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Boundedness of Sublinear Operators and Commutators on Generalized Morrey Spaces

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Dedicated to 70th birthday of Prof. S. Samko

Abstract. In this paper the authors study the boundedness for a large class of sublinear operators $T_\alpha, \alpha \in [0, n)$ generated by Calderón–Zygmund operators ($\alpha = 0$) and generated by Riesz potential operator ($\alpha > 0$) on generalized Morrey spaces $M_{p,\varphi}$. As an application of the above result, the boundeness of the commutator of sublinear operators $T_{b,\alpha}, \alpha \in [0, n)$ on generalized Morrey spaces is also obtained. In the case $b \in BMO$ and $T_{b,\alpha}$ is a sublinear operator, we find the sufficient conditions on the pair (φ_1, φ_2) which ensures the boundedness of the operators $T_{b,\alpha}, \alpha \in [0, n)$ from one generalized Morrey space M_{p,φ_1} to another M_{q,φ_2} with $1/p - 1/q = \alpha/n$. In all the cases the conditions for the boundedness are given in terms of Zygmund-type integral inequalities on (φ_1, φ_2), which do not assume any assumption on monotonicity of φ_1, φ_2 in r. Conditions of these theorems are satisfied by many important operators in analysis, in particular, Littlewood–Paley operator, Marcinkiewicz operator and Bochner–Riesz operator.

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Keywords. Sublinear operator, generalized Morrey space, Calderón–Zygmund operator, Riesz potential operator, fractional maximal operator, commutator, BMO, Littlewood–Paley operator, Marcinkiewicz operator, Bochner–Riesz operator.

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1. Introduction

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at x of radius r, ${}^{\complement}B(x,r)$ denote its complement and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$.

Let $f \in L_1^{\rm loc}(\mathbb{R}^n)$. The fractional maximal operator M_α and the Riesz potential I_α are defined by

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int\limits_{B(x,t)} |f(y)|dy, \qquad 0 \le \alpha < n,$$

$$I_\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha}}, \qquad 0 < \alpha < n.$$

If $\alpha = 0$, then $M \equiv M_0$ is the Hardy–Littlewood maximal operator.

Let K be a Calderón–Zygmund singular integral operator, briefly a Calderón–Zygmund operator, i.e., a linear operator bounded from $L_2(\mathbb{R}^n)$ to $L_2(\mathbb{R}^n)$ taking all infinitely continuously differentiable functions f with compact support to the functions $f \in L_1^{\rm loc}(\mathbb{R}^n)$ represented by

$$Kf(x) = \int\limits_{\mathbb{R}^n} k(x,y) f(y)\, dy \qquad \text{a.e. off supp} f.$$

Here $k(x,y)$ is a continuous function away from the diagonal which satisfies the standard estimates: there exist $c_1 > 0$ and $0 < \varepsilon \le 1$ such that

$$|k(x,y)| \le c_1 |x-y|^{-n}$$

for all $x, y \in \mathbb{R}^n$, $x \ne y$, and

$$|k(x,y) - k(x',y)| + |k(y,x) - k(y,x')| \le c_1 \left(\frac{|x-x'|}{|x-y|} \right)^{\varepsilon} |x-y|^{-n}$$

whenever $2|x-x'| \le |x-y|$. Such operators were introduced in [13].

It is well known that fractional maximal operator, Riesz potential and Calderón–Zygmund operators play an important role in harmonic analysis (see [19,32,42,44]).

Suppose that $T \equiv T_0$ represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin suppf$

$$|Tf(x)| \le c_0 \int\limits_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy, \tag{1.1}$$

where c_0 is independent of f and x. Similarly, we assume that $T_\alpha, \alpha \in (0,n)$ represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin suppf$

$$|T_\alpha f(x)| \le c_1 \int\limits_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \tag{1.2}$$

for some $\alpha \in (0,n)$, where c_1 is independent of f and x.

For a function b, suppose that the commutator operator $T_b \equiv T_{b,0}$ represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin suppf$

$$|T_b f(x)| \leq c_0 \int\limits_{\mathbb{R}^n} |b(x) - b(y)|\, |x-y|^{-n} |f(y)| dy, \tag{1.3}$$

where c_0 is independent of f and x. Similarly, we assume that the commutator operator $T_{b,\alpha}, \alpha \in (0,n)$ represents a linear or a sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin suppf$

$$|T_{b,\alpha} f(x)| \leq c_0 \int\limits_{\mathbb{R}^n} |b(x) - b(y)|\, |x-y|^{-n+\alpha} |f(y)| dy \tag{1.4}$$

for some $\alpha \in (0,n)$, where c_0 is independent of f and x.

In the first part of this work, we prove the boundedness of the sublinear operators T satisfying condition (1.1) generated by Calderón–Zygmund operators from one generalized Morrey space M_{p,φ_1} to another $M_{p,\varphi_2}, 1 < p < \infty$, and from the space M_{1,φ_1} to the weak space WM_{1,φ_2}. In the case $b \in BMO$ and T_b is a sublinear operator, we find the sufficient conditions on the pair (φ_1, φ_2) which ensures the boundedness of the operators T_b from M_{p,φ_1} to $M_{p,\varphi_2}, 1 < p < \infty$. In the second part of this work, we prove the boundedness of the sublinear operators $T_\alpha, \alpha \in (0,n)$ satisfying condition (1.2) generated by Riesz potential operator from one generalized Morrey space M_{p,φ_1} to $M_{q,\varphi_2}, 1 < p < q < \infty, 1/p - 1/q = \alpha/n$, and from the space M_{1,φ_1} to the weak space $WM_{q,\varphi_2}, 1 < q < \infty, 1 - 1/q = \alpha/n$. In the case $b \in BMO$ and $T_{b,\alpha}$ is a sublinear operator, we find the sufficient conditions on the pair (φ_1, φ_2) which ensures the boundedness of the operators $T_{b,\alpha}$ from M_{p,φ_1} to $M_{q,\varphi_2}, 1 < p < q < \infty, 1/p - 1/q = \alpha/n$.

We point out that the condition (1.1) was first introduced by Soria and Weiss in [39]. The conditions (1.1) and (1.2) are satisfied by many interesting operators in harmonic analysis, such as the Calderón–Zygmund operators, Carleson's maximal operators, Hardy–Littlewood maximal operators, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Ricci–Stein's oscillatory singular integrals, the Bochner–Riesz means and so on (see [31,39] for details).

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Morrey Spaces

The classical Morrey spaces $M_{p,\lambda}$ were originally introduced by Morrey in [34] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [34,37].

We denote by $M_{p,\lambda} \equiv M_{p,\lambda}(\mathbb{R}^n)$ the Morrey space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\lambda}} \equiv \|f\|_{M_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

where $1 \le p < \infty$ and $0 \le \lambda \le n$.

Note that $M_{p,0} = L_p(\mathbb{R}^n)$ and $M_{p,n} = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $M_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

We also denote by $WM_{p,\lambda} \equiv WM_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}} \equiv \|f\|_{WM_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(B(x,r))$ denotes the weak L_p-space of measurable functions f for which

$$\begin{aligned}\|f\|_{WL_p(B(x,r))} &\equiv \|f\chi_{B(x,r)}\|_{WL_p(\mathbb{R}^n)}\\ &= \sup_{t>0} t\,|\{y \in B(x,r) :\, |f(y)| > t\}|^{1/p}\\ &= \sup_{0<t\le|B(x,r)|} t^{1/p}\left(f\chi_{B(x,r)}\right)^*(t) < \infty.\end{aligned}$$

Here g^* denotes the non-increasing rearrangement of the function g.

Chiarenza and Frasca [10] studied the boundedness of the maximal operator M in these spaces. Their results can be summarized as follows:

Theorem 2.1. *Let $1 \le p < \infty$ and $0 \le \lambda < n$. Then for $p > 1$ the operator M is bounded on $M_{p,\lambda}$ and for $p = 1 M$ is bounded from $M_{1,\lambda}$ to $WM_{1,\lambda}$.*

The classical result by Hardy–Littlewood–Sobolev states that if $1 < p < q < \infty$, then I_α is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if and only if $\alpha = n\left(\frac{1}{p} - \frac{1}{q}\right)$ and for $p = 1 < q < \infty, I_\alpha$ is bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ if and only if $\alpha = n\left(1 - \frac{1}{q}\right)$. S. Spanne (published by Peetre [37]) and Adams [1] studied boundedness of the Riesz potential in Morrey spaces. Their results, can be summarized as follows.

