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The boundedness of the generalized anisotropic potentials with rough kernels in the Lorentz spaces

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In this paper, we study the generalized anisotropic potential integral $K_{\alpha,\gamma} \otimes f$ and anisotropic fractional integral $I_{\Omega,\alpha,\gamma} f$ with rough kernels, associated with the Laplace–Bessel differential operator Δ_B. We prove that the operator $f \to K_{\alpha,\gamma} \otimes f$ is bounded from the Lorentz spaces $L_{p,r,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,s,\gamma}(\mathbb{R}^n_{k,+})$ for $1 \le p < q \le \infty$, $1 \le r \le s \le \infty$. As a result of this, we get the necessary and sufficient conditions for the boundedness of $I_{\Omega,\alpha,\gamma}$ from the Lorentz spaces $L_{p,s,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,r,\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < q < \infty$, $1 \le r \le s \le \infty$ and from $L_{1,r,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\infty,\gamma}(\mathbb{R}^n_{k,+}) \equiv WL_{q,\gamma}(\mathbb{R}^n_{k,+})$, $1 < q < \infty$, $1 \le r \le \infty$. Furthermore, for the limiting case $p = Q/\alpha$, we give an analogue of Adams' theorem on the exponential integrability of $I_{\Omega,\alpha,\gamma}$ in $L_{Q/\alpha,\gamma}(\mathbb{R}^n_{k,+})$.

Keywords: Laplace–Bessel differential operator; generalized anisotropic potential integral; rough anisotropic fractional integral; Lorentz spaces

2000 Mathematics Subject Classifications: 42B20; 42B25; 42B35

1. Introduction

Let $\mathbb{R}^n_{k,+}$ be the part of the Euclidean space $\mathbb{R}^n$ of points $x = (x', x'')$ defined by the inequalities $x_1 > 0, \dots, x_k > 0$, $x' = (x_1, \dots, x_k)$, $x'' = (x_{k+1}, \dots, x_n)$, $1 \le k \le n$, and $\gamma = (\gamma_1, \dots, \gamma_k)$ is a multi-index consisting of fixed positive numbers such that $|\gamma| = \gamma_1 + \cdots + \gamma_k$ and $(x')^\gamma = x_1^{\gamma_1} \cdot \ldots \cdot x_k^{\gamma_k}$. Note that in the case $k = n$ we assume $x = x'$.

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x, r)$ denote the open ball centred at x of radius r. Let $d = (d_1, \dots, d_n)$, $d_i \ge 1$, $i = 1, \dots, n$, $|d| = \sum_{i=1}^n d_i$ and $t^d x \equiv (t^{d_1} x_1, \dots, t^{d_n} x_n)$. By [3,5], the function $F(x, \rho) = \sum_{i=1}^n x_i^2 \rho^{-2d_i}$, considered for any fixed $x \in \mathbb{R}^n$, is a decreasing one with respect to $\rho > 0$ and the equation $F(x, \rho) = 1$ is uniquely solvable. This unique solution will be denoted by $\rho(x)$. It is a simple matter to check that $\rho(x - y)$ defines a distance between any two points $x, y \in \mathbb{R}^n$. Thus $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous metric

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space [3–5]. The balls with respect to ρ, centered at x of radius r, are just the ellipsoids

$$\mathcal{E}_d(x,r)=\left\{y\in\mathbb{R}^n:\frac{(y_1-x_1)^2}{r^{2d_1}}+\cdots+\frac{(y_n-x_n)^2}{r^{2d_n}}<1\right\},$$

with the Lebesgue measure $|\mathcal{E}_d(0,r)|_\gamma=\int_{\mathcal{E}_d(0,r)}(x')^\gamma\,\mathrm{d}x=\omega(n,k,\gamma)r^Q$, $\omega(n,k,\gamma)=|B(0,1)|_\gamma$, $Q=|d|+(d,\gamma)$ and $(d,\gamma)=\sum_{i=1}^n d_i\gamma_i$. If $d=\mathbf{1}\equiv(1,\dots,1)$, then clearly $\rho(x)=|x|$ and $\mathcal{E}_\mathbf{1}(x,r)=B(x,r)$.

In this paper, we obtain some inequalities on the generalized anisotropic potential integrals with rough kernels generated by the generalized shift operator of the form [15–17]

$$T^y f(x)=C_{k,\gamma}\int_0^\pi\cdots\int_0^\pi f((x',y')_\alpha,x''-y'')\,\mathrm{d}\nu(\alpha),$$

where $C_{k,\gamma}=\pi^{-k/2}\prod_{i=1}^k(\Gamma((\gamma_i+1)/2))/(\Gamma(\gamma_i/2))$, $x=(x',x'')\in\mathbb{R}^n_{k,+}$, $(x',y')_\alpha=((x_1,y_1)_{\alpha_1},\dots,(x_k,y_k)_{\alpha_k})$,

$$(x_i,y_i)_{\alpha_i}=\sqrt{x_i^2-2x_iy_i\cos\alpha_i+y_i^2},\quad 1\le i\le k,\qquad \mathrm{d}\nu(\alpha)=\prod_{i=1}^k\sin^{\gamma_i-1}\alpha_i\,\mathrm{d}\alpha_i,\quad 1\le k\le n.$$

Note that the generalized shift operator T^y is closely related to the Δ_B Laplace–Bessel differential operator [15]

$$\Delta_B=\sum_{i=1}^k B_i+\sum_{i=k+1}^n\frac{\partial^2}{\partial x_i^2},\qquad k=1,\dots,n,$$

where $B_i=\partial^2/\partial x_i^2+\gamma_i/x_i\partial/\partial x_i$, $\gamma_i>0$, $i=1,\dots,k$.

Furthermore, T^y generates the corresponding convolution

$$(f\otimes g)(x)=\int_{\mathbb{R}^n_{k,+}}f(y)T^yg(x)(y')^\gamma\,\mathrm{d}y.$$

The fractional integrals and related topics associated with the Laplace–Bessel differential operator have been research areas for many mathematicians such as Kipriyanov [15], Lyakhov [17], Aliev and Gadjiev [2], Gadjiev and Guliyev [6], Serbetci and Ekincioglu [21], Guliyev [7–11], Guliyev *et al.* [12] and Guliyev and Garakhanova [14].

Suppose $K_{\alpha,\gamma}$ belongs to the weak $L_{Q/(Q-\alpha),\gamma}(\mathbb{R}^n_{k,+})$, $\Omega\in L_{Q/(Q-\alpha),\gamma}(S^{n-1}_{k,+})$, and let Ω be d-homogeneous of degree zero on $\mathbb{R}^n_{k,+}$, i.e. $\Omega(t^dx)=\Omega(x)$ for all $t>0$, $x\in\mathbb{R}^n_{k,+}$, where $S^{n-1}_{k,+}=\{x\in\mathbb{R}^n_{k,+}:|x|^2\equiv x_1^2+\cdots+x_n^2=1\}$, and $0<\alpha<Q$.

We define the generalized anisotropic potential integral by

$$(K_{\alpha,\gamma}\otimes f)(x)=\int_{\mathbb{R}^n_{k,+}}K_\alpha(y)T^yf(x)(y')^\gamma\,\mathrm{d}y,$$

and the anisotropic fractional integral by

$$I_{\Omega,\alpha,\gamma}f(x)=\int_{\mathbb{R}^n_{k,+}}\frac{\Omega(y)}{\rho(y)^{Q-\alpha}}T^yf(x)(y')^\gamma\,\mathrm{d}y$$

with rough kernels associated with the Laplace–Bessel differential operator Δ_B. It is clear that when $\Omega\equiv1$, $I_{\Omega,\alpha,\gamma}$ is the usual anisotropic Riesz potential $I_{\alpha,\gamma}$, associated with Δ_B [8,9].

