alpha|}}))} \\
&= \|f\|_{\mathcal{M}_{1,\omega_1}(\mathbb{R}^n)}. \qquad \square
\end{aligned}$$

Corollary 4.3 ([12]) *Let $p \in [1,\infty)$ and $(\omega_1,\omega_2) \in \widetilde{\mathcal{Z}}_{p,|\alpha|}(\mathbb{R}^n)$. Then for $p > 1$ the operator T is bounded from $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}_{p,\omega_2}(\mathbb{R}^n)$, and for $p = 1$ the operator T is bounded from $\mathcal{M}_{1,\omega_1}$ to $W\mathcal{M}_{1,\omega_2}$.*

Note that Theorem 2.6 and Corollary 4.3 coincide.

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