Theorem 2.2 (Spanne, but published by Peetre [37]). *Let $0 < \alpha < n, 1 < p < \frac{n}{\alpha}, 0 < \lambda < n - \alpha p$. Moreover, let $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $\frac{\lambda}{p} = \frac{\mu}{q}$. Then for $p > 1$ the operator I_α is bounded from $M_{p,\lambda}$ to $M_{q,\lambda}$ and for $p = 1 I_\alpha$ is bounded from $M_{1,\lambda}$ to $WM_{q,\lambda}$.*

Theorem 2.3 (Adams [1]). *Let $0 < \alpha < n, 1 < p < \frac{n}{\alpha}, 0 < \lambda < n - \alpha p$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$. Then for $p > 1$ the operator I_α is bounded from $M_{p,\lambda}$ to $M_{q,\lambda}$ and for $p = 1 I_\alpha$ is bounded from $M_{1,\lambda}$ to $WM_{q,\lambda}$.*

Recall that, for $0 < \alpha < n$,

$$M_\alpha f(x) \le v_n^{\frac{\alpha}{n}-1} I_\alpha(|f|)(x),$$

hence Theorems 2.2 and 2.3 also implies boundedness of the fractional maximal operator M_α, where v_n is the volume of the unit ball in $\mathbb{R}^n$.

Di Fazio and Ragusa [17] studied the boundedness of the Calderón–Zygmund operators in Morrey spaces, and their results imply the following statement for Calderón–Zygmund operators K.

Theorem 2.4. *Let $1 \le p < \infty, 0 < \lambda < n$. Then for $1 < p < \infty$ Calderón–Zygmund operator K is bounded on $M_{p,\lambda}$ and for $p = 1 K$ is bounded from $M_{1,\lambda}$ to $WM_{1,\lambda}$.*

Note that in the case of the classical Calderón–Zygmund singular integral operators Theorem 2.4 was proved by Peetre [37].

3. Generalized Morrey Spaces

We find it convenient to define the generalized Morrey spaces in the form as follows.

Definition 3.1. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\varphi}} \equiv \|f\|_{M_{p,\varphi}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\begin{aligned}\|f\|_{WM_{p,\varphi}} &\equiv \|f\|_{WM_{p,\varphi}(\mathbb{R}^n)} \\ &= \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{WL_p(B(x,r))} < \infty.\end{aligned}$$

According to this definition, we recover the spaces $M_{p,\lambda}$ and $WM_{p,\lambda}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$M_{p,\lambda} = M_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}},$$
$$WM_{p,\lambda} = WM_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

In [20–23, 25, 33] and [35] there were obtained sufficient conditions on φ_1 and φ_2 for the boundedness of the maximal operator M and Calderón–Zygmund operator K from M_{p,φ_1} to M_{p,φ_2}, $1 < p < \infty$ and of the fractional maximal operator M_α and Riesz potential operator I_α from M_{p,φ_1} to M_{q,φ_2}, $1 < p < q < \infty$ (see also [3–7]). In [35, 36] the following condition was imposed on $\varphi(x,r)$:

$$c^{-1}\varphi(x,r) \le \varphi(x,t) \le c\,\varphi(x,r) \tag{3.1}$$

whenever $r \le t \le 2r$, where $c(\ge 1)$ does not depend on t, r and $x \in \mathbb{R}^n$, jointly with the condition:

$$\int_r^\infty \varphi(x,t)^p \frac{dt}{t} \le C\,\varphi(x,r)^p, \tag{3.2}$$

for the sublinear operator T satisfying condition (1.1), and the condition

$$\int_r^\infty t^{\alpha p}\varphi(x,t)^p \frac{dt}{t} \le C\, r^{\alpha p}\varphi(x,r)^p \tag{3.3}$$

for the sublinear operator T_α satisfying condition (1.2), where $C(>0)$ does not depend on r and $x \in \mathbb{R}^n$.

4. Sublinear Operators Generated by Calderón–Zygmund Operators in the Spaces $M_{p,\varphi}$

In [15] the following statements was proved by sublinear operator T satisfying condition (1.1), containing the result in [33,35,36].

Theorem 4.1. *Let $1 < p < \infty$ and $\varphi(x,r)$ satisfy conditions (3.1)–(3.2). Let T be a sublinear operator satisfying condition (1.1) and bounded on $L_p(\mathbb{R}^n)$. Then the operator T is bounded on $M_{p,\varphi}$.*

The following statements, containing results obtained in [33,35] was proved in [20] (see also [21,22]).

Theorem 4.2. *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\int_t^\infty \varphi_1(x,r)\frac{dr}{r} \le C\,\varphi_2(x,t), \tag{4.1}$$

where C does not depend on x and t. Then the operators M and K are bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2}.

In this section we are going to use the following statement on the boundedness of the Hardy operator

$$(Hg)(t) := \frac{1}{t}\int_0^t g(r)dr, \quad 0 < t < \infty.$$

Theorem 4.3 ([9]). *The inequality*

$$\operatorname*{ess\,sup}_{t>0} w(t)Hg(t) \le c \operatorname*{ess\,sup}_{t>0} v(t)g(t)$$

holds for all non-negative and non-increasing g on $(0,\infty)$ if and only if

$$A := \sup_{t>0}\frac{w(t)}{t}\int_0^t \frac{dr}{\operatorname*{ess\,sup}_{0<s<r} v(s)} < \infty,$$

and $c \approx A$.

Lemma 4.4. *Let $1 \le p < \infty, T$ be a sublinear operator satisfying condition (1.1), and bounded on $L_p(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$.*

Then, for $1 < p < \infty$ *the inequality*

$$\|Tf\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int\limits_{2r}^{\infty} t^{-\frac{n}{p}-1} \|f\|_{L_p(B(x_0,t))} dt$$

holds for any ball $B(x_0, r)$ *and for all* $f \in L_p^{\rm loc}(\mathbb{R}^n)$*.*

Moreover, for $p = 1$ *the inequality*

$$\|Tf\|_{WL_1(B(x_0,r))} \lesssim r^n \int\limits_{2r}^{\infty} t^{-n-1} \|f\|_{L_1(B(x_0,t))} dt, \tag{4.2}$$

holds for any ball $B(x_0, r)$ *and for all* $f \in L_1^{\rm loc}(\mathbb{R}^n)$*.*

Proof. Let $p \in (1, \infty)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius $r, 2B = B(x_0, 2r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2B}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2B)}}(y), \quad r > 0, \tag{4.3}$$

and have

$$\|Tf\|_{L_p(B)} \le \|Tf_1\|_{L_p(B)} + \|Tf_2\|_{L_p(B)}.$$

Since $f_1 \in L_p(\mathbb{R}^n), Tf_1 \in L_p(\mathbb{R}^n)$ and from the boundedness of T in $L_p(\mathbb{R}^n)$ it follows that:

$$\|Tf_1\|_{L_p(B)} \le \|Tf_1\|_{L_p(\mathbb{R}^n)} \le C\|f_1\|_{L_p(\mathbb{R}^n)} = C\|f\|_{L_p(2B)},$$

where constant $C > 0$ is independent of f.

It's clear that $x \in B, y \in {}^{\complement}(2B)$ implies $\frac{1}{2}|x_0 - y| \le |x - y| \le \frac{3}{2}|x_0 - y|$. We get

$$|Tf_2(x)| \le 2^n c_0 \int\limits_{\complement_{(2B)}} \frac{|f(y)|}{|x_0 - y|^n} dy.$$

By Fubini's theorem we have

$$\begin{aligned}
\int\limits_{\complement_{(2B)}} \frac{|f(y)|}{|x_0 - y|^n} dy &\approx \int\limits_{\complement_{(2B)}} |f(y)| \int\limits_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy \\
&\approx \int\limits_{2r}^{\infty} \int\limits_{2r \le |x_0-y| < t} |f(y)| dy \frac{dt}{t^{n+1}} \\
&\lesssim \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |f(y)| dy \frac{dt}{t^{n+1}}.
\end{aligned}$$

Applying Hölder's inequality, we get

$$\int\limits_{\complement_{(2B)}} \frac{|f(y)|}{|x_0 - y|^n} dy \lesssim \int\limits_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}. \tag{4.4}$$

Moreover, for all $p \in [1,\infty)$ the inequality

$$\|Tf_2\|_{L_p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}} \tag{4.5}$$

is valid. Thus

$$\|Tf\|_{L_p(B)} \lesssim \|f\|_{L_p(2B)} + r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.$$

On the other hand,

$$\begin{aligned} \|f\|_{L_p(2B)} &\approx r^{\frac{n}{p}} \|f\|_{L_p(2B)} \int_{2r}^{\infty} \frac{dt}{t^{\frac{n}{p}+1}} \\ &\lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}. \end{aligned} \tag{4.6}$$