In this paper, we obtain a pointwise rearrangement estimate of the generalized anisotropic potential integral $K_{\alpha,\gamma} \otimes f$ by using the O'Neil inequality for the convolution given in [11] by the authors. Then we prove that $K_{\alpha,\gamma} \otimes f$ is bounded from the Lorentz spaces $L_{p,r,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,s,\gamma}(\mathbb{R}^n_{k,+})$, $1 \le p \le Q/\alpha$, $1 \le r \le s \le \infty$, and $1/p - 1/q = \alpha/Q$, $Q = |d| + (d,\gamma)$, where $(d,\gamma) = \sum_{i=1}^n d_i\gamma_i$. As a result of this, we obtain the necessary and sufficient conditions for the rough anisotropic fractional integral operator $I_{\Omega,\alpha,\gamma}$ to be bounded from the Lorentz spaces $L_{p,s,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,r,\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < q < \infty$, $1 \le r \le s \le \infty$ and from the spaces $L_{1,r,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$, $1 < q < \infty$, $1 \le r \le \infty$. Finally, we give an analogue of Adams' theorem on the exponential integrability of anisotropic potential integrals with rough kernel $I_{\Omega,\alpha,\gamma} f$ for the limiting case $p = Q/\alpha$ in $L_{Q/\alpha,\gamma}(\mathbb{R}^n_{k,+})$.

2. Preliminaries

Denote by $L_{p,\gamma} \equiv L_{p,\gamma}(\mathbb{R}^n_{k,+})$ the set of all classes of measurable functions f with finite norm

$$\|f\|_{L_{p,\gamma}} = \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma \, \mathrm{d}x\right)^{1/p}, \quad 1 \le p < \infty.$$

If $p = \infty$, we assume

$$L_{\infty,\gamma}(\mathbb{R}^n_{k,+}) = L_\infty(\mathbb{R}^n_{k,+}) = \left\{ f : \|f\|_{L_{\infty,\gamma}} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n_{k,+}} |f(x)| < \infty \right\}.$$

Suppose $f : \mathbb{R}^n_{k,+} \to \mathbb{R}$ is a measurable function, then the decreasing γ-rearrangement of f defined on $[0,\infty)$ by

$$f^*_\gamma(t) = \inf\{s > 0 \ : \ f_{*,\gamma}(s) \le t\}, \quad (t \ge 0)$$

where $f_{*,\gamma}$ is the γ-distribution function of f [11,18] defined by

$$\begin{aligned} f_{*,\gamma}(s) &\equiv |\{x \in \mathbb{R}^n_{k,+} : |f(x)| > s\}|_\gamma \\ &= \int_{\{x\in\mathbb{R}^n_{k,+}:|f(x)|>s\}} (x')^\gamma \, \mathrm{d}x, \quad s \ge 0. \end{aligned}$$

We denote by $WL_{p,\gamma}(\mathbb{R}^n_{k,+})$ the weak $L_{p,\gamma}$ (Marcinkiewicz) space of all measurable functions f with finite norm

$$\|f\|_{WL_{p,\gamma}} = \sup_{t>0} t^{1/p} f^*_\gamma(t) < \infty, \quad 1 \le p < \infty.$$

We define a function f^{**}_γ on $(0,\infty)$ by $f^{**}_\gamma(t) = (1/t)\int_0^t f^*_\gamma(s)\,\mathrm{d}s$, $\quad t > 0$.

DEFINITION 1 *If $0 < p, q < \infty$, then the Lorentz space $L_{p,q,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,q}(\mathbb{R}^n_{k,+}, (x')^\gamma \,\mathrm{d}x)$ is the set of all classes of measurable functions f with the finite quasi-norm*

$$\|f\|_{p,q,\gamma} \equiv \|f\|_{L_{p,q,\gamma}} = \left(\int_0^\infty (t^{1/p} f^*_\gamma(t))^q \frac{\mathrm{d}t}{t}\right)^{1/q}.$$

If $0 < p \le \infty$, $q = \infty$, then $L_{p,\infty,\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,\gamma}(\mathbb{R}^n_{k,+})$.
If $1 \le q \le p$ or $p = q = \infty$, then the functional $\|f\|_{p,q,\gamma}$ is a norm. If $p = q = \infty$, then the space $L_{\infty,\infty,\gamma}(\mathbb{R}^n_{k,+})$ is denoted by $L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$.

In the case $1 < p, q < \infty$, we define

$$\|f\|_{(p,q),\gamma} = \left(\int_0^\infty (t^{1/p} f_\gamma^{**}(t))^q \frac{\mathrm{d}t}{t}\right)^{1/q},$$

(with the usual modification if $0 < p \leq \infty$, $q = \infty$) which is a norm on $L_{p,q,\gamma}(R^n_{k,+})$ for $1 < p < \infty$, $1 \leq q \leq \infty$ or $p = q = \infty$.

If $1 < p \leq \infty$ and $1 \leq q \leq \infty$, then

$$\|f\|_{p,q,\gamma} \leq \|f\|_{(p,q),\gamma} \leq p'\|f\|_{p,q,\gamma},$$

where $p' = p/(p-1)$. That is, the quasi-norms $\|f\|_{p,q,\gamma}$ and $\|f\|_{(p,q),\gamma}$ are equivalent.

3. Main results

In the following theorem, we give a pointwise rearrangement estimate of the generalized anisotropic potential integral $K_{\alpha,\gamma} \otimes f$ associated with the Laplace–Bessel differential operator Δ_B by using the O'Neil inequality for the convolutions obtained in Section 4 (see Theorem 5).

THEOREM 1 *Suppose that $K_{\alpha,\gamma} \in WL_{Q/(Q-\alpha),\gamma}(\mathbb{R}^n_{k,+})$, $0 < \alpha < Q$. Then for $K_{\alpha,\gamma} \otimes f$ the following inequalities hold*

$$(K_{\alpha,\gamma} \otimes f)^*_\gamma(t) \leq (K_{\alpha,\gamma} \otimes f)^{**}_\gamma(t) \leq A_1 \left(\frac{Q}{\alpha} t^{\alpha/Q-1} \int_0^t f^*_\gamma(s)\,\mathrm{d}s + \int_t^\infty s^{\alpha/Q-1} f^*_\gamma(s)\,\mathrm{d}s\right), \tag{1}$$

where $A_1 = C_{k,\gamma}(Q/\alpha)\|K_\alpha\|_{WL_{Q/(Q-\alpha),\gamma}}$.

COROLLARY 1 *Suppose that Ω is d-homogeneous of degree zero on $\mathbb{R}^n_{k,+}$ and $\Omega \in L_{Q/(Q-\alpha),\gamma}(S^{n-1}_{k,+})$, $0 < \alpha < Q$. Then for the rough anisotropic fractional integral $I_{\Omega,\alpha,\gamma} f$ the following inequalities hold*

$$(I_{\Omega,\alpha,\gamma} f)^*_\gamma(t) \leq (I_{\Omega,\alpha,\gamma} f)^{**}_\gamma(t) \leq A_2 \left(\frac{Q}{\alpha} t^{\alpha/Q-1} \int_0^t f^*_\gamma(s)\,\mathrm{d}s + \int_t^\infty s^{\alpha/Q-1} f^*_\gamma(s)\,\mathrm{d}s\right),$$

where

$$A_2 = C_{k,\gamma}\left(\frac{Q}{\alpha}\right)\left(\frac{A}{Q}\right)^{(Q-\alpha)/Q}, \quad A = \|\Omega\|^{Q/(Q-\alpha)}_{L_{Q/(Q-\alpha),\gamma}(S^{n-1}_{k,+})}.$$

COROLLARY 2 *For the anisotropic Riesz potential*

$$I_{\alpha,\gamma} f(x) = \int_{\mathbb{R}^n_{k,+}} T^y \rho(x)^{\alpha-Q} f(y)(y')^\gamma\,\mathrm{d}y, \quad 0 < \alpha < Q,$$

the following inequalities hold

$$(I_{\alpha,\gamma} f)^*_\gamma(t) \leq (I_{\alpha,\gamma} f)^{**}_\gamma(t) \leq A_3 \left(\frac{Q}{\alpha} t^{\alpha/Q-1} \int_0^t f^*_\gamma(s)\,\mathrm{d}s + \int_t^\infty s^{\alpha/Q-1} f^*_\gamma(s)\,\mathrm{d}s\right),$$

where $A_3 = C_{k,\gamma}(Q/\alpha)\omega(n,k,\gamma)^{(Q-\alpha)/Q}$.