Thus

$$\|Tf\|_{L_p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.$$

Let $p=1$. From the weak $(1,1)$ boundedness of T and (4.6) it follows that:

$$\begin{aligned} \|Tf_1\|_{WL_1(B)} \le \|Tf_1\|_{WL_1(\mathbb{R}^n)} \lesssim \|f_1\|_{L_1(\mathbb{R}^n)} \\ = \|f\|_{L_1(2B)} \lesssim r^n \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| dy \frac{dt}{t^{n+1}}. \end{aligned} \tag{4.7}$$

Then by (4.5) and (4.7) we get the inequality (4.2). □

Theorem 4.5. *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\int_{r}^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\, \varphi_2(x,r), \tag{4.8}$$

where C does not depend on x and r. Let T be a sublinear operator satisfying condition (1.1) bounded on $L_p(\mathbb{R}^n)$ for $p>1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. Then the operator T is bounded from M_{p,φ_1} to M_{p,φ_2} for $p>1$ and from M_{1,φ_1} to WM_{1,φ_2}. Moreover, for $p>1$

$$\|Tf\|_{M_{p,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}},$$

and for $p=1$

$$\|Tf\|_{WM_{1,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}}.$$

Proof. By Lemma 4.4 and Theorem 4.3 we have for $p > 1$

$$\begin{aligned}
\|Tf\|_{M_{p,\varphi_2}} &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \|f\|_{L_p(B(x,t))} \frac{dt}{t^{\frac{n}{p}+1}} \\
&\approx \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_0^{r^{-\frac{n}{p}}} \|f\|_{L_p(B(x,t^{-\frac{p}{n}}))} dt \\
&= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r^{-\frac{p}{n}})^{-1} \int_0^r \|f\|_{L_p(B(x,t^{-\frac{p}{n}}))} dt \\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r^{-\frac{p}{n}})^{-1}\, r\, \|f\|_{L_p(B(x,r^{-\frac{p}{n}}))} = \|f\|_{M_{p,\varphi_1}}
\end{aligned}$$

and for $p = 1$

$$\begin{aligned}
\|Tf\|_{WM_{1,\varphi_2}} &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \|f\|_{L_1(B(x,t))} \frac{dt}{t^{n+1}} \\
&\approx \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_0^{r^{-n}} \|f\|_{L_1(B(x,t^{-\frac{1}{n}}))} dt \\
&= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r^{-\frac{1}{n}})^{-1} \int_0^r \|f\|_{L_1(B(x,t^{-\frac{1}{n}}))} dt \\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r^{-\frac{1}{n}})^{-1}\, r\, \|f\|_{L_1(B(x,r^{-\frac{1}{n}}))} = \|f\|_{M_{1,\varphi_1}}.
\end{aligned}$$

□

Corollary 4.6. *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfy condition (4.8). Then the operators M and K are bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2}.*

Remark 4.7. Corollary 4.6 was proved in [2]. Note that condition (4.8) in Theorem 4.5 is weaker than condition (4.1) in Theorem 4.2. Indeed, if condition (4.1) holds, then

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le \int_r^\infty \varphi_1(t) \frac{dt}{t}, \quad r \in (0,\infty),$$

so condition (4.8) holds.

On the other hand the functions

$$\varphi_1(r) = r^{\beta-\frac{n}{p}} \left| \sin\Big(\max\{1, \frac{\pi}{r}\}\Big)\right|, \quad \varphi_2(r) = r^{2\beta-\frac{n}{p}}, \quad 0 < \beta < \frac{n}{2p}$$

satisfy condition (4.8), how us, in the case $r \in (0,1)$ $\operatorname*{ess\,inf}_{r<s<\infty} \varphi_1(s)s^{\frac{n}{p}} = 0$ and,

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(s)s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \approx \begin{cases} 0, & r \in (0,1), \\ r^{\beta-\frac{n}{p}}, & r \in (1,\infty) \end{cases} \lesssim \varphi_2(r), \quad r \in (0,\infty),$$

but do not satisfy condition (4.1).

Second example, the functions

$$\varphi_1(r) = \frac{1}{\chi_{(1,\infty)}(r) r^{\frac{n}{p}-\beta}}, \quad \varphi_2(r) = r^{-\frac{n}{p}}\left(1 + r^\beta\right), \quad 0 < \beta < \frac{n}{p}$$

satisfy condition (4.8) but do not satisfy condition (4.1).

5. Sublinear Operators Generated by Riesz Potential in the Spaces $M_{p,\varphi}$

In [15] the following statements was proved by sublinear operator T_α satisfying condition (1.2), containing the result in [33,35].

Theorem 5.1. *Let $1 < p < \infty, 0 < \alpha < \frac{n}{p}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $\varphi(x,r)$ satisfy conditions (3.1) and (3.3). Let T_α be a sublinear operator satisfying condition (1.2) and bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$. Then the operator T_α is bounded from $M_{p,\varphi}$ to $M_{q,\varphi}$.*

The following statements, containing results obtained in [33,35] was proved in [20,22] (see also [3–7,21,23]).

Theorem 5.2. *Let $1 \le p < \infty, 0 < \alpha < \frac{n}{p}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and (φ_1, φ_2) satisfy the condition*

$$\int_t^\infty r^\alpha \varphi_1(x,r) \frac{dr}{r} \le C\, \varphi_2(x,t), \tag{5.1}$$

where C does not depend on x and t. Then the operators M_α and I_α are bounded from M_{p,φ_1} to M_{q,φ_2} for $p>1$ and from M_{1,φ_1} to WM_{q,φ_2} for $p=1$.

Lemma 5.3. *Let $1 \le p < \infty, 0 < \alpha < \frac{n}{p}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, T_\alpha$ be a sublinear operator satisfying condition (1.2), and bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ for $p = 1$.*

Then, for $p > 1$ the inequality

$$\|T_\alpha f\|_{L_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^\infty t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x_0,t))} dt$$

holds for any ball $B(x_0,r)$ and for all $f \in L_p^{\rm loc}(\mathbb{R}^n)$.

Moreover, for $p = 1$ the inequality

$$\|T_\alpha f\|_{WL_q(B(x_0,r))} \lesssim r^{\frac{n}{q}} \int_{2r}^\infty t^{-\frac{n}{q}-1} \|f\|_{L_1(B(x_0,t))} dt \tag{5.2}$$

holds for any ball $B(x_0,r)$ and for all $f \in L_1^{\rm loc}(\mathbb{R}^n)$.

Proof. Let $1 < p < \infty, 0 < \alpha < \frac{n}{p}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement(2B)}$. Hence

$$\|T_\alpha f\|_{L_q(B)} \leq \|T_\alpha f_1\|_{L_q(B)} + \|T_\alpha f_2\|_{L_q(B)}.$$

Since $f_1 \in L_p(\mathbb{R}^n), T_\alpha f_1 \in L_q(\mathbb{R}^n)$ and from the boundedness of T_α from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ it follows that:

$$\|T_\alpha f_1\|_{L_q(B)} \leq \|T_\alpha f_1\|_{L_q(\mathbb{R}^n)} \leq C\|f_1\|_{L_p(\mathbb{R}^n)} = C\|f\|_{L_p(2B)},$$

where constant $C > 0$ is independent of f.

It's clear that $x \in B, y \in \complement(2B)$ implies $\frac{1}{2}|x_0 - y| \leq |x - y| \leq \frac{3}{2}|x_0 - y|$. We get

$$|T_\alpha f_2(x)| \leq 2^{n-\alpha} c_1 \int\limits_{\complement(2B)} \frac{|f(y)|}{|x_0 - y|^{n-\alpha}} dy.$$

By Fubini's theorem we have

$$\begin{aligned}\int\limits_{\complement(2B)} \frac{|f(y)|}{|x_0 - y|^{n-\alpha}} dy &\approx \int\limits_{\complement(2B)} |f(y)| \int\limits_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1-\alpha}} dy \\ &\approx \int\limits_{2r}^{\infty} \int\limits_{2r \leq |x_0-y| \leq t} |f(y)| dy \frac{dt}{t^{n+1-\alpha}} \\ &\lesssim \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |f(y)| dy \frac{dt}{t^{n+1-\alpha}}.\end{aligned}$$

Applying Hölder's inequality, we get

$$\int\limits_{\complement(2B)} \frac{|f(y)|}{|x_0 - y|^{n-\alpha}} dy \lesssim \int\limits_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}. \tag{5.3}$$

Moreover, for all $p \in [1, \infty)$ the inequality

$$\|T_\alpha f_2\|_{L_q(B)} \lesssim r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}. \tag{5.4}$$

is valid. Thus

$$\|T_\alpha f\|_{L_q(B)} \lesssim \|f\|_{L_p(2B)} + r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

On the other hand,

$$\|f\|_{L_p(2B)} \approx r^{\frac{n}{q}} \|f\|_{L_p(2B)} \int_{2r}^{\infty} \frac{dt}{t^{\frac{n}{q}+1}}$$
$$\leq r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}. \tag{5.5}$$