One of the main purposes of this paper is to give the following Hardy–Littlewood–Sobolev inequality for the generalized anisotropic potential integral $K_{\alpha,\gamma} \otimes f$ associated with the Laplace–Bessel differential operator Δ_B in the Lorentz spaces.

THEOREM 2 (Hardy–Littlewood–Sobolev theorem for $K_{\alpha,\gamma} \otimes f$ in the Lorentz spaces) *Let* $0 < \alpha < Q$, $1 \le p < q < \infty$, *and* $K_{\alpha,\gamma} \in WL_{Q/(Q-\alpha),\gamma}(\mathbb{R}^n_{k,+})$. *Then*

(1) *If* $1 < p < Q/\alpha$, $1 \le r \le s \le \infty$, $f \in L_{p,r,\gamma}(\mathbb{R}^n_{k,+})$ *and* $1/p - 1/q = \alpha/Q$, *then* $K_{\alpha,\gamma} \otimes f \in L_{q,s,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|K_{\alpha,\gamma} \otimes f\|_{L_{q,s,\gamma}} \le A_1 K(p,q,r,s)\|f\|_{L_{p,r,\gamma}},$$

where $K(p,q,r,s) = ((Q/\alpha)(p')^{1/s}(p's'/r')^{1/r'} + (qr/s)^{1/s}q^{1/r'})$, $p' = p/(p-1)$.

(2) *If* $p = 1$, $1 \le r \le \infty$, $f \in L_{1,r,\gamma}(\mathbb{R}^n_{k,+})$ *and* $1 - 1/q = \alpha/Q$, *then* $K_{\alpha,\gamma} \otimes f \in WL_{q,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|K_{\alpha,\gamma} \otimes f\|_{WL_{q,\gamma}} \le A_1 \left(\frac{Q}{\alpha} + 1\right) \|f\|_{L_{1,r,\gamma}}.$$

(3) *If* $p = Q/\alpha$, $r = 1$, *and* $f \in L_{Q/\alpha,1,\gamma}(\mathbb{R}^n_{k,+})$, *then* $K_{\alpha,\gamma} \otimes f \in L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|K_{\alpha,\gamma} \otimes f\|_{L_{\infty,\gamma}} \le A_1 \frac{Q}{\alpha} \|f\|_{L_{Q/\alpha,1,\gamma}}.$$

As a consequence of Theorem 2, we have the following corollaries.

COROLLARY 3 (Hardy–Littlewood–Sobolev theorem for $I_{\Omega,\alpha,\gamma} f$ in the Lorentz spaces) *Let* $0 < \alpha < Q$, $1 \le p < q < \infty$, *and let* Ω *be* d*-homogeneous of degree zero on* $\mathbb{R}^n_{k,+}$ *and* $\Omega \in L_{Q/(Q-\alpha),\gamma}(S^{n-1}_{k,+})$. *Then*

(1) *If* $1 < p < Q/\alpha$, $1 \le r \le s \le \infty$, $f \in L_{p,r,\gamma}(\mathbb{R}^n_{k,+})$ *and* $1/p - 1/q = \alpha/Q$, *then* $I_{\Omega,\alpha,\gamma} f \in L_{q,s,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|I_{\Omega,\alpha,\gamma} f\|_{L_{q,s,\gamma}} \le A_2 K(p,q,r,s)\|f\|_{L_{p,r,\gamma}}.$$

(2) *If* $p = 1$, $1 \le r \le \infty$, $f \in L_{1,r,\gamma}(\mathbb{R}^n_{k,+})$ *and* $1 - 1/q = \alpha/Q$, *then* $I_{\Omega,\alpha,\gamma} f \in WL_{q,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|I_{\Omega,\alpha,\gamma} f\|_{WL_{q,\gamma}} \le A_2 \left(\frac{Q}{\alpha} + 1\right) \|f\|_{L_{1,r,\gamma}}.$$

(3) *If* $p = Q/\alpha$, $r = 1$, *and* $f \in L_{Q/\alpha,1,\gamma}(\mathbb{R}^n_{k,+})$, *then* $I_{\Omega,\alpha,\gamma} f \in L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|I_{\Omega,\alpha,\gamma} f\|_{L_{\infty,\gamma}} \le A_2 \frac{Q}{\alpha} \|f\|_{L_{Q/\alpha,1,\gamma}}.$$

COROLLARY 4 (Hardy–Littlewood–Sobolev theorem for $I_{\alpha,\gamma} f$ in the Lorentz spaces) *Let* $0 < \alpha < Q$ *and* $1 \le p < q < \infty$.

(1) *If* $1 < p < Q/\alpha$, $1 \le r \le s \le \infty$, $f \in L_{p,r,\gamma}(\mathbb{R}^n_{k,+})$ *and* $1/p - 1/q = \alpha/Q$, *then* $I_{\alpha,\gamma} f \in L_{q,s,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|I_{\alpha,\gamma} f\|_{L_{q,s,\gamma}} \le A_3 K(p,q,r,s)\|f\|_{L_{p,r,\gamma}}.$$

(2) *If* $p = 1$, $1 \le r \le \infty$, $f \in L_{1,r,\gamma}(\mathbb{R}^n_{k,+})$ *and* $1 - 1/q = \alpha/Q$, *then* $I_{\alpha,\gamma} f \in WL_{q,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|I_{\alpha,\gamma} f\|_{WL_{q,\gamma}} \le A_3 \left(\frac{Q}{\alpha} + 1\right) \|f\|_{L_{1,r,\gamma}}.$$

(3) *If* $p = Q/\alpha$, $r = 1$, *and* $f \in L_{Q/\alpha,1,\gamma}(\mathbb{R}^n_{k,+})$, *then* $I_{\alpha,\gamma} f \in L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|I_{\alpha,\gamma} f\|_{L_{\infty,\gamma}} \le A_3 \frac{Q}{\alpha} \|f\|_{L_{Q/\alpha,1,\gamma}}.$$

COROLLARY 5 *Let* $0 < \alpha < Q$, Ω *be* d*-homogeneous of degree zero on* $\mathbb{R}^n_{k,+}$ *and* $\Omega \in L_{Q/(Q-\alpha),\gamma}(S^{n-1}_{k,+})$.

(1) *If* $1 < p < Q/\alpha$, $1 \le r \le s \le \infty$, $f \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$ *and* $1/p - 1/q = \alpha/Q$, *then* $I_{\Omega,\alpha,\gamma} f \in L_{q,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|I_{\Omega,\alpha,\gamma} f\|_{L_{q,\gamma}} \le A_4 K(p,q) \|f\|_{L_{p,\gamma}},$$

where $A_4 = C_{k,\gamma}(Q/\alpha)(A/Q)^{(Q-\alpha)/Q}$, $K(p,q) \equiv K(p,q,p,q) = ((Q/\alpha) p^{1/q} q^{1/p'} + (p')^{1/q}(q')^{1/p'})$.

(2) *If* $p = 1$, $f \in L_{1,\gamma}(\mathbb{R}^n_{k,+})$ *and* $1 - 1/q = \alpha/Q$, *then* $I_{\Omega,\alpha,\gamma} f \in WL_{q,\gamma}(\mathbb{R}^n_{k,+})$ *and*

$$\|I_{\Omega,\alpha,\gamma} f\|_{WL_{q,\gamma}} \le A_4 \frac{Q}{\alpha} \|f\|_{L_{1,\gamma}}.$$

Note that in the case $\Omega \equiv 1$, Corollary 5 was proved in [9].

In the following theorem, we get the necessary and sufficient conditions for the boundedness of the rough anisotropic fractional integral operator $I_{\Omega,\alpha,\gamma}$ in the Lorentz spaces.

THEOREM 3 *Let* $1 \le p < q < \infty$ *and let* Ω *be* d*-homogeneous of degree zero on* $\mathbb{R}^n_{k,+}$ *and* $\Omega \in L_{Q/(Q-\alpha),\gamma}(S^{n-1}_{k,+})$, $0 < \alpha < Q$.

(1) *If* $1 < p < Q/\alpha$, $1 \le r \le s \le \infty$, *then the condition* $1/p - 1/q = \alpha/Q$ *is necessary and sufficient for the boundedness of* $I_{\Omega,\alpha,\gamma}$ *from* $L_{p,r,\gamma}(\mathbb{R}^n_{k,+})$ *to* $L_{q,s,\gamma}(\mathbb{R}^n_{k,+})$.
(2) *If* $p = 1$, $1 \le r \le \infty$, *then the condition* $1 - 1/q = \alpha/Q$ *is necessary and sufficient for the boundedness of* $I_{\Omega,\alpha,\gamma}$ *from* $L_{1,r,\gamma}(\mathbb{R}^n_{k,+})$ *to* $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$.

We can give the following corollaries from Theorem 3.

COROLLARY 6 *Let* $1 \le p < q < \infty$ *and* $0 < \alpha < Q$. *Let also* Ω *be* d*-homogeneous of degree zero on* $\mathbb{R}^n_{k,+}$ *and* $\Omega \in L_{Q/(Q-\alpha),\gamma}(S^{n-1}_{k,+})$.

(1) *If* $1 < p < Q/\alpha$, *then the condition* $1/p - 1/q = \alpha/Q$ *is necessary and sufficient for the boundedness of* $I_{\Omega,\alpha,\gamma}$ *from* $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ *to* $L_{q,\gamma}(\mathbb{R}^n_{k,+})$.
(2) *If* $p = 1$, *then the condition* $1 - 1/q = \alpha/Q$ *is necessary and sufficient for the boundedness of* $I_{\Omega,\alpha,\gamma}$ *from* $L_{1,\gamma}(\mathbb{R}^n_{k,+})$ *to* $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$.

COROLLARY 7 *Let* $1 \le p < q < \infty$ *and* $0 < \alpha < Q$.

(1) *If* $1 < p < Q/\alpha$, $1 \le r \le s \le \infty$, *then the condition* $1/p - 1/q = \alpha/Q$ *is necessary and sufficient for the boundedness of* $I_{\alpha,\gamma}$ *from the Lorentz spaces* $L_{p,r,\gamma}(\mathbb{R}^n_{k,+})$ *to* $L_{q,s,\gamma}(\mathbb{R}^n_{k,+})$.
(2) *If* $p = 1$, $1 \le r \le \infty$, *then the condition* $1 - 1/q = \alpha/Q$ *is necessary and sufficient for the boundedness of* $I_{\alpha,\gamma}$ *from* $L_{1,r,\gamma}(\mathbb{R}^n_{k,+})$ *to* $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$.

In the limiting case $p = Q/\alpha$ the boundedness of the rough anisotropic fractional integral operator $I_{\Omega,\alpha,\gamma}$ in $L_{Q/\alpha,\gamma}(\mathbb{R}^n_{k,+})$ does not hold. However, the following theorem can be regarded as the substitute of the boundedness for $I_{\Omega,\alpha,\gamma}$ in this case. This theorem is an analogue of the Adams theorem given in [1] by the exponential integrability for the Riesz potential of order α $(0 < \alpha < n)$.

THEOREM 4 *Let $0 < \alpha < Q$, Ω be d-homogeneous of degree zero on $\mathbb{R}^n_{k,+}$ and $\Omega \in L_{Q/(Q-\alpha),\gamma}(S^{n-1}_{k,+})$. Then there is a constant $C_0 = C_0(n, k, \gamma, \alpha)$ depending only on n, k, γ and α such that for all $f \in L_{Q/\alpha,\gamma}(\mathcal{E}_d(0, r))$*

$$\frac{1}{|\mathcal{E}_d(0, r)|_\gamma} \int_{\mathcal{E}_d(0,r)} \exp\left(Q \left|\frac{I_{\Omega,\alpha,\gamma} f(x)}{\|\Omega\|_{L_{Q/(Q-\alpha),\gamma}} \|f\|_{L_{Q/\alpha,\gamma}}}\right|^{Q/(Q-\alpha)}\right) (x')^\gamma \, \mathrm{d}x \le C_0.$$

In the isotropic case, Theorem 4 was provided in [13].

4. Some auxiliary lemmas

LEMMA 1 *Let f and g be measurable functions on $\mathbb{R}^n_{k,+}$ such that* $\sup\{f(x) : x \in \mathbb{R}^n_{k,+}\} \le \lambda$ *and f vanishes outside of a measurable set E with $|E|_\gamma = \tau$. Then, for all $t > 0$,*

$$(f \otimes g)^{**}_\gamma(t) \le \lambda\tau \min\{g^{**}_\gamma(\tau), g^{**}_\gamma(t)\}. \tag{2}$$

Proof For $a > 0$, define

$$g_a = \begin{cases} g(x), & \text{if } |g(x)| \le a \\ 0, & \text{if } |g(x)| > a \end{cases}$$

and let

$$g^a(x) = g(x) - g_a(x).$$

Then, we can write

$$f \otimes g = f \otimes g_a + f \otimes g^a.$$

If $s > a$, then $g^a_{*,\gamma}(s) = g_{*,\gamma}(s) = 0$. If $s \le a$, then we have

$$\begin{aligned} g^a_{*,\gamma}(s) &= \int_{\{y:g^a(y)>s\}} (y')^\gamma \, \mathrm{d}y \\ &= \int_{\{y:s<g^a(y)\le a\}} (y')^\gamma \, \mathrm{d}y \\ &= g_{*,\gamma}(a), \end{aligned}$$

and we have

$$\begin{aligned} (f \otimes g^a)^{**}_\gamma(t) &\le \sup_{\mathbb{R}^n_{k,+}} |(f \otimes g^a)(y)| \\ &\le \sup_E f(y) ||g^a||_{L_{1,\gamma}} \\ &\le \lambda \int_a^\infty g^a_{*,\gamma}(s) \, \mathrm{d}s \\ &\le \lambda\tau a = \lambda\tau g^{**}_\gamma(t). \end{aligned}$$

The last inequality follows from the equality

$$f^{**}_\gamma(t) = f^*_\gamma(t) + \frac{1}{t} \int_{f^*_\gamma(t)}^\infty f_{*,\gamma}(s) \, \mathrm{d}s, \tag{3}$$

and thus, the first inequality of the lemma is established.