Thus

$$\|T_\alpha f\|_{L_q(B)} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}.$$

Let $p = 1$. From the weak $(1, q)$ boundedness of T_α and (5.5) it follows that:

$$\|T_\alpha f_1\|_{WL_q(B)} \leq \|T_\alpha f_1\|_{WL_q(\mathbb{R}^n)} \lesssim \|f_1\|_{L_1(\mathbb{R}^n)}$$
$$= \|f\|_{L_1(2B)} \lesssim r^{\frac{n}{q}} \int_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} \frac{dt}{t^{\frac{n}{q}+1}}. \tag{5.6}$$

Then from (5.4) and (5.6) we get the inequality (5.2). $\square$

Theorem 5.4. *Let $1 \leq p < \infty, 0 < \alpha < \frac{n}{p}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, and (φ_1, φ_2) satisfy the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} dt \leq C\, \varphi_2(x,r), \tag{5.7}$$

where C does not depend on x and r. Let T_α be a sublinear operator satisfying condition (1.2) bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ for $p = 1$. Then the operator T_α is bounded from M_{p,φ_1} to M_{q,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{q,φ_2} for $p = 1$. Moreover, for $p > 1$

$$\|T_\alpha f\|_{M_{q,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}},$$

and for $p = 1$

$$\|T_\alpha f\|_{WM_{q,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}}.$$

Proof. By Lemma 5.3 and Theorem 4.3 we have for $p > 1$

$$\begin{aligned}\|T_\alpha f\|_{M_{q,\varphi_2}} &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \|f\|_{L_p(B(x,t))} \frac{dt}{t^{\frac{n}{q}+1}} \\ &\approx \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_0^{r^{-\frac{n}{q}}} \|f\|_{L_p(B(x,t^{-\frac{q}{n}}))} dt \\ &= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r^{-\frac{q}{n}})^{-1} \int_0^{r} \|f\|_{L_p(B(x,t^{-\frac{q}{n}}))} dt \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r^{-\frac{q}{n}})^{-1}\, r^{\frac{q}{p}} \|f\|_{L_p(B(x,r^{-\frac{q}{n}}))} = \|f\|_{M_{p,\varphi_1}}\end{aligned}$$

and for $p = 1$

$$\begin{aligned}\|T_\alpha f\|_{WM_{q,\varphi_2}} &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \|f\|_{L_1(B(x,t))} \frac{dt}{t^{\frac{n}{q}+1}} \\ &\approx \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r)^{-1} \int_0^{r^{-\frac{n}{q}}} \|f\|_{L_1(B(x,t^{-\frac{n}{q}}))} dt \\ &= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_2(x,r^{-\frac{q}{n}})^{-1} \int_0^{r} \|f\|_{L_1(B(x,t^{-\frac{q}{n}}))} dt \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r^{-\frac{q}{n}})^{-1}\, r^{q} \|f\|_{L_1(B(x,r^{-\frac{q}{n}}))} = \|f\|_{M_{1,\varphi_1}}.\end{aligned}$$

□

Corollary 5.5. *Let $1 \le p < \infty, 0 < \alpha < \frac{n}{p}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and (φ_1, φ_2) satisfy condition (5.7). Then the operators M_α and I_α are bounded from M_{p,φ_1} to M_{q,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{q,φ_2} for $p = 1$.*

Remark 5.6. It is obvious that if condition (5.1) holds, then condition (5.7) holds too. In general, condition (5.7) does not imply condition (5.1). For example, the functions

$$\varphi_1(r) = \frac{1}{\chi_{(1,\infty)}(r) r^{\frac{n}{p}-\beta}},\ \varphi_2(r) = r^{-\frac{n}{q}}\left(1 + r^\beta\right),\ 0 < \beta < \frac{n}{q}$$

satisfy condition (5.7) but do not satisfy condition (5.1).

6. Commutators of Sublinear Operators Generated by Calderón–Zygmund Operators in the Spaces $M_{p,\varphi}$

Let T be a Calderón–Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$. A well known result of Coifman et al. [14] states that the commutator operator $[a,T]f = T(af) - a\,Tf$ is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. The commutator of Calderón–Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [10–12, 17]).

First we introduce the definition of the space of $BMO(\mathbb{R}^n)$.

Definition 6.1. Suppose that $f \in L_1^{\rm loc}(\mathbb{R}^n)$, let

$$\|f\|_* = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y) - f_{B(x,r)}|dy < \infty,$$

where

$$f_{B(x,r)} = \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} f(y)dy.$$

Define

$$BMO(\mathbb{R}^n) = \{f \in L_1^{\rm loc}(\mathbb{R}^n) \ : \ \|f\|_* < \infty\}.$$

If one regards two functions whose difference is a constant as one, then space $BMO(\mathbb{R}^n)$ is a Banach space with respect to norm $\|\cdot\|_*$.

Remark 6.2. (1) The John–Nirenberg inequality: there are constants C_1, $C_2 > 0$, such that for all $f \in BMO(\mathbb{R}^n)$ and $\beta > 0$

$$|\{x \in B \ : \ |f(x) - f_B| > \beta\}| \le C_1|B|e^{-C_2\beta/\|f\|_*}, \ \forall B \subset \mathbb{R}^n.$$

(2) The John–Nirenberg inequality implies that

$$\|f\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y) - f_{B(x,r)}|^p dy \right)^{\frac{1}{p}} \tag{6.1}$$

for $1 < p < \infty$.

(3) Let $f \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$\left|f_{B(x,r)} - f_{B(x,t)}\right| \le C\|f\|_* \ln\frac{t}{r} \text{ for } 0 < 2r < t, \tag{6.2}$$

where C is independent of f, x, r and t.

In [15] the following statement was proved for the commutators of sublinear operators, containing the result in [33, 35].

Theorem 6.3. *Let $1 < p < \infty, \varphi(x,r)$ satisfy conditions (3.1)–(3.2) and $b \in BMO(\mathbb{R}^n)$. Let also T be a linear operator satisfying condition (1.1) and the commutator operator $[b,T]$ bounded on $L_p(\mathbb{R}^n)$. Then the operator $[b,T]$ is bounded on $M_{p,\varphi}$.*

Remark 6.4. Note that, Theorem 6.3 in the following form also valid. Let $1 < p < \infty, b \in BMO(\mathbb{R}^n), \varphi(x,r)$ satisfy the conditions (3.1) and (3.2). Suppose that T_b is a sublinear operator satisfies the condition (1.3) and bounded on $L_p(\mathbb{R}^n)$, then the operator T_b is bounded on $M_{p,\varphi}$.

Lemma 6.5. *Let $1 < p < \infty, b \in BMO(\mathbb{R}^n)$, and T_b be a sublinear operator satisfying condition (1.3) and bounded on $L_p(\mathbb{R}^n)$.*

Then the inequality

$$\|T_b f\|_{L_p(B(x_0,r))} \lesssim \|a\|_* \, r^{\frac{n}{p}} \int\limits_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) t^{-\frac{n}{p}-1} \|f\|_{L_p(B(x_0,t))} dt$$

holds for any ball $B(x_0,r)$ and for all $f \in L_p^{\rm loc}(\mathbb{R}^n)$.

Proof. Let $p \in (1,\infty)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{(2B)}}$. Hence

$$\|T_b f\|_{L_p(B)} \le \|T_b f_1\|_{L_p(B)} + \|T_b f_2\|_{L_p(B)}.$$

From the boundedness of T_b in $L_p(\mathbb{R}^n)$ it follows that:

$$\begin{aligned}\|T_b f_1\|_{L_p(B)} &\le \|T_b f_1\|_{L_p(\mathbb{R}^n)} \\ &\lesssim \|a\|_* \, \|f_1\|_{L_p(\mathbb{R}^n)} = \|a\|_* \, \|f\|_{L_p(2B)}.\end{aligned}$$

For $x \in B$ we have

$$\begin{aligned}|T_b f_2(x)| &\lesssim \int\limits_{\mathbb{R}^n} \frac{|a(y)-a(x)|}{|x-y|^n} |f(y)| dy \\ &\approx \int\limits_{\complement_{(2B)}} \frac{|a(y)-a(x)|}{|x_0-y|^n} |f(y)| dy.\end{aligned}$$

Then

$$\begin{aligned}\|T_b f_2\|_{L_p(B)} &\lesssim \left(\int\limits_B \left(\int\limits_{\complement_{(2B)}} \frac{|a(y)-a(x)|}{|x_0-y|^n} |f(y)| dy \right)^p dx \right)^{\frac{1}{p}} \\ &\lesssim \left(\int\limits_B \left(\int\limits_{\complement_{(2B)}} \frac{|a(y)-a_B|}{|x_0-y|^n} |f(y)| dy \right)^p dx \right)^{\frac{1}{p}} \\ &\quad + \left(\int\limits_B \left(\int\limits_{\complement_{(2B)}} \frac{|a(x)-a_B|}{|x_0-y|^n} |f(y)| dy \right)^p dx \right)^{\frac{1}{p}} \\ &= I_1 + I_2.\end{aligned}$$