To prove the second inequality, set $a = g^*(\tau)$ to obtain

$$\begin{aligned}(f \otimes g)^{**}_\gamma(t) &= \frac{1}{t} \sup_{|A|_\gamma = t} \int_A |(f \otimes g)(y)| (y')^\gamma \, \mathrm{d}y \\ &\le \sup_{\mathbb{R}^n_{k,+}} |(f \otimes g)(y)| \\ &\le \sup_{\mathbb{R}^n_{k,+}} |(f \otimes g_a)(y)| + \sup_{\mathbb{R}^n_{k,+}} |(f \otimes g^a)(y)| \\ &\le \lambda \tau g^*_\gamma(t) + \lambda \int_{g^*_\gamma(\tau)}^\infty g_{*,\gamma}(s) \, \mathrm{d}s \\ &\le \lambda \tau \left[g^*_\gamma(t) + \frac{1}{\tau} \int_{g^*_\gamma(\tau)}^\infty g_{*,\gamma}(s) \, \mathrm{d}s \right] \\ &\le \lambda \tau g^{**}_\gamma(t)\end{aligned}$$

by Equation (3). ∎

In the following theorem, we show that the O'Neil inequality for rearrangements of the convolution associated with the Laplace–Bessel differential operator Δ_B holds. The methods of the proof used here are close to those in [22].

THEOREM 5 (O'Neil inequality for rearrangements of convolutions associated with Δ_B)
If f and g are measurable functions, then for any $t > 0$

$$(f \otimes g)^{**}_\gamma(t) \le t f^{**}_\gamma(t) g^{**}_\gamma(t) + \int_t^\infty f^*_\gamma(u) g^*_\gamma(u) \, \mathrm{d}u. \tag{4}$$

Proof Fix $t > 0$ and select a doubly infinite sequence $\{y_i\}$ whose indices ranges from $-\infty$ to ∞ such that

$$\begin{aligned} y_0 &= f^*_\gamma(t) \\ y_i &\le y_{i+1} \\ \lim_{i\to\infty} y_i &= \infty \\ \lim_{i\to-\infty} y_i &= 0. \end{aligned}$$

Let

$$f(z) = \sum_{i=-\infty}^{\infty} f_i(z),$$

where

$$f_i(z) = \begin{cases} 0, & \text{if } |f(z)| \le y_{i-1}; \\ f(z) - y_{i-1} \operatorname{sgn} f(z), & \text{if } y_{i-1} < |f(z)| \le y_i; \\ y_i - y_{i-1} \operatorname{sgn} f(z), & \text{if } y_i < |f(z)|. \end{cases}$$

Clearly, the series converges absolutely, and therefore,

$$\begin{aligned} f \otimes g &= \left(\sum_{i=-\infty}^{\infty} f_i \right) \otimes g \\ &= \left(\sum_{i=-\infty}^{0} f_i \right) \otimes g + \left(\sum_{i=1}^{\infty} f_i \right) \otimes g \\ &= h_1 + h_2 \end{aligned}$$

with

$$(f \otimes g)_\gamma^{**}(t) \le (h_1)_\gamma^{**}(t) + (h_2)_\gamma^{**}(t).$$

To evaluate $(h_2)_\gamma^{**}(t)$, we use inequality (2) with $E_i \equiv \{z : |f(z)| > y_{i-1}\} = E$ and $a = y_i - y_{i-1}$ to obtain

$$\begin{aligned} (h_2)_\gamma^{**}(t) &\le \sum_{i=1}^{\infty} (y_i - y_{i-1}) f_{*,\gamma}(y_{i-1}) g_\gamma^{**}(t) \\ &= g_\gamma^{**}(t) \sum_{i=1}^{\infty} f_{*,\gamma}(y_{i-1})(y_i - y_{i-1}). \end{aligned}$$

The series on the right is an infinite Riemann sum for the integral

$$\int_{f_\gamma^*(t)}^{\infty} f_{*,\gamma}(y)\,\mathrm{d}y,$$

and provides an arbitrarily close approximation with an appropriate choice of the sequence $\{y_i\}$. Therefore,

$$(h_2)_\gamma^{**}(t) \le g_\gamma^{**}(t) \int_{f_\gamma^*(t)}^{\infty} f_{*,\gamma}(y)\,\mathrm{d}y. \tag{5}$$

From inequality (2),

$$(h_1)_\gamma^{**}(t) \le \sum_{i=1}^{\infty} (y_i - y_{i-1}) f_{*,\gamma}(y_{i-1}) g_\gamma^{**}(f_{*,\gamma}(y_{i-1})).$$

Similarly as in [22, Lemma 1.8.8], we have that

$$\begin{aligned} (h_1)_\gamma^{**}(t) &\le \int_0^{f_\gamma^*(t)} f_{*,\gamma}(y) g_\gamma^{**}(f_{*,\gamma}(y))\,\mathrm{d}y \\ &= -\int_t^{\infty} u g_\gamma^{**}(u)\,\mathrm{d}f_\gamma^*(u) \\ &= -u g_\gamma^{**}(u) f_\gamma^*(u)|_t^{\infty} + \int_t^{\infty} f_\gamma^*(u) g_\gamma^*(u)\,\mathrm{d}u \\ &\le t g_\gamma^{**}(t) f_\gamma^*(t) + \int_t^{\infty} f_\gamma^*(u) g_\gamma^*(u)\,\mathrm{d}u \end{aligned} \tag{6}$$

Thus, from (3), (5) and (6),

$$(h_1)_\gamma^{**}(t) + (h_2)_\gamma^{**}(t) \le g_\gamma^{**}(t)\left[t f_\gamma^*(t) + \int_{f_\gamma^*(t)}^\infty f_{*,\gamma}(y)\,\mathrm{d}y\right] + \int_t^\infty f_\gamma^*(u) g_\gamma^*(u)\,\mathrm{d}u$$
$$\le t f_\gamma^{**}(t) g_\gamma^{**}(t) + \int_t^\infty f_\gamma^*(u) g_\gamma^*(u)\,\mathrm{d}u. \qquad \blacksquare$$

We need the following two generalized Hardy inequalities [19] which are to be used in the proof of Theorem 2.

LEMMA 2 *Let* $1 \le r \le s \le \infty$ *and let* v *and* w *be two functions such that measurable and positive a.e. on* $(0,\infty)$. *Then there exists a constant* C *independent of the function* φ *such that*

$$\left(\int_0^\infty \left(\int_0^t \varphi(\tau)\,\mathrm{d}\tau\right)^s w(t)\,\mathrm{d}t\right)^{1/s} \le C\left(\int_0^\infty \varphi(t)^r v(t)\,\mathrm{d}t\right)^{1/r}, \tag{7}$$

if and only if

$$K = \sup_{r>0}\left(\int_r^\infty w(t)\,\mathrm{d}t\right)^{1/s}\left(\int_0^r v(t)^{1-r'}\,\mathrm{d}t\right)^{1/r'} < \infty. \tag{8}$$

Moreover, if C *is the best constant in* (7) *and* K *is defined by* (8), *then*

$$K \le C \le k(r,s)K. \tag{9}$$

Here the constant $k(r,s)$ *in* (9) *can be written in various forms. For example* [20],

$$k(r,s) = r^{1/s}(r')^{1/r'} \quad or \quad k(r,s) = s^{1/s}(s')^{1/r'} \quad or \quad k(r,s) = \left(1+\frac{s}{r'}\right)^{1/s}\left(1+\frac{r'}{s}\right)^{1/r'}.$$

LEMMA 3 *Let* $1 \le r \le s \le \infty$, *and let* v *and* w *be two functions such that measurable and positive a.e. on* $(0,\infty)$. *Then there exists a constant* C *independent of the function* φ *such that*

$$\left(\int_0^\infty \left(\int_t^\infty \varphi(\tau)\,\mathrm{d}\tau\right)^s w(t)\,\mathrm{d}t\right)^{1/s} \le C\left(\int_0^\infty \varphi(t)^r v(t)\,\mathrm{d}t\right)^{1/r} \tag{10}$$

if and only if

$$K_1 = \sup_{r>0}\left(\int_0^r w(t)\,\mathrm{d}t\right)^{1/s}\left(\int_r^\infty v(t)^{1-r'}\,\mathrm{d}t\right)^{1/r'} < \infty.$$

Moreover, the best constant C *in* (10) *satisfies the inequalities* $K_1 \le C \le k(r,s)K_1$.