Let us estimate I_1.

$$\begin{aligned}
I_1 &\approx r^{\frac{n}{p}} \int\limits_{\complement_{(2B)}} \frac{|a(y)-a_B|}{|x_0-y|^n}|f(y)|dy \\
&\approx r^{\frac{n}{p}} \int\limits_{\complement_{(2B)}} |a(y)-a_B||f(y)| \int\limits_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}}dy \\
&\approx r^{\frac{n}{p}} \int\limits_{2r}^{\infty} \int\limits_{2r\le|x_0-y|\le t} |a(y)-a_B||f(y)|dy\frac{dt}{t^{n+1}} \\
&\lesssim r^{\frac{n}{p}} \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |a(y)-a_B||f(y)|dy\frac{dt}{t^{n+1}}.
\end{aligned}$$

Applying Hölder's inequality and by (6.1), (6.2), we get

$$\begin{aligned}
I_1 &\lesssim r^{\frac{n}{p}} \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |a(y)-a_{B(x_0,t)}||f(y)|dy\frac{dt}{t^{n+1}} \\
&\quad + r^{\frac{n}{p}} \int\limits_{2r}^{\infty} |a_{B(x_0,r)}-a_{B(x_0,t)}| \int\limits_{B(x_0,t)} |f(y)|dy\frac{dt}{t^{n+1}} \\
&\lesssim r^{\frac{n}{p}} \int\limits_{2r}^{\infty} \left(\int\limits_{B(x_0,t)} |a(y)-a_{B(x_0,t)}|^{p'}dy \right)^{\frac{1}{p'}} \|f\|_{L_p(B(x_0,t))}\frac{dt}{t^{n+1}} \\
&\quad + r^{\frac{n}{p}} \int\limits_{2r}^{\infty} |a_{B(x_0,r)}-a_{B(x_0,t)}|\|f\|_{L_p(B(x_0,t))}\frac{dt}{t^{\frac{n}{p}+1}} \\
&\lesssim \|a\|_* \, r^{\frac{n}{p}} \int\limits_{2r}^{\infty} \left(1+\ln\frac{t}{r}\right)\|f\|_{L_p(B(x_0,t))}\frac{dt}{t^{\frac{n}{p}+1}}.
\end{aligned}$$

In order to estimate I_2 note that

$$I_2 = \left(\int\limits_B |a(x)-a_B|^p dx \right)^{\frac{1}{p}} \int\limits_{\complement_{(2B)}} \frac{|f(y)|}{|x_0-y|^n}dy.$$

By (6.1), we get

$$I_2 \lesssim \|a\|_* \, r^{\frac{n}{p}} \int\limits_{\complement_{(2B)}} \frac{|f(y)|}{|x_0-y|^n}dy.$$

Thus, by (4.4)

$$I_2 \lesssim \|a\|_* \, r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.$$

Summing up I_1 and I_2, for all $p \in (1,\infty)$ we get

$$\|T_b f_2\|_{L_p(B)} \lesssim \|a\|_* \, r^{\frac{n}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}. \tag{6.3}$$

Finally,

$$\|T_b f\|_{L_p(B)} \lesssim \|a\|_* \, \|f\|_{L_p(2B)} + \|a\|_* \, r^{\frac{n}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}},$$

and statement of Lemma 6.5 follows by (4.6). □

The following theorem is true.

Theorem 6.6. *Let $1 < p < \infty, b \in BMO(\mathbb{R}^n)$ and (φ_1, φ_2) satisfy the condition*

$$\int_{r}^{\infty} \Big(1+\ln\frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\, \varphi_2(x,r), \tag{6.4}$$

where C does not depend on x and r. Let T_b be a sublinear operator satisfying condition (1.3) and bounded on $L_p(\mathbb{R}^n)$.

Then the operator T_b is bounded from M_{p,φ_1} to M_{p,φ_2}. Moreover

$$\|T_b f\|_{M_{p,\varphi_2}} \lesssim \|a\|_* \, \|f\|_{M_{p,\varphi_1}}.$$

Proof. The statement of Theorem 6.6 follows by Lemma 6.5 and Theorem 4.3 in the same manner as in the proof of Theorem 4.5. □

For the sublinear commutator of the maximal operator

$$M_b(f)(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |b(x)-b(y)||f(y)|dy$$

and for the linear commutator of the singular integral $[b,K]$ from Theorem 6.6 we get the following new results.

Corollary 6.7. *Let $1 < p < \infty, (\varphi_1, \varphi_2)$ satisfy condition (6.4) and $b \in BMO(\mathbb{R}^n)$. Then the operators M_b and $[b,K]$ are bounded from M_{p,φ_1} to M_{p,φ_2}.*

7. Commutators of Sublinear Operators Generated by Riesz Potential in the Spaces $M_{p,\varphi}$

In [15] the following statement was proved for the commutators of sublinear operators, containing the result in [33,35].

Theorem 7.1. *Let* $1 < p < \infty, 0 < \alpha < \frac{n}{p}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, b \in BMO(\mathbb{R}^n), \varphi(x,r)$ *which satisfies the conditions* (3.1) *and* (3.3). *Let also* T_α *be a linear operator and satisfies the condition* (1.2). *If the operator* $[b, T_\alpha]$ *is bounded from* $L_p(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$, *then the operator* $[b, T_\alpha]$ *is bounded from* $M_{p,\varphi}$ *to* $M_{q,\varphi}$.

Remark 7.2. Note that, Theorem 7.1 in the following form also valid. Let $1 < p < \infty, 0 < \alpha < \frac{n}{p}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, b \in BMO(\mathbb{R}^n), \varphi(x,r)$ satisfy the conditions (3.1) and (3.3). Suppose that $T_{b,\alpha}$ is a sublinear operator satisfies the condition (1.4) and bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$, then the operator $T_{b,\alpha}$ is bounded from $M_{p,\varphi}$ to $M_{q,\varphi}$.

Lemma 7.3. *Let* $1 < p < \infty, 0 < \alpha < \frac{n}{p}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, b \in BMO(\mathbb{R}^n)$, *and a sublinear operator* $T_{b,\alpha}$ *satisfies the condition* (1.4) *and bounded from* $L_p(\mathbb{R}^n)$ *to* $L_q(\mathbb{R}^n)$. *Then the inequality*

$$\|T_{b,\alpha} f\|_{L_q(B(x_0,r))} \lesssim \|b\|_* \, r^{\frac{n}{q}} \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big) t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x_0,t))} dt$$

holds for any ball $B(x_0, r)$ *and for all* $f \in L_p^{\rm loc}(\mathbb{R}^n)$.

Proof. Let $1 < p < \infty, 0 < \alpha < \frac{n}{p}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. As in the proof of Lemma 5.3, we represent function f in form (4.3) and have

$$\|T_{b,\alpha} f\|_{L_q(B)} \le \|T_{b,\alpha} f_1\|_{L_q(B)} + \|T_{b,\alpha} f_2\|_{L_q(B)}.$$

From the boundedness of $T_{b,\alpha}$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ it follows that:

$$\begin{aligned}\|T_{b,\alpha} f_1\|_{L_q(B)} &\le \|T_{b,\alpha} f_1\|_{L_q(\mathbb{R}^n)} \\ &\lesssim \|a\|_* \, \|f_1\|_{L_p(\mathbb{R}^n)} = \|a\|_* \, \|f\|_{L_p(2B)}.\end{aligned}$$

For $x \in B$ we have

$$\begin{aligned}|T_{b,\alpha} f_2(x)| &\lesssim \int_{\mathbb{R}^n} \frac{|a(y) - a(x)|}{|x - y|^{n-\alpha}} |f(y)| dy \\ &\approx \int_{\complement_{(2B)}} \frac{|a(y) - a(x)|}{|x_0 - y|^{n-\alpha}} |f(y)| dy.\end{aligned}$$

Then

$$\begin{aligned}\|T_{b,\alpha}f_2\|_{L_q(B)} &\lesssim \left(\int\limits_B \left(\int\limits_{\complement_{(2B)}} \frac{|a(y)-a(x)|}{|x_0-y|^{n-\alpha}}|f(y)|dy\right)^q dx\right)^{\frac{1}{q}}\\ &\le \left(\int\limits_B \left(\int\limits_{\complement_{(2B)}} \frac{|a(y)-a_B|}{|x_0-y|^{n-\alpha}}|f(y)|dy\right)^q dx\right)^{\frac{1}{q}}\\ &\quad + \left(\int\limits_B \left(\int\limits_{\complement_{(2B)}} \frac{|a(x)-a_B|}{|x_0-y|^{n-\alpha}}|f(y)|dy\right)^q dx\right)^{\frac{1}{q}}\\ &= J_1 + J_2.\end{aligned}$$

Let us estimate J_1.

$$\begin{aligned}J_1 &= r^{\frac{n}{q}} \int\limits_{\complement_{(2B)}} \frac{|a(y)-a_B|}{|x_0-y|^{n-\alpha}}|f(y)|dy\\ &\approx r^{\frac{n}{q}} \int\limits_{\complement_{(2B)}} |a(y)-a_B||f(y)| \int\limits_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1-\alpha}}dy\\ &\approx r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \int\limits_{2r\le|x_0-y|\le t} |a(y)-a_B||f(y)|dy\frac{dt}{t^{n+1-\alpha}}\\ &\lesssim r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |a(y)-a_B||f(y)|dy\frac{dt}{t^{n+1-\alpha}}.\end{aligned}$$

Applying Hölder's inequality and by (6.1), (6.2), we get

$$\begin{aligned}J_1 &\lesssim r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \int\limits_{B(x_0,t)} |a(y)-a_{B(x_0,t)}||f(y)|dy\frac{dt}{t^{n+1-\alpha}}\\ &\quad + r^{\frac{n}{q}} \int\limits_{2r}^{\infty} |a_{B(x_0,r)}-a_{B(x_0,t)}| \int\limits_{B(x_0,t)} |f(y)|dy\frac{dt}{t^{n+1-\alpha}}\\ &\lesssim r^{\frac{n}{q}} \int\limits_{2r}^{\infty} \left(\int\limits_{B(x_0,t)} |a(y)-a_{B(x_0,t)}|^{p'}dy\right)^{\frac{1}{p'}} \|f\|_{L_p(B(x_0,t))}\frac{dt}{t^{n+1-\alpha}}\end{aligned}$$

$$+r^{\frac{n}{q}}\int_{2r}^{\infty}|a_{B(x_0,r)}-a_{B(x_0,t)}|\|f\|_{L_p(B(x_0,t))}\frac{dt}{t^{\frac{n}{p}+1-\alpha}}$$

$$\lesssim \|a\|_*\,r^{\frac{n}{q}}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_p(B(x_0,t))}\frac{dt}{t^{\frac{n}{q}+1}}.$$

In order to estimate J_2 note that

$$J_2=\bigg(\int_B|a(x)-a_B|^q dx\bigg)^{\frac{1}{q}}\int_{\complement(2B)}\frac{|f(y)|}{|x_0-y|^{n-\alpha}}dy.$$

By (6.1), we get

$$J_2\lesssim \|a\|_*\,r^{\frac{n}{q}}\int_{\complement(2B)}\frac{|f(y)|}{|x_0-y|^{n-\alpha}}dy.$$

Thus, by (5.3)

$$J_2\lesssim \|a\|_*\,r^{\frac{n}{q}}\int_{2r}^{\infty}\|f\|_{L_p(B(x_0,t))}\frac{dt}{t^{\frac{n}{q}+1}}.$$

Summing up J_1 and J_2, for all $p\in(1,\infty)$ we get

$$\|T_{b,\alpha}f_2\|_{L_q(B)}\lesssim \|a\|_*\,r^{\frac{n}{q}}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_p(B(x_0,t))}\frac{dt}{t^{\frac{n}{q}+1}}. \tag{7.1}$$

Finally,

$$\|T_{b,\alpha}f\|_{L_q(B)}\lesssim \|a\|_*\,\|f\|_{L_p(2B)}+\|a\|_*\,r^{\frac{n}{q}}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_p(B(x_0,t))}\frac{dt}{t^{\frac{n}{q}+1}},$$

and statement of Lemma 7.3 follows by (5.5). □

The following theorem is true.

Theorem 7.4. *Let $1<p<\infty, 0<\alpha<\frac{n}{p}, \frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}, b\in BMO(\mathbb{R}^n)$ and (φ_1,φ_2) satisfy the condition*

$$\int_r^{\infty}\Big(1+\ln\frac{t}{r}\Big)\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{q}+1}}dt\le C\,\varphi_2(x,r), \tag{7.2}$$

where C does not depend on x and r. Suppose that $T_{b,\alpha}$ is a sublinear operator which satisfies the condition (1.4) and bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$.

Then, the operator $T_{b,\alpha}$ is bounded from M_{p,φ_1} to M_{q,φ_2}. Moreover

$$\|T_{b,\alpha}f\|_{M_{q,\varphi_2}}\lesssim \|b\|_*\,\|f\|_{M_{p,\varphi_1}}.$$

Proof. The statement of Theorem 7.4 follows by Lemma 7.3 and Theorem 4.3 in the same manner as in the proof of Theorem 5.4. □

For the sublinear commutator of the fractional maximal operator

$$M_{b,\alpha}(f)(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int\limits_{B(x,t)} |b(x)-b(y)||f(y)|dy$$

and for the linear commutator of the Riesz potential $[b, I_\alpha]$ from Theorem 7.4 we get the following new results.

Corollary 7.5. *Let* $1 < p < \infty, 0 < \alpha < \frac{n}{p}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, (\varphi_1, \varphi_2)$ *satisfies the condition* (7.2) *and* $b \in BMO(\mathbb{R}^n)$. *Then, the operators* $M_{b,\alpha}$ *and* $[b, I_\alpha]$ *are bounded from* M_{p,φ_1} *to* M_{q,φ_2}.

8. Some Applications

In this section, we shall apply Theorems 4.5, 5.4, 6.6 and 7.4 to several particular operators such as the Littlewood–Paley operator, Marcinkiewicz operator, Bochner–Riesz operator, Schrödinger type operators $V^\gamma(-\Delta + V)^{-\beta}, V^\gamma\nabla(-\Delta + V)^{-\beta}$ and fractional powers of the some analytic semi-groups.

8.1. Littlewood–Paley Operator

The Littlewood–Paley functions play an important role in classical harmonic analysis, for example in the study of non-tangential convergence of Fatou type and boundedness of Riesz transforms and multipliers [40–42,44]. The Littlewood–Paley operator is defined as follows.

Definition 8.1. Suppose that $\psi \in L_1(\mathbb{R}^n)$ satisfies

$$\int\limits_{\mathbb{R}^n} \psi(x)dx = 0. \tag{8.1}$$

Then the generalized Littlewood–Paley g function g_ψ is defined by

$$g_\psi(f)(x) = \left(\int\limits_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$ and $F_t(f) = \psi_t * f$.

The following theorem is valid (see [32], Theorem 5.1.2).

Theorem 8.2. *Suppose that* $\psi \in L_1(\mathbb{R}^n)$ *satisfies* (8.1) *and the following properties*:

$$|\psi(x)| \leq \frac{C}{(1+|x|)^{n+1}}, \quad x \in \mathbb{R}^n \tag{8.2}$$

$$\int\limits_{\mathbb{R}^n} |\psi(x+h) - \psi(x)|dx \leq C|h|^\alpha, \quad x \in \mathbb{R}^n \tag{8.3}$$

where C *and* $\alpha > 0$ *are both independent of* x *and* h. *Then* g_ψ *is bounded on* $L_p(\mathbb{R}^n)$ *for all* $1 < p < \infty$, *and bounded from* $L_1(\mathbb{R}^n)$ *to* $WL_1(\mathbb{R}^n)$.

Let H be the space $H = \{h : \|h\| = (\int_0^\infty |h(t)|^2 dt/t^2)^{1/2}\}$, then, for each fixed $x \in \mathbb{R}^n, F_t(f)(x)$ may be viewed as a mapping from $[0, \infty)$ to H, and it is clear that $g_\psi(f)(x) = \|F_t(f)(x)\|$. In fact, by Minkowski inequality and the conditions on ψ, we get

$$g_\psi(f)(x) \le \int\limits_{\mathbb{R}^n} |f(y)| \left(\int\limits_0^\infty |\psi_t(x-y)|^2 \frac{dt}{t} \right)^{1/2} dy$$
$$\le C \int\limits_{\mathbb{R}^n} |f(y)| \left(\int\limits_0^\infty \frac{t^{-2n}}{(1+|x-y|/t)^{2(n+1)}} \frac{dt}{t} \right)^{1/2} dy$$
$$\le C \int\limits_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy.$$

Thus we get

Corollary 8.3. *Let $1 \le p < \infty, (\varphi_1, \varphi_2)$ satisfy condition (4.8) and $\psi \in L_1(\mathbb{R}^n)$ satisfies (8.1)–(8.3). Then the Littlewood–Paley operator g_ψ is bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and the operator g_ψ is bounded from M_{1,φ_1} to WM_{1,φ_2}.*

Corollary 8.4. *Let $1 < p < \infty, (\varphi_1, \varphi_2)$ satisfy condition (6.4), $b \in BMO(\mathbb{R}^n)$ and $\psi \in L_1(\mathbb{R}^n)$ satisfies (8.1)–(8.3). Then the commutator of Littlewood–Paley operator $[a, g_\psi]$ is bounded from M_{p,φ_1} to M_{p,φ_2}.*

8.2. Marcinkiewicz Operator

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$ equipped with the Lebesgue measure $d\sigma$. Suppose that Ω satisfies the following conditions.