LEMMA 4 [1] *Let $a(s,t)$ be a non-negative measurable function on $(-\infty,+\infty)\times[0,+\infty)$ such that $0<s<t$*

$$a(s,t)\le 1,\quad \text{a.e. if } 0<s<t, \tag{11}$$

$$\operatorname*{ess\,sup}_{t>0}\left(\int_{-\infty}^{0}+\int_{t}^{\infty} a(s,t)^{p'}\,\mathrm{d}s\right)^{1/p'} = b<\infty. \tag{12}$$

Then there is a constant $C_0=C_0(p,b)$, such that for $\phi\ge 0$ with

$$\int_{-\infty}^{\infty}\phi(s)^p\,\mathrm{d}s\le 1, \tag{13}$$

we have

$$\int_0^{\infty} \mathrm{e}^{-F(t)}\,\mathrm{d}t\le C_0, \tag{14}$$

where

$$F(t)=t-\left(\int_{-\infty}^{\infty} a(s,t)\phi(s)\,\mathrm{d}s\right)^{p'}. \tag{15}$$

5. Proof of the theorems

Proof of Theorem 1 Since $K_{\alpha,\gamma}\in WL_{Q/(Q-\alpha),\gamma}(\mathbb{R}^n_{k,+})$, we have

$$(K_\alpha)^*_\gamma(t)\le \|K_\alpha\|_{WL_{Q/(Q-\alpha),\gamma}} t^{\alpha/Q-1},\quad (K_\alpha)^{**}_\gamma(t)\le \frac{Q}{\alpha}\|K_\alpha\|_{WL_{Q/(Q-\alpha),\gamma}} t^{\alpha/Q-1}.$$

By using inequality (4), we get inequality (1). Hence, the proof of the theorem is completed. ■

Proof of Theorem 2 The proof of the theorem is based on the pointwise rearrangement estimate of $K_{\alpha,\gamma}\otimes f$ obtained in Theorem 1.

(1) Let $1<p<Q/\alpha$, $1\le r\le s\le\infty$, $f\in L_{p,r,\gamma}(\mathbb{R}^n_{k,+})$ and $1/p-1/q=\alpha/Q$. By using inequality (1), we have

$$\begin{aligned}\|K_{\alpha,\gamma}\otimes f\|_{L_{q,s,\gamma}} &= \|(K_{\alpha,\gamma}\otimes f)^*_\gamma(t)t^{1/q-1/s}\|_{L_s(0,\infty)}\\ &\le A_1\frac{Q}{\alpha}\left(\int_0^{\infty} t^{s(\alpha/Q-1)+s/q-1}\left(\int_0^t f^*_\gamma(s)\,\mathrm{d}s\right)^s \mathrm{d}t\right)^{1/s}\\ &\quad + A_1\left(\int_0^{\infty}\left(\int_t^{\infty} s^{\alpha/Q-1} f^*_\gamma(s)\,\mathrm{d}s\right)^s t^{s/q-1}\,\mathrm{d}t\right)^{1/s}.\end{aligned}$$

From Lemma 2, for the validity of the inequality

$$\left(\int_0^{\infty} t^{s(\alpha/Q-1)+s/q-1}\left(\int_0^t f^*_\gamma(\tau)\,\mathrm{d}\tau\right)^s \mathrm{d}t\right)^{1/s}\le C_1\left(\int_0^{\infty}\left(t^{1/p}f^*_\gamma(t)\right)^r\frac{\mathrm{d}t}{t}\right)^{1/r},$$

the necessary and sufficient condition is

$$\sup_{t>0}\left(\int_t^\infty \tau^{s(\alpha/Q-1)+s/q-1}\,\mathrm{d}\tau\right)^{1/s}\left(\int_0^t \tau^{(r/p-1)(1-r')}\,\mathrm{d}\tau\right)^{1/r'}$$
$$= s^{-1/s}\left(1-\frac{\alpha}{Q}-\frac{1}{q}\right)^{-1/s}\left(\frac{p'}{r'}\right)^{1/r'}\sup_{t>0} t^{\alpha/Q-1+1/q+1-1/p} < \infty$$
$$\Longleftrightarrow \frac{1}{p}-\frac{1}{q}=\frac{\alpha}{Q},$$

where

$$C_1 \le s^{-1/s}\left(1-\frac{\alpha}{Q}-\frac{1}{q}\right)^{-1/s}\left(\frac{p'}{r'}\right)^{1/r'} s^{1/s}(s')^{1/r'} = (p')^{1/s}\left(\frac{p's'}{r'}\right)^{1/r'}.$$

Furthermore, from Lemma 3, for the validity of the inequality

$$\left(\int_0^\infty\left(\int_t^\infty \tau^{\alpha/Q-1} f_\gamma^*(\tau)\,\mathrm{d}\tau\right)^s t^{s/q-1}\,\mathrm{d}t\right)^{1/s} \le C_2\left(\int_0^\infty \left(t^{1/p} f_\gamma^*(t)\right)^r \frac{\mathrm{d}t}{t}\right)^{1/r},$$

the necessary and sufficient condition is

$$\sup_{t>0}\left(\int_0^t \tau^{s/q-1}\,\mathrm{d}\tau\right)^{1/s}\left(\int_t^\infty \tau^{(\alpha/Q-1)r'-r'/p+r'/r}\,\mathrm{d}\tau\right)^{1/r'}$$
$$= \left(\frac{q}{s}\right)^{1/s}(r')^{-1/r'}\left(\frac{1}{p}-\frac{\alpha}{Q}\right)^{-1/r'}\sup_{t>0} t^{\alpha/Q-(1/p-1/q)} < \infty \Longleftrightarrow \frac{1}{p}-\frac{1}{q}=\frac{\alpha}{Q},$$

where $C_2 \le (q/s)^{1/s}(r')^{-1/r'}(1/p-\alpha/Q)^{-1/r'} r^{1/s}(r')^{1/r'} = (qr/s)^{1/s} q^{1/r'}$.

By using these inequalities, we obtain

$$\|K_{\alpha,\gamma}\otimes f\|_{L_{q,s,\gamma}} \le A_1\left(C_1\frac{Q}{\alpha}+C_2\right)\|f\|_{L_{p,r,\gamma}}.$$

(2) Let $p=1$, $1-1/q=\alpha/Q$, $1\le r\le\infty$ and $f\in L_{1,r,\gamma}(\mathbb{R}^n_{k,+})$.
From inequality (1), we have

$$\begin{aligned}\|K_{\alpha,\gamma}\otimes f\|_{WL_{q,\gamma}} &= \sup_{t>0} t^{1/q}(K_{\alpha,\gamma}\otimes f)_\gamma^*(t)\\ &\le A_1\sup_{t>0} t^{1/q}\left(\frac{Q}{\alpha}t^{\alpha/Q-1}\int_0^t f_\gamma^*(s)\,\mathrm{d}s+\int_t^\infty s^{\alpha/Q-1}f_\gamma^*(s)\,\mathrm{d}s\right)\\ &= A_1\frac{Q}{\alpha}\sup_{t>0}\int_0^t f_\gamma^*(s)\,\mathrm{d}s + A_1\sup_{t>0} t^{1/q}\int_t^\infty s^{-1/q}f_\gamma^*(s)\,\mathrm{d}s\\ &\le A_1\left(\frac{Q}{\alpha}+1\right)\|f_\gamma^*\|_{L_1(0,\infty)} = A_1\left(\frac{Q}{\alpha}+1\right)\|f\|_{L_{1,\gamma}}.\end{aligned}$$

(3) Let $p=Q/\alpha$, $r=1$ and $f\in L_{Q/\alpha,1,\gamma}(\mathbb{R}^n_{k,+})$.