(a) Ω is the homogeneous function of degree zero on $\mathbb{R}^n \backslash \{0\}$, that is,

$$\Omega(\mu x) = \Omega(x), \quad \text{for any } \mu > 0, x \in \mathbb{R}^n \backslash \{0\}.$$

(b) Ω has mean zero on S^{n-1}, that is,

$$\int\limits_{S^{n-1}} \Omega(x') d\sigma(x') = 0.$$

(c) $\Omega \in \mathrm{Lip}_\gamma(S^{n-1}), 0 < \gamma \le 1$, that is there exists a constant $M > 0$ such that,

$$|\Omega(x') - \Omega(y')| \le M|x' - y'|^\gamma \quad \text{for any } x', y' \in S^{n-1}.$$

In 1958, Stein [41] defined the Marcinkiewicz integral of higher dimension μ_Ω as

$$\mu_\Omega(f)(x) = \left(\int\limits_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int\limits_{|x-y| \le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy.$$

Since Stein's work in 1958, the continuity of Marcinkiewicz integral has been extensively studied as a research topic and also provides useful tools in harmonic analysis [32,40,42,44].

The Marcinkiewicz operator is defined by (see [45])

$$\mu_{\Omega,\alpha}(f)(x) = \left(\int_0^\infty |F_{\Omega,\alpha,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,\alpha,t}(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\alpha}} f(y)dy.$$

Note that $\mu_\Omega f = \mu_{\Omega,0} f$.

Let H be the space $H = \{h : \|h\| = (\int_0^\infty |h(t)|^2 dt/t^3)^{1/2} < \infty\}$. Then, it is clear that $\mu_{\Omega,\alpha}(f)(x) = \|F_{\Omega,\alpha,t}(x)\|$.

By Minkowski inequality and the conditions on Ω, we get

$$\mu_{\Omega,\alpha}(f)(x) \le \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1-\alpha}} |f(y)| \left(\int_{|x-y|}^\infty \frac{dt}{t^3} \right)^{1/2} dy \le C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy.$$

Thus, $\mu_{\Omega,\alpha}$ satisfies condition (1.2). It is known that $\mu_{\Omega,\alpha}$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ for $p = 1$ (see [45]), then from Theorems 5.4 and 7.4 we get

Corollary 8.5. *Let $1 \le p < \infty, 0 < \alpha < \frac{n}{p}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and (φ_1, φ_2) satisfy condition (5.7) and Ω satisfies conditions* (a)–(c). *Then $\mu_{\Omega,\alpha}$ is bounded from M_{p,φ_1} to M_{q,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{q,φ_2} for $p = 1$.*

Corollary 8.6. *Let $1 < p < \infty, 0 \le \alpha < \frac{n}{p}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, (\varphi_1, \varphi_2)$ satisfy condition (7.2), $b \in BMO(\mathbb{R}^n)$ and Ω be satisfies conditions (a)–(c). Then $[a, \mu_{\Omega,\alpha}]$ is bounded from M_{p,φ_1} to M_{q,φ_2}.*

8.3. Bochner–Riesz Operator

Let $\delta > (n-1)/2, B_t^\delta(f)\hat{}(\xi) = (1 - t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(x) = t^{-n} B^\delta(x/t)$ for $t > 0$. The maximal Bochner–Riesz operator is defined by (see [27,28])

$$B_{\delta,*}(f)(x) = \sup_{t>0} |B_t^\delta(f)(x)|.$$

Let H be the space $H = \{h : \|h\| = \sup_{t>0} |h(t)| < \infty\}$, then it is clear that $B_{\delta,*}(f)(x) = \|B_t^\delta(f)(x)\|$.

By the condition on B_r^δ (see [19]), we have

$$\begin{aligned} |B_r^\delta(x-y)| &\le C r^{-n}(1 + |x-y|/r)^{-(\delta+(n+1)/2)} \\ &= C \left(\frac{r}{r + |x-y|} \right)^{\delta-(n-1)/2} \frac{1}{(r + |x-y|)^n} \\ &\le |x-y|^{-n}, \end{aligned}$$

and

$$B_{\delta,*}(f)(x) \le C \int\limits_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy.$$

Thus, $B_{\delta,*}$ satisfies condition (1.1). It is known that $B_{\delta,*}$ is bounded on $L_p(\mathbb{R}^n)$ for $p>1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$, then from Theorems 4.5 and 6.6 we get

Corollary 8.7. *Let $1 \le p < \infty, (\varphi_1, \varphi_2)$ satisfy condition (4.8) and $\delta > (n-1)/2$. Then the Bochner–Riesz operator $B_{\delta,*}$ is bounded from M_{p,φ_1} to M_{p,φ_2} for $p>1$ and the operator $B_{\delta,*}$ is bounded from M_{1,φ_1} to WM_{1,φ_2}.*

Corollary 8.8. *Let $1 < p < \infty, (\varphi_1, \varphi_2)$ satisfy condition (6.4), $\delta > (n-1)/2$ and $b \in BMO(\mathbb{R}^n)$. Then the commutator of Bochner–Riesz operator $[a, B_{\delta,*}]$ is bounded from M_{p,φ_1} to M_{p,φ_2}.*

Remark 8.9. Recall that, under the assumptions $\varphi(x,r)$ satisfy conditions (3.1)–(3.2), the Corollaries 8.7 and 8.8 were proved in [26].

8.4. Schrödinger Type Operators $V^\gamma(-\Delta+V)^{-\beta}$ and $V^\gamma\nabla(-\Delta+V)^{-\beta}$

In this subsection we consider the Schrödinger operator $-\Delta+V$ on $\mathbb{R}^n$, where the nonnegative potential V belongs to the reverse Hölder class $B_\infty(\mathbb{R}^n)$ for some $q_1 \ge n$. The generalized Morrey $M_{p,\varphi_1} \to M_{q,\varphi_2}$ estimates for the commutator of operators $V^\gamma(-\Delta+V)^{-\beta}$ and $V^\gamma\nabla(-\Delta+V)^{-\beta}$ are obtained.

The investigation of Schrödinger operators on the Euclidean space $\mathbb{R}^n$ with nonnegative potentials which belong to the reverse Hölder class has attracted attention of a number of authors (cf. [18,38,46]). Shen [38] studied the Schrödinger operator $-\Delta+V$, assuming the nonnegative potential V belongs to the reverse Hölder class $B_q(\mathbb{R}^n)$ for $q \ge n/2$ and he proved the L_p boundedness of the operators $(-\Delta+V)^{i\gamma}, \nabla^2(-\Delta+V)^{-1}, \nabla(-\Delta+V)^{-\frac{1}{2}}$ and $\nabla(-\Delta+V)^{-1}$. Kurata and Sugano generalized Shen's results to uniformly elliptic operators in [24]. Sugano [43] also extended some results of Shen to the operator $V^\gamma(-\Delta+V)^{-\beta}, 0 \le \gamma \le \beta \le 1$ and $V^\gamma\nabla(-\Delta+V)^{-\beta}, 0 \le \gamma \le \frac{1}{2} \le \beta \le 1$ and $\beta-\gamma \ge \frac{1}{2}$. Later, Lu [30] and Li [29] investigated the Schrödinger operators in a more general setting.

We investigate the generalized Morrey $M_{p,\varphi_1} \to M_{q,\varphi_2}$ boundedness of the operators

$$\mathcal{T}_1 = V^\gamma(-\Delta+V)^{-\beta}, \quad 0 \le \gamma \le \beta \le 1,$$
$$\mathcal{T}_2 = V^\gamma\nabla(-\Delta+V)^{-\beta}, \quad 0 \le \gamma \le \frac{1}{2} \le \beta \le 1, \quad \beta-\gamma \ge \frac{1}{2}.$$

Note that the operators $V(-\Delta+V)^{-1}$ and $V^{\frac{1}{2}}\nabla(-\Delta+V)^{-1}$ in [29] are the special case of $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively.