By using inequality (1), we have

$$\begin{aligned}\|K_{\alpha,\gamma}\otimes f\|_{L_{\infty,\gamma}} &= \sup_{t>0}(K_{\alpha,\gamma}\otimes f)^*_\gamma(t)\\ &\le A_1\sup_{t>0}\left(\frac{Q}{\alpha}t^{\alpha/Q-1}\int_0^t f^*_\gamma(s)\,\mathrm{d}s+\int_t^\infty s^{\alpha/Q-1}f^*_\gamma(s)\,\mathrm{d}s\right)\\ &\le A_1\frac{Q}{\alpha}\int_0^\infty s^{\alpha/Q-1}f^*_\gamma(s)\,\mathrm{d}s = A_1\frac{Q}{\alpha}\|f\|_{L_{Q/\alpha,1,\gamma}}.\end{aligned}$$

Thus, the proof of Theorem 2 is completed. ■

Proof of Theorem 3 Sufficiency of the theorem follows from Corollary 3.

Necessity. (1) Suppose that the operator $I_{\Omega,\alpha,\gamma}$ is bounded from $L_{p,r,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,s,\gamma}(\mathbb{R}^n_{k,+})$ and $1<p<Q/\alpha$.

Define $f_t(x) =: f(t^d x)$ for $t>0$. Then it can be easily shown that

$$\|f_t\|_{L_{p,r,\gamma}} = t^{-Q/p}\|f\|_{L_{p,r,\gamma}},\quad I_{\Omega,\alpha,\gamma}f_t(x) = t^{-\alpha}I_{\Omega,\alpha,\gamma}f(t^d x),$$

and

$$\|I_{\Omega,\alpha,\gamma}f_t\|_{L_{q,s,\gamma}} = t^{-\alpha-Q/q}\|I_{\Omega,\alpha,\gamma}f\|_{L_{q,s,\gamma}}.$$

Since the operator $I_{\Omega,\alpha,\gamma}$ is bounded from $L_{p,r,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,s,\gamma}(\mathbb{R}^n_{k,+})$, we have

$$\|I_{\Omega,\alpha,\gamma}f\|_{L_{q,s,\gamma}} \le C\|f\|_{L_{p,r,\gamma}},$$

where C is independent of f. Then we get

$$\|I_{\Omega,\alpha,\gamma}f\|_{L_{q,s,\gamma}} = t^{\alpha+Q/q}\|I_{\Omega,\alpha,\gamma}f_t\|_{L_{q,s,\gamma}} \le Ct^{\alpha+Q/q}\|f_t\|_{L_{p,r,\gamma}} = Ct^{\alpha+Q/q-Q/p}\|f\|_{L_{p,r,\gamma}}.$$

If $1/p<1/q+\alpha/Q$, then for all $f\in L_{p,r,\gamma}(\mathbb{R}^n_{k,+})$ we have $\|I_{\Omega,\alpha,\gamma}f\|_{L_{q,s,\gamma}}=0$ as $t\to 0$. If $1/p>1/q+\alpha/Q$, then for all $f\in L_{p,r,\gamma}(\mathbb{R}^n_{k,+})$, we have $\|I_{\Omega,\alpha,\gamma}f\|_{L_{q,s,\gamma}}=0$ as $t\to\infty$. Therefore, we get $1/p=1/q+\alpha/Q$.

(2) Suppose that the operator $I_{\Omega,\alpha,\gamma}$ is bounded from $L_{1,r,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$. It is easy to show that

$$\|f_t\|_{L_{1,r,\gamma}} = t^{-Q}\|f\|_{L_{1,r,\gamma}}$$

and

$$\|I_{\Omega,\alpha,\gamma}f_t\|_{WL_{q,\gamma}} = t^{-\alpha-Q/q}\|I_{\Omega,\alpha,\gamma}f\|_{WL_{q,\gamma}}.$$

By the boundedness of $I_{\Omega,\alpha,\gamma}$ from $L_{1,r,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$, we have

$$\|I_{\Omega,\alpha,\gamma}f\|_{WL_{q,\gamma}} \le C\|f\|_{L_{1,r,\gamma}},$$

where C is independent of f. Then we have

$$\begin{aligned}(I_{\Omega,\alpha,\gamma}f_t)_{*,\gamma}(\tau) &= t^{-Q}(I_{\Omega,\alpha,\gamma}f)_{*,\gamma}(t^\alpha\tau),\\ \|I_{\Omega,\alpha,\gamma}f_t\|_{WL_{q,\gamma}} &= t^{-\alpha-Q/q}\|I_{\Omega,\alpha,\gamma}f\|_{WL_{q,\gamma}},\end{aligned}$$

and

$$\|I_{\Omega,\alpha,\gamma}f\|_{WL_{q,\gamma}} = t^{\alpha+Q/q}\|I_{\Omega,\alpha,\gamma}f_t\|_{WL_{q,\gamma}} \le Ct^{\alpha+Q/q}\|f_t\|_{L_{1,r,\gamma}} = Ct^{\alpha+Q/q-Q}\|f\|_{L_{1,r,\gamma}}.$$

If $1<1/q+\alpha/Q$, then for all $f\in L_{1,r,\gamma}(\mathbb{R}^n_{k,+})$ we have $\|I_{\Omega,\alpha,\gamma}f\|_{WL_{q,\gamma}}=0$ as $t\to 0$.

If $1 > 1/q + \alpha/Q$, then for all $f \in L_{1,\gamma}(\mathbb{R}^n_{k,+})$ we have $\|I_{\Omega,\alpha,\gamma} f\|_{WL_{q,\gamma}} = 0$ as $t \to \infty$. Therefore, we get the equality $1 = 1/q + \alpha/Q$ and the proof of the theorem is completed. ■

Proof of Theorem 4 First, assume that $\|f\|_{L_{Q/\alpha,\gamma}} = 1$. By using the O'Neil inequality (Corollary 1) for the rearrangement of a convolution, we have

$$(I_{\Omega,\alpha,\gamma} f)^*_\gamma(t) \le (I_{\Omega,\alpha,\gamma} f)^{**}_\gamma(t) \le C_{k,\gamma} \left(\frac{A}{Q}\right)^{1/p'} \left(pt^{-1/p'} \int_0^t f^*_\gamma(s)\,\mathrm{d}s + \int_t^D s^{-1/p'} f^*_\gamma(s)\,\mathrm{d}s\right), \tag{16}$$

where $p = Q/\alpha$, $p' = Q/(Q-\alpha)$ and $D = |\mathcal{E}_d(0,r)|_\gamma$.