It is worth pointing out that we need to establish pointwise estimates for $\mathcal{T}_1, \mathcal{T}_2$ and their adjoint operators by using the estimates of fundamental solution for the Schrödinger operator on $\mathbb{R}^n$ in [29]. And we prove the generalized Morrey estimates by using $M_{p,\varphi_1} \to M_{q,\varphi_2}$ boundedness of the fractional maximal operators.

Let $V \geq 0$. We say $V \in B_\infty$, if there exists a constant $C > 0$ such that

$$\|V\|_{L_\infty(B)} \leq \frac{C}{|B|} \int\limits_B V(x)dx$$

holds for every ball B in $\mathbb{R}^n$ (see [29]).

By the functional calculus, we may write, for all $0 < \beta < 1$,

$$(-\Delta + V)^{-\beta} = \frac{1}{\pi} \int\limits_0^\infty \lambda^{-\beta} \, (-\Delta + V + \lambda)^{-1} \, d\lambda.$$

Let $f \in C_0^\infty(\mathbb{R}^n)$. From $(-\Delta + V + \lambda)^{-1} f(x) = \int_{\mathbb{R}^n} \Gamma(x,y,\lambda) f(y)dy$, it follows that

$$\mathcal{T}_1 f(x) = \int\limits_{\mathbb{R}^n} K_1(x,y) V(x)^\gamma f(y) dy,$$

where

$$K_1(x,y) = \begin{cases} \frac{1}{\pi} \int_0^\infty \lambda^{-\beta} \, \Gamma(x,y,\lambda) \, d\lambda & \text{for } 0 < \beta < 1 \\ \Gamma(x,y,0) & \text{for } \beta = 1. \end{cases}$$

The following two pointwise estimates for $\mathcal{T}_1$ and $\mathcal{T}_2$ which proven in [46], Lemma 3.2 with the potential $V \in B_\infty$.

Theorem A. *Suppose $V \in B_\infty$ and $0 \leq \gamma \leq \beta \leq 1$. Then, for any $f \in C_0^\infty(\mathbb{R}^n)$*

$$|\mathcal{T}_1 f(x)| \lesssim M_\alpha f(x), \quad |[b, \mathcal{T}_1] f| \lesssim M_{b,\alpha} f(x),$$

where $\alpha = 2(\beta - \gamma)$.

Theorem B. *Suppose $V \in B_\infty, 0 \leq \gamma \leq \frac{1}{2} \leq \beta \leq 1$ and $\beta - \gamma \geq \frac{1}{2}$. Then, for any $f \in C_0^\infty(\mathbb{R}^n)$*

$$|\mathcal{T}_2 f(x)| \lesssim M_\alpha f(x), \quad |[b, \mathcal{T}_2] f| \lesssim M_{b,\alpha} f(x),$$

where $\alpha = 2(\beta - \gamma) - 1$.

The above theorems will yield the generalized Morrey estimates for $\mathcal{T}_1$ and $\mathcal{T}_2$.

Corollary 8.10. *Assume that $V \in B_\infty$, and $0 \leq \gamma \leq \beta \leq 1$. Let $1 \leq p \leq q < \infty, 2(\beta - \gamma) = n\left(\frac{1}{p} - \frac{1}{q}\right)$ and the condition (5.7) be satisfied for $\alpha = 2(\beta - \gamma)$.*

Then, for any $f \in C_0^\infty(\mathbb{R}^n)$

$$\|\mathcal{T}_1 f\|_{M_{q,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}}, \quad \textit{for } p > 1$$

and

$$\|\mathcal{T}_1 f\|_{WM_{q,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}} \; \textit{ for } p = 1$$

Corollary 8.11. *Assume that $V \in B_\infty, b \in BMO(\mathbb{R}^n)$ and $0 \leq \gamma \leq \beta \leq 1$. Let $1 < p \leq q < \infty, 2(\beta - \gamma) = n\left(\frac{1}{p} - \frac{1}{q}\right)$ and the condition (7.2) be satisfied for $\alpha = 2(\beta - \gamma)$.*

Then, for any $f \in C_0^\infty(\mathbb{R}^n)$

$$\|[b, \mathcal{T}_1]f\|_{M_{q,\varphi_2}} \lesssim \|b\|_* \, \|f\|_{M_{p,\varphi_1}}.$$

Corollary 8.12. *Assume that* $V \in B_\infty, 0 \le \gamma \le \frac{1}{2} \le \beta \le 1$ *and* $\beta - \gamma \ge \frac{1}{2}$. *Let* $1 \le p \le q < \infty, 2(\beta-\gamma)-1 = n\left(\frac{1}{p} - \frac{1}{q}\right)$ *and the condition* (5.7) *be satisfied for* $\alpha = 2(\beta-\gamma)-1$.

Then, for any $f \in C_0^\infty(\mathbb{R}^n)$

$$\|\mathcal{T}_2 f\|_{M_{q,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}}, \quad \text{for } p > 1$$

and

$$\|\mathcal{T}_2 f\|_{WM_{q,\varphi_2}} \lesssim \|f\|_{M_{1,\varphi_1}} \quad \text{for } p = 1$$

Corollary 8.13. *Assume that* $V \in B_\infty, 0 \le \gamma \le \frac{1}{2} \le \beta \le 1, b \in BMO(\mathbb{R}^n)$ *and* $\beta - \gamma \ge \frac{1}{2}$. *Let* $1 < p \le q < \infty, 2(\beta-\gamma)-1 = n\left(\frac{1}{p} - \frac{1}{q}\right)$ *and the condition* (7.2) *be satisfied for* $\alpha = 2(\beta-\gamma)-1$.

Then, for any $f \in C_0^\infty(\mathbb{R}^n)$

$$\|[b, \mathcal{T}_2]f\|_{M_{q,\varphi_2}} \lesssim \|b\|_* \, \|f\|_{M_{p,\varphi_1}}.$$

8.5. Fractional Powers of the Some Analytic Semigroups

The theorems of the previous sections can be applied to various operators which are estimated from above by Riesz potentials. We give some examples.

Suppose that L is a linear operator on L_2 which generates an analytic semigroup e^{-tL} with the kernel $p_t(x,y)$ satisfying a Gaussian upper bound, that is,

$$|p_t(x,y)| \le \frac{c_1}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}} \tag{8.4}$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$, where c_1, $c_2 > 0$ are independent of x, y and t.

For $0 < \alpha < n$, the fractional powers $L^{-\alpha/2}$ of the operator L are defined by

$$L^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL} f(x) \frac{dt}{t^{-\alpha/2+1}}.$$

Note that if $L = -\triangle$ is the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the Riesz potential I_α. See, for example, Chapter 5 in [40].

Theorem 8.14. *Let condition* (8.4) *be satisfied. Moreover, let* $1 \le p < \infty, 0 < \alpha < \frac{n}{p}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, (\varphi_1, \varphi_2)$ *satisfy condition* (5.7). *Then* $L^{-\alpha/2}$ *is bounded from* M_{p,φ_1} *to* M_{q,φ_2} *for* $p > 1$ *and from* M_{1,φ_1} *to* WM_{q,φ_2} *for* $p = 1$.

Proof. Since the semigroup e^{-tL} has the kernel $p_t(x,y)$ which satisfies condition (8.4), it follows that

$$|L^{-\alpha/2} f(x)| \lesssim I_\alpha(|f|)(x)$$

(see [16]). Hence by the aforementioned theorems we have

$$\|L^{-\alpha/2} f\|_{M_{q,\varphi_2}} \lesssim \|I_\alpha(|f|)\|_{M_{q,\varphi_2}} \lesssim \|f\|_{M_{p,\varphi_1}}.$$

□

Let b be a locally integrable function on $\mathbb{R}^n$, the commutator of b and $L^{-\alpha/2}$ is defined as follows

$$[b, L^{-\alpha/2}]f(x) = b(x)L^{-\alpha/2}f(x) - L^{-\alpha/2}(bf)(x).$$

In [16] extended the result of [8] from $(-\Delta)$ to the more general operator L defined above. More precisely, they showed that when $b \in BMO(\mathbb{R}^n)$, then the commutator operator $[b, L^{-\alpha/2}]$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ for $1 < p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then from Theorem 7.4 we get

Theorem 8.15. *Let condition* (8.4) *be satisfied. Moreover, let* $1 < p < \infty, 0 < \alpha < \frac{n}{p}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, b \in BMO(\mathbb{R}^n)$, *and* (φ_1, φ_2) *satisfies the condition* (7.2). *Then* $[b, L^{-\alpha/2}]$ *is bounded from* M_{p,φ_1} *to* M_{q,φ_2}.

Property (8.4) is satisfied for large classes of differential operators (see, for example [5]). In [5] also other examples of operators which are estimates from above by Riesz potentials are given. In these cases Theorems 5.4 and 7.4 are also applicable for proving boundedness of those operators and commutators from M_{p,φ_1} to M_{q,φ_2}.

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