Let

$$a(s,t) = \begin{cases} 1, & 0 < s < t, \\ p\,\mathrm{e}^{(t-s)/p'}, & t < s < \infty, \\ 0, & -\infty < s \le 0, \end{cases}$$

and

$$\phi(s) = D^{1/p} f^*_\gamma(D\,\mathrm{e}^{-s})\,\mathrm{e}^{-s/p}.$$

Then we have

$$\sup_{t>0} \left(\int_{-\infty}^0 + \int_t^\infty a(s,t)^{p'}\,\mathrm{d}s\right)^{1/p'} = \sup_{t>0} \left(\int_t^\infty (p\,\mathrm{e}^{(t-s)/p'})^{p'}\,\mathrm{d}s\right)^{1/p'} = p < \infty,$$

and

$$\begin{aligned}\int_{-\infty}^\infty \phi(s)^p\,\mathrm{d}s &= \int_{-\infty}^\infty D f^*_\gamma(D\,\mathrm{e}^{-s})^p\,\mathrm{e}^{-s}\,\mathrm{d}s = \int_0^\infty f^*_\gamma(t)^p\,\mathrm{d}t \\ &= \int_0^D f^*_\gamma(t)^p\,\mathrm{d}t = \int_{\mathcal{E}_d(0,r)} |f(x)|^p (x')^\gamma\,\mathrm{d}x \le 1.\end{aligned}$$

Thus, $a(s,t)$ and $\phi(s)$ satisfy (11)–(13). By Lemma 4, there is a constant C_0 depending only on p such that

$$\int_0^\infty \mathrm{e}^{-F(t)}\,\mathrm{d}t \le C_0, \tag{17}$$

where

$$F(t) = t - \left(\int_{-\infty}^\infty a(s,t)\phi(s)\,\mathrm{d}s\right)^{p'}.$$

On the other hand, from the definitions of $a(s,t)$ and $\phi(s)$, it follows that

$$\begin{aligned}F(t) &= t - \left(\int_0^t \phi(s)\,\mathrm{d}s + \int_t^\infty p\,\mathrm{e}^{(t-s)/p'}\phi(s)\,\mathrm{d}s\right)^{p'} \\ &= t - \left(\int_0^t D^{1/p} f^*_\gamma(D\,\mathrm{e}^{-s})\,\mathrm{e}^{-s/p}\,\mathrm{d}s + \int_t^\infty p\,\mathrm{e}^{(t-s)/p'} D^{1/p} f^*_\gamma(D\,\mathrm{e}^{-s})\,\mathrm{e}^{-s/p}\,\mathrm{d}s\right)^{p'}.\end{aligned}$$

By the change of variables, we have

$$F\left(\ln\frac{D}{t}\right) = \ln\frac{D}{t} - \left(\int_0^{\ln(D/t)} D^{1/p} f_\gamma^*(D\,\mathrm{e}^{-s})\,\mathrm{e}^{-s/p}\,\mathrm{d}s\right.$$
$$\left. + \int_{\ln(D/t)}^{\infty} p\,\mathrm{e}^{(\ln(D/t)-s)/p'} D^{1/p}\, f_\gamma^*(D\,\mathrm{e}^{-s})\,\mathrm{e}^{-s/p}\,\mathrm{d}s\right)^{p'}$$
$$= \ln\frac{D}{t} - (I_1 + I_2)^{p'}.$$

Here I_1 and I_2 can be written in the following form:

$$I_1 = \int_0^{\ln(D/t)} D^{1/p}\, f_\gamma^*(D\,\mathrm{e}^{-s})\,\mathrm{e}^{-s/p}\,\mathrm{d}s = \int_t^D f_\gamma^*(\tau)\tau^{-1/p'}\,\mathrm{d}\tau,$$
$$I_2 = \int_{\ln(D/t)}^{\infty} p\,\mathrm{e}^{(\ln(D/t)-s)/p'} D^{1/p}\, f_\gamma^*(D\,\mathrm{e}^{-s})\,\mathrm{e}^{-s/p}\,\mathrm{d}s$$
$$= \int_{\ln(D/t)}^{\infty} p\,\mathrm{e}^{\ln(D/t)/p'}\,\mathrm{e}^{-s/p'}\,\mathrm{e}^{-s/p} D^{1/p}\, f_\gamma^*(D\,\mathrm{e}^{-s})\,\mathrm{d}s$$
$$= \int_{\ln(D/t)}^{\infty} p\,D\,t^{-1/p'}\,\mathrm{e}^{-s}\, f_\gamma^*(D\,\mathrm{e}^{-s}$$
$$= pt^{-1/p'}\int_0^t f_\gamma^*(\tau)\,\mathrm{d}\tau.$$

Then we have

$$F\left(\ln\frac{D}{t}\right) = \ln\frac{D}{t}\left(p\,t^{-1/p'}\int_0^t f_\gamma^*(\tau)\,\mathrm{d}\tau + \int_t^D f_\gamma^*(\tau)\tau^{-1/p'}\,\mathrm{d}\tau\right)^{p'}. \tag{18}$$

Combining (16) and (17) with (18), we get

$$C_0 \ge \int_0^\infty \mathrm{e}^{-F(t)}\,\mathrm{d}t = \int_0^D t^{-1}\,\mathrm{e}^{-F(\ln(D/t))}\,\mathrm{d}t$$
$$= \int_0^D t^{-1}\exp\left\{\left(p\,t^{-1/p'}\int_0^t f_\gamma^*(\tau)\,\mathrm{d}\tau + \int_t^D f_\gamma^*(\tau)\,\tau^{-1/p'}\,\mathrm{d}\tau\right)^{p'} - \ln\frac{D}{t}\right\}\mathrm{d}t$$
$$= \frac{1}{D}\int_0^D \exp\left\{\left(p\,t^{-1/p'}\int_0^t f_\gamma^*(\tau)\,\mathrm{d}\tau + \int_t^D f_\gamma^*(\tau)\,\tau^{-1/p'}\,\mathrm{d}\tau\right)^{p'}\right\}\mathrm{d}t$$
$$\ge \frac{1}{D}\int_0^D \exp\left\{\frac{Q}{A}[(I_{\Omega,\alpha,\gamma}f)_\gamma^*(t)]^{p'}\right\}\mathrm{d}t$$
$$= \frac{1}{D}\int_{\mathcal{E}_d(0,r)} \exp\left(\frac{Q}{A}|I_{\Omega,\alpha,\gamma}f(x)|^{p'}\right)(x')^\gamma\,\mathrm{d}x,$$

i.e.

$$\frac{1}{|\mathcal{E}_d(0,r)|_\gamma}\int_{\mathcal{E}_d(0,r)} \exp\left(Q\left|\frac{I_{\Omega,\alpha,\gamma}f(x)}{\|\Omega\|_{L_{Q/(Q-\alpha),\gamma}}}\right|^{Q/(Q-\alpha)}\right)(x')^\gamma\,\mathrm{d}x \le C_0, \tag{19}$$

where

$$\|f\|_{L_{Q/\alpha,\gamma}} = 1.$$

Now consider the general case. If $\|f\|_{L_{Q/\alpha,\gamma}} \neq 1$, then we denote $g = f/\|f\|_{L_{Q/\alpha,\gamma}}$. Thus,

$$I_{\Omega,\alpha,\gamma} g(x) = \frac{I_{\Omega,\alpha,\gamma} f(x)}{\|f\|_{L_{Q/\alpha,\gamma}}}$$

and $\|g\|_{L_{Q/\alpha,\gamma}} = 1$. From (19), it follows that

$$\frac{1}{|\mathcal{E}_d(0,r)|_\gamma} \int_{\mathcal{E}_d(0,r)} \exp\left(Q \left| \frac{I_{\Omega,\alpha,\gamma} f(x)}{\|\Omega\|_{L_{Q/(Q-\alpha),\gamma}} \|f\|_{L_{Q/\alpha,\gamma}}} \right|^{Q/(Q-\alpha)} \right) (x')^\gamma \, \mathrm{d}x \leq C_0,$$

This finishes the proof of Theorem 4. ■